# QCD at finite temperature and density: criticality

Joaquín Grefa

September 2nd - 5th, 2025

**From Colliders to the Cosmos:**
exploring the extremes of matter with experimental and astrophysical observations

# Outline

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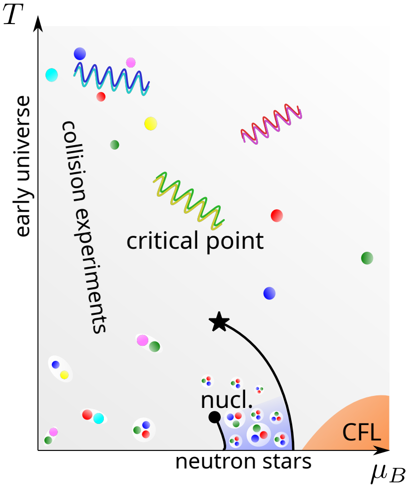
# QCD Phase Diagram

We can explore the QCD phase diagram by changing $\sqrt{s}$ in relativistic heavy ion collisions

Models predict a first order phase transition line with a critical point

Lattice QCD is the most reliable theoretical tool to study the QCD phase diagram.



# QCD Phase Diagram

We can explore the QCD phase diagram by changing $\sqrt{s}$ in relativistic heavy ion collisions

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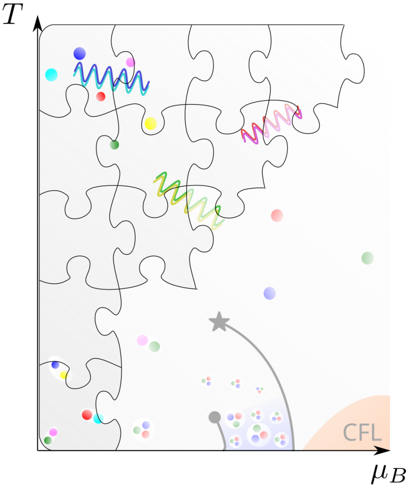
Lattice QCD is the most reliable theoretical tool to study the QCD phase diagram.

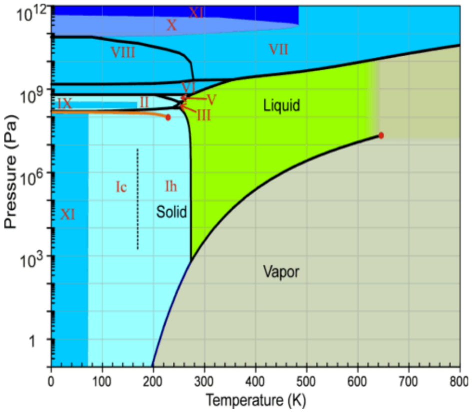
**Sign problem:**

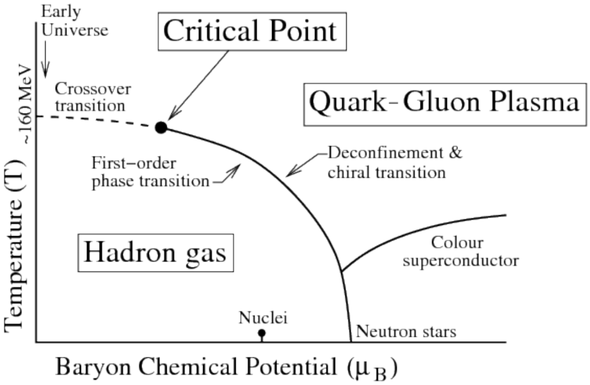
Equation of state for low to moderate $\mu_B/T$.

Borsányi, Fodor, Guenther et al., PRL **126** (2021)



# Why a critical point?



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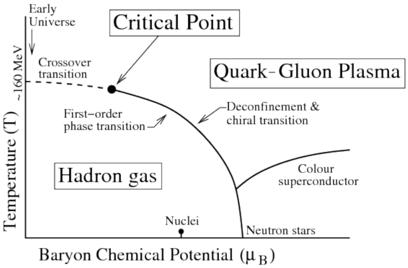


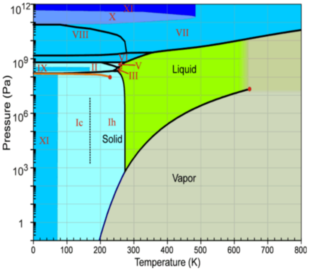


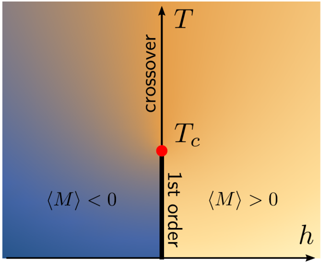

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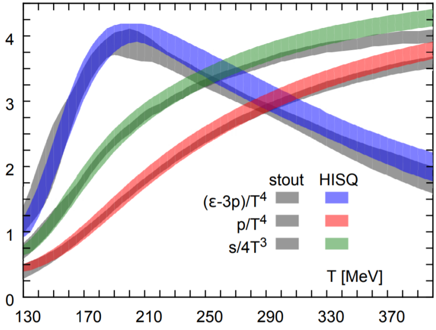
# Model Requirements

We need a simplified theoretical framework that describes QCD in the desired energy range.

Interpret data $\Longleftrightarrow$ make predictions

**Requirements:**

- QCD symmetries, degrees of freedom, thermodynamics, and/or interactions.
- Agreement with Lattice EoS at $\mu_B = 0$
- Agreement with lattice susceptibilities at $\mu = 0$




stout: S. Borsanyi et al., 1309.5258, PBL (2014)
HISQ: A. Bazavov et al., 1407.6387, PRD (2014)

# Critical point predictions as of some years ago

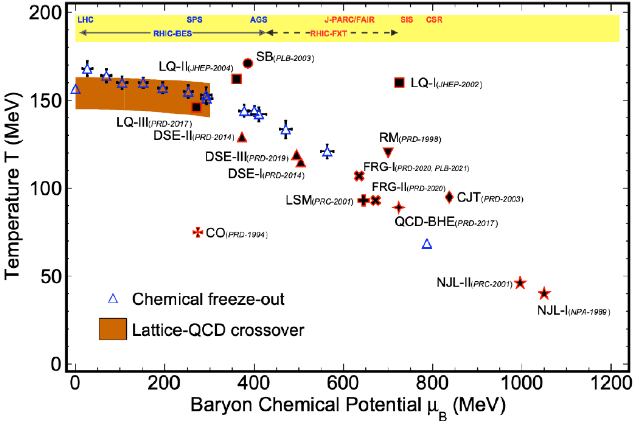



Figure adapted from A. Pandav, D. Mallick, B. Mohanty, Prog. Part. Nucl. Phys. 125 (2022)

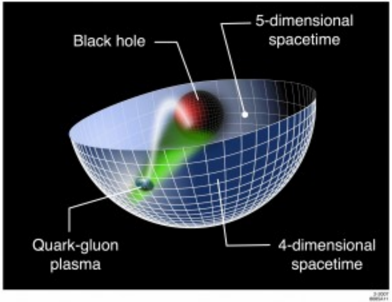
- Including the scenario of no critical point at all.
  de Forcrand, Philipsen, JHEP 01, 077 (2007); VV, Steinheimer, Philipsen, Stoecker, PRD 97, 114030 (2018)

# Holography (Gauge/String duality)

**Holographic gauge/gravity correspondence**

5D Classical Gravity with asymptotically anti-de Sitter geometry $\Longleftrightarrow$ 3+1D Strongly coupled QFT in Minkowski spacetime

Maldacena 1997; Witten 1998; Gubser, Polyakov, Klebanov 1998




- Strongly coupled, nearly perfect fluid behavior of the QGP.

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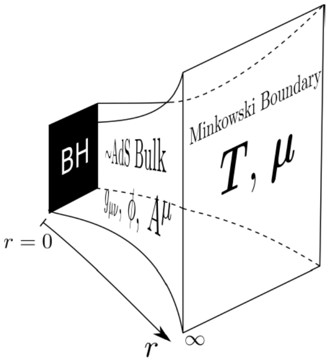


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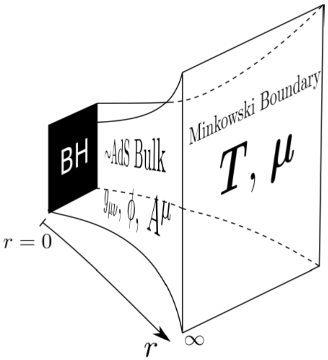


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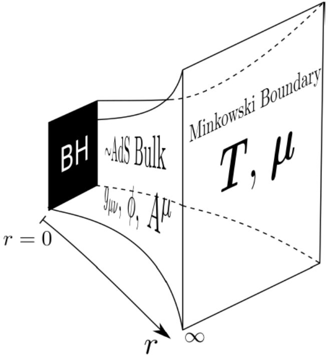


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  J. G., et al. PRD 104 (2021)

- Able to handle near and out-of-equilibrium calculations.

  S. S. Gubser et al. PRL 101, (2008), J. G. et al. PRD 106 (2022)

# Dualities

Gauge theory in 4d: $J_B^\mu(x)$ $T_{\mu\nu}(x)$ $\mathrm{Tr}\, F^2(x)$

Gravity in 5d: $A^M(x,r)$ $g_{MN}(x,r)$ $\phi(x,r)$

**Most general gravitational effective action**

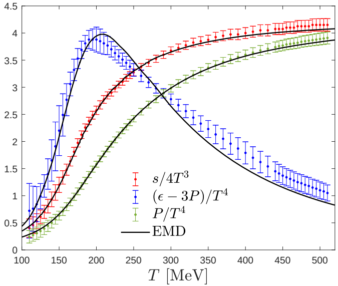
$$\mathcal{S} = \frac{1}{2\kappa^2}\int d^5x\sqrt{-g}\left[R - \frac{1}{2}(\partial_M\phi)^2 - V(\phi) - \frac{1}{4}f(\phi)F_{MN}^2\right]$$

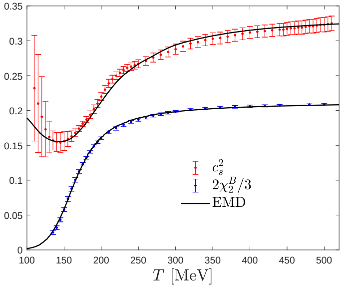
Thermodynamics in QFT ⟷ Solutions of Einstein's equations with a black hole

# Holography (Black Hole engineering - EMD model - gauge/gravity duality )

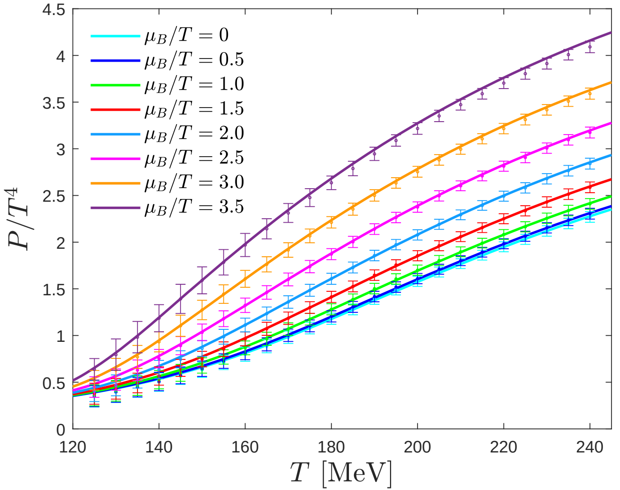
O DeWolfe et al. Phys.Rev.D 83, (2011). R Rougemont et al. JHEP(2016)102. R. Critelli et al., Phys.Rev.D96(2017).

$$S = \frac{1}{2\kappa_5^2} \int_{M_5} d^5x \sqrt{-g} \left[ R - \frac{(\partial_\mu \phi)^2}{2} - \underbrace{V(\phi)}_{\text{nonconformal}} - \underbrace{\frac{f(\phi) F_{\mu\nu}^2}{4}}_{\mu_B \neq 0} \right]$$



# Comparison with the state-of-the-art lattice QCD thermodynamics

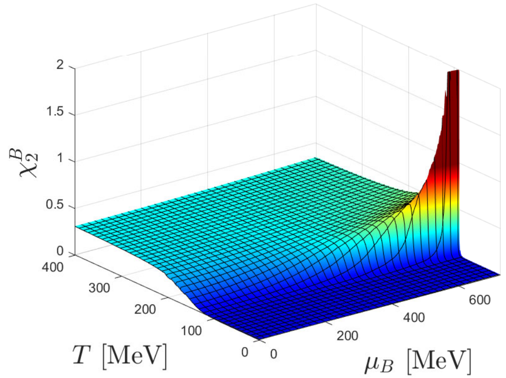


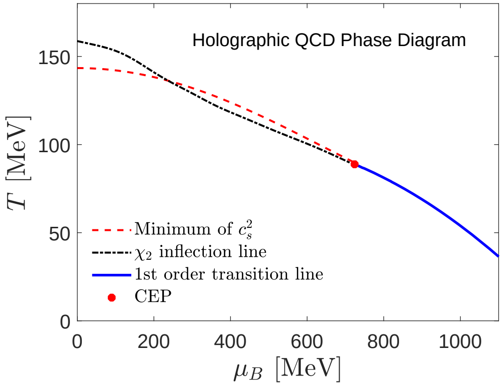

Lattice results: S. Borsanyi et al. 10.1103/PhysRevLett.126.232001

# Locating the Critical End Point (CEP)

$T_{CEP} = 89$ MeV

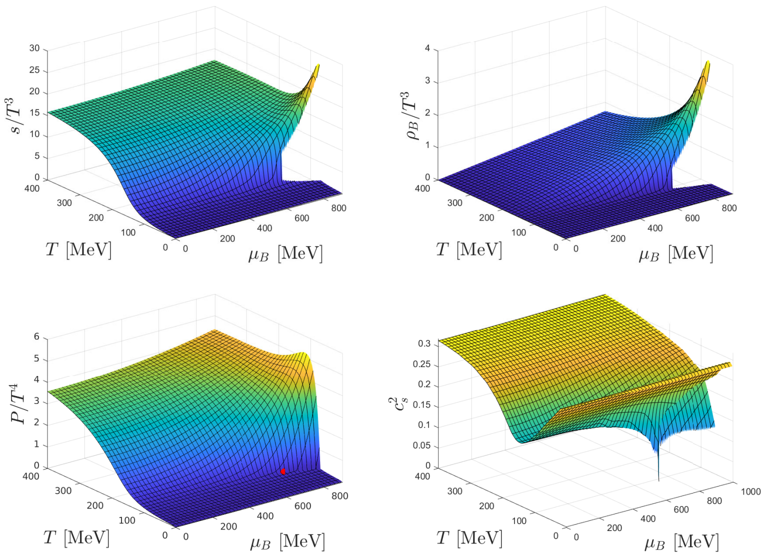
$\mu_B^{CEP} = 724$ MeV




BH curves: J. G et al. PRD.104 (2021)

# Equation of State

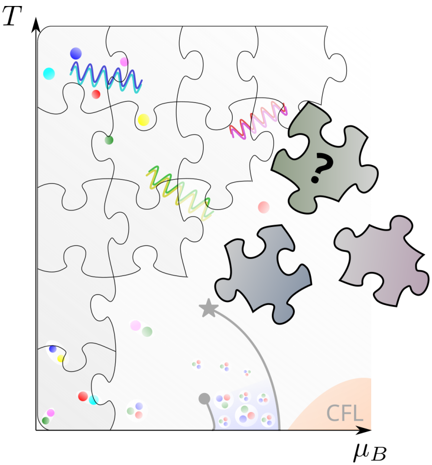



- Entropy density and baryon density exhibit a gap that corresponds to the line of first order phase transition.
- Critical point localized (see pressure)
- The minimum on $c_2^2$ corresponds to the location of the critical point.

plots: J. G et al. PRD.104 (2021)

# Bayesian black-hole engineering

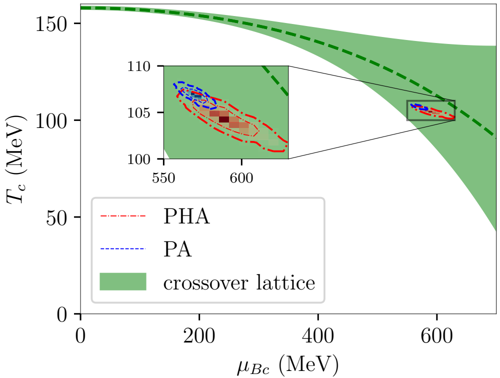
- How do lattice results constrain phase diagram/critical point?
- Systematic scan over possible extrapolations to higher densities.
- Bayesian black-hole engineering: what scenarios described by model compatible with the lattice results + error bars.

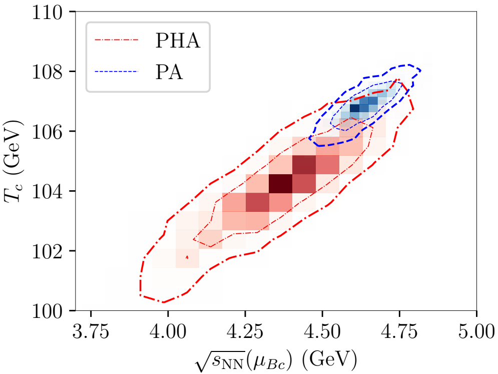



M. Hippert, J.G., T.A. Manning. J. Noronha, J. Noronha-Hostler, I. Portillo, C. Ratti, R. Rougemont, M. Trujillo, arXiv:2309.00579.

# Holographic Bayesian Analysis: posterior critical points

$(T_c,\, \mu_{Bc})_{PHA} = (104 \pm 3,\, 589^{+36}_{-26})$ MeV, $(T_c,\, \mu_{Bc})_{PA} = (107 \pm 1,\, 571 \pm 11)$ MeV.




- Both Ansätze overlap at $1\,\sigma$. **Robust results!**

M. Hippert, J.G., T.A. Manning. J. Noronha, J. Noronha-Hostler, I. Portillo, C. Ratti, R. Rougemont, M. Trujillo, arXiv:2309.00579.

# Functional methods

Based on the truncated expansion of the QCD functional, and requires control of errors to compute thermodynamics.

**Dyson–Schwinger Equations (DSE)**

- integral equations derived from the QCD action that relate the propagators (Green's functions) of quarks and gluons to each other.
- are solved approximately using truncation schemes and modeling of the interaction vertices.

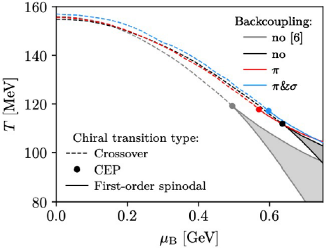
**Functional renormalization group (FRG)**

- describing how the effective QCD action changes as one varies the energy scale.
- allows for a continuous evolution from microscopic physics to macroscopic phenomena, capturing quantum, thermal, and density fluctuations along the way.

# Effective QCD theories prediction

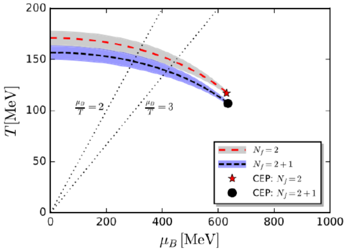
- Different effective approaches, all in excellent agreement with lattice QCD at $\mu_B = 0$ (and $\mu_B/T \sim 3.5$), predict the location of the critical point in a similar region.
- If true, reachable in heavy ion collisions at $\sqrt{s_{NN}} \sim 3 - 5$ GeV.

**Dyson-Schwinger equations**



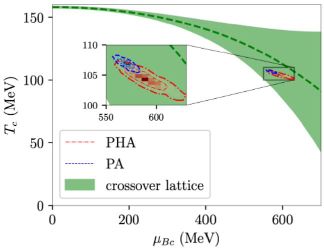

Gunkel, Fischer, PRD 104, 054202 (2021)

**Functional renormalization group**




Fu, Pawlowski, Rennecke, PRD 101, 053032 (2020)

**Black-hole engineering**




Hippert, J.G., et al., PRD 110, 094006 (2024)

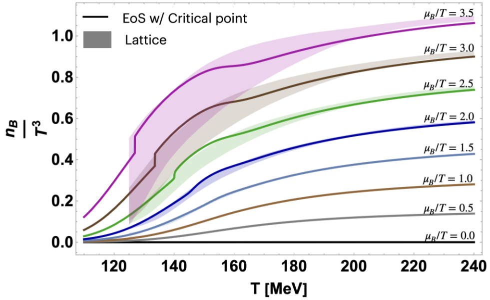
# EoS constructions that includes a critical point

## $T'$ alternative expansion scheme

$$T\frac{\chi_1^B(T,\mu_B)}{\mu_B} = \chi_2^B(T',0)$$

$$T'(T,\mu_B) = T\left[1 + \kappa_2^{BB}(T)\left(\frac{\mu_B}{T}\right)^2 + \kappa_4^{BB}(T)\left(\frac{\mu_B}{T}\right)^4 + \dots\right]$$

Borsanyi et al., PRL 126, 232001 (2021)

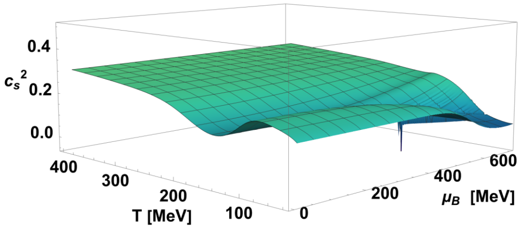



## 2D Ising T.Ex.S

$10 < T < 800$ MeV; $\mu_B < 700$ MeV

M. Kahangirwe, et al., PRD 109 (2024)

- Using the the new alt. exp., a CP can be mapped from the 3D Ising model to to study criticality in a QCD EoS.



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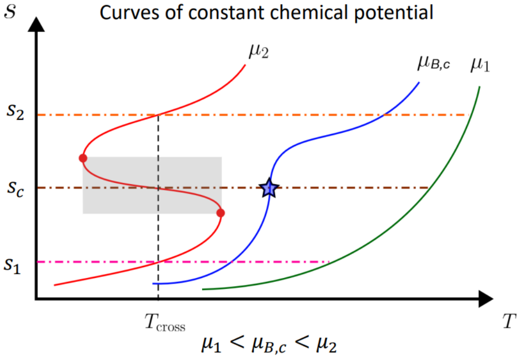
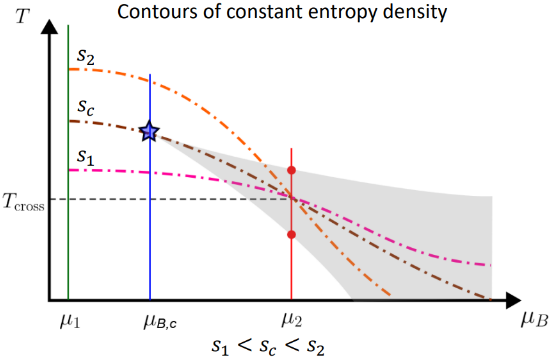
# Entropy density contours

- If lines of constant $s$ cross, it suggests a first order phase transition with a CP; otherwise, the EoS only exhibits a crossover.

$$T_s\left(\mu_B; T_0\right) = T_0 + \alpha_2\left(T_0\right)\frac{\mu_B^2}{2}; \qquad \alpha_2\left(T_0\right) = -\frac{2T_0\chi_2^B(T_0)+T_0^2\chi_2^{B\prime}(T_0)}{s'(T_0)}; \qquad \chi_2^B = \left[\frac{\partial^2(p/T)^4}{\partial(\mu_B/T)^2}\right]_T$$



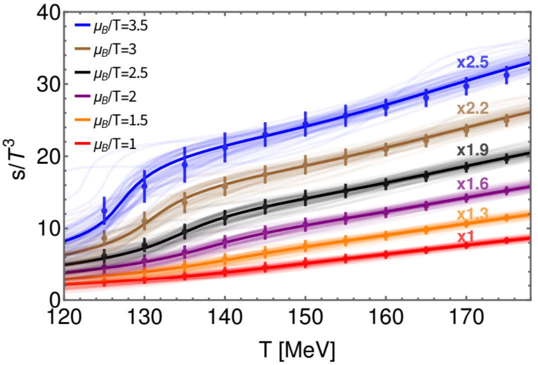



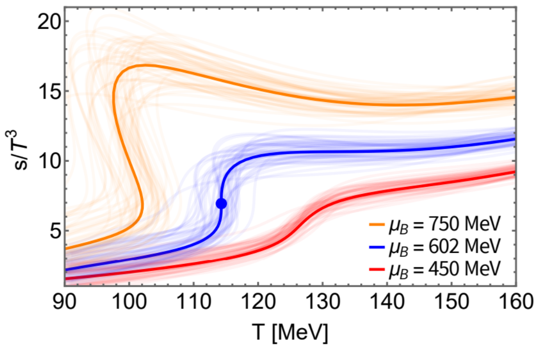

H. Shah et al. arXiv:2410.16026

# Entropy density contours

- Excelent agreement with state-of-the-art lattice QCD data up to $\mu_B/T = 3.5$
  Borsanyi et al., PRL 126, 232001 (2021)

$$\mu_B^c = 602 \pm 62 \text{ MeV} \qquad T^c = 114 \pm 7 \text{ MeV}$$

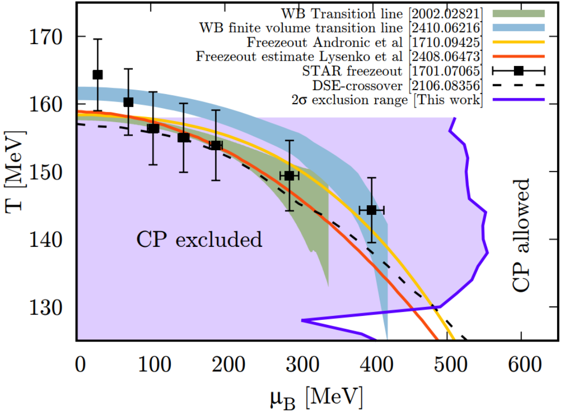



H. Shah et al. arXiv:2410.16026

# Constraints with improved lattice data

- New continuum extrapolated equation of state at zero density with improved precision & new data at imaginary chemical potential
- CP excluded at $\mu_B <\sim 450$ MeV




Borsanyi et al. arXiv:2502.1026

# Critical point predictions as of some years ago

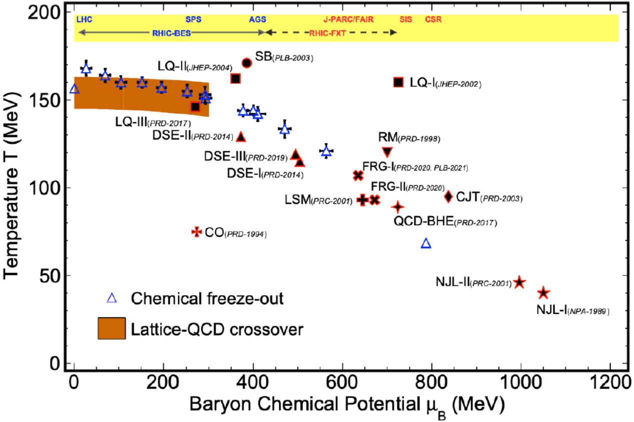
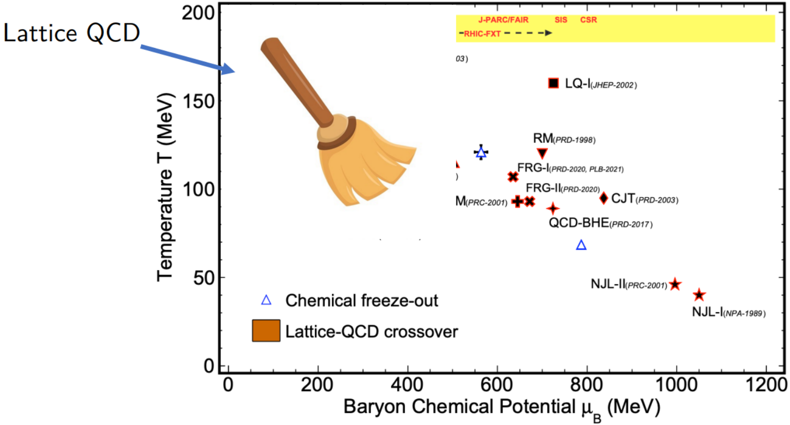



Figure adapted from A. Pandav, D. Mallick, B. Mohanty, Prog. Part. Nucl. Phys. 125 (2022)

- Including the scenario of no critical point at all.
  de Forcrand, Philipsen, JHEP 01, 077 (2007); VV, Steinheimer, Philipsen, Stoecker, PRD 97, 114030 (2018)

# Critical point with new lattice constraints: no CP at $mu_B < 450$ MeV

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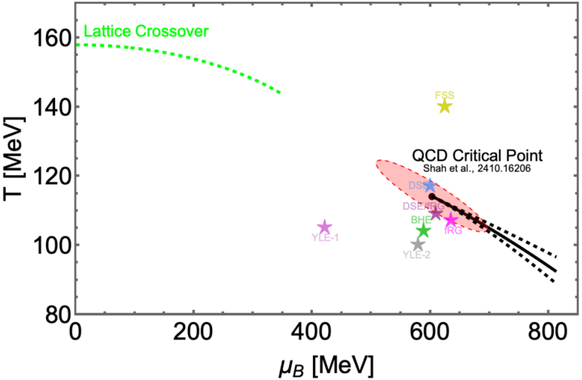
# Current scenario

- Predictions converge to the same region...




Adapted from H. Shah et al. arXiv:2410.16026

Critical point estimate at $O(\mu_B^2)$:

$T_c = 114 \pm 7$ MeV, $\quad \mu_B = 602 \pm 62$ MeV

**Estimates from recent literature:**

YLE-1: D.A. Clarke et al. (Bielefeld-Parma), arXiv:2405.10196

YLE-2: G. Basar, PRC 110, 015203 (2024)

BHE: M. Hippert et al., arXiv:2309.00579

fRG: W-J. Fu et al., PRD 101, 054032 (2020)

DSE/fRG: Gao, Pawlowski., PLB 820, 136584 (2021)

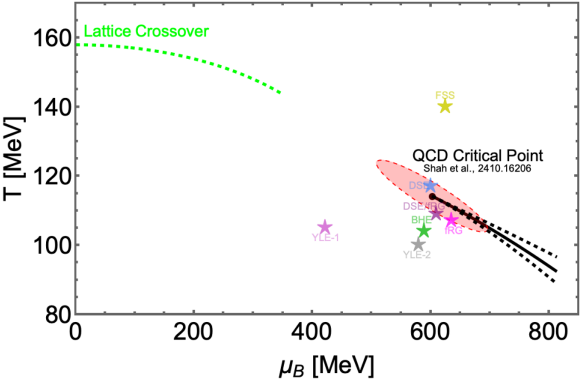
DSE: P.J. Gunkel et al., PRD 104, 052022 (2021)

FSS: A. Sorensen et al., arXiv:2405.10278

# Current scenario

- Predictions converge to the same region...

  because lattice QCD has not ruled out that region yet?

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# Fluctuations

Cumulants measure the chemical potential derivatives of the QCD equation of state

**Cumulants as moments of the particle number distribution**

variance: $\kappa_2 = \langle (\Delta N)^2 \rangle = \sigma$

skewness: $\kappa_3 = \langle (\Delta N)^3 \rangle$

kurtosis: $\kappa_4 = \langle (\Delta N)^4 \rangle - 3 \langle (\Delta N)^2 \rangle^2$

$$\kappa_2 \sim \xi^2, \quad \kappa_3 \sim \xi^{4.5}, \quad \kappa_4 \sim \xi^7$$

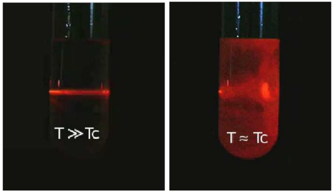
$$\xi \to \infty$$

**Cumulants as chemical potential derivatives of the EoS**

$$\ln Z(T, V, \mu) = \ln \left[ \sum_N e^{\mu N/T} Z^{ce}(T, V, N) \right]$$

$$\kappa_n \propto \frac{\partial^n (\ln Z)}{\partial \mu^n}$$

**Critical opalescence**



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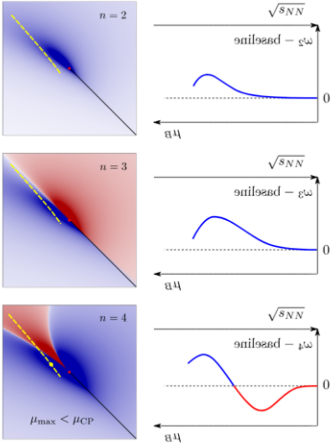
kurtosis: $\kappa_4 = \langle (\Delta N)^4 \rangle - 3 \langle (\Delta N)^2 \rangle^2$

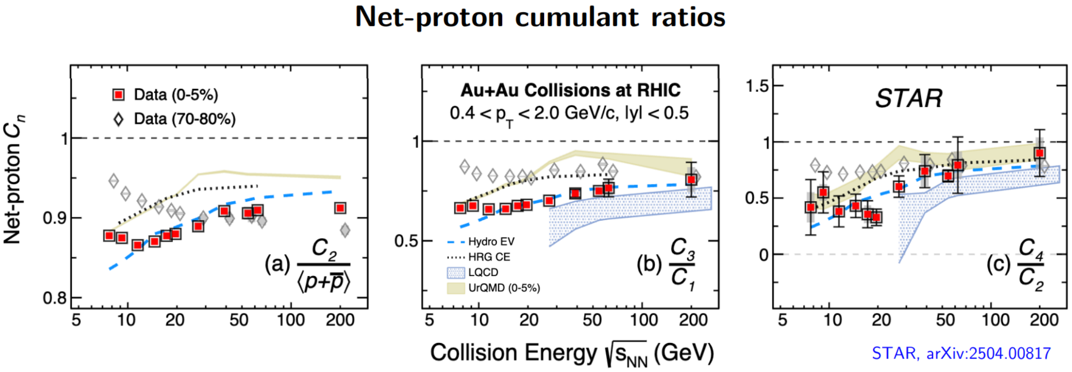
$$\kappa_2 \sim \xi^2, \quad \kappa_3 \sim \xi^{4.5}, \quad \kappa_4 \sim \xi^7$$

$$\xi \to \infty$$




M. Stephanov. SQM 2024

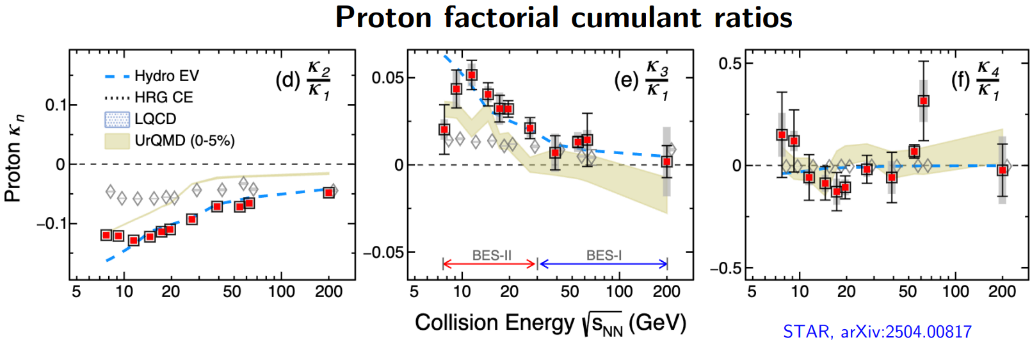
- Overall agreement with the baseline for $\sqrt{s_{NN}} \sim 10 - 20$ GeV




STAR, arXiv:2504.00817

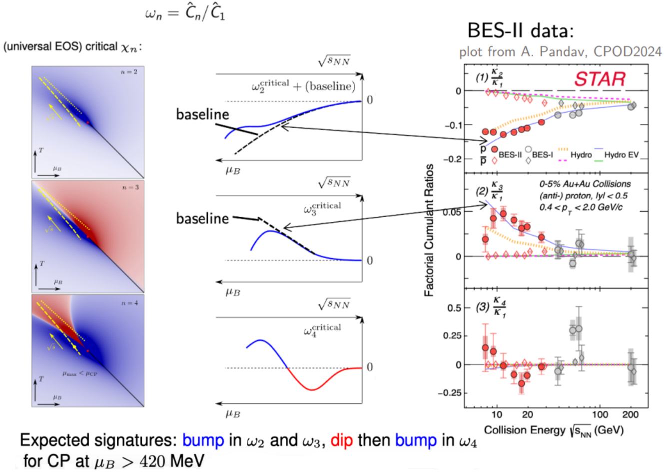
# Factorial cumulants!

- Exhibit more structure



# factorial cumulants

V. Vovchenko, arXiv:2504.01368, and adapted from Stephanov, arXiv:2410.02861




non monotonic $\kappa_2/\kappa_1$, $\kappa_3/\kappa_1$ and maybe $\kappa_4/\kappa_1$

Factorial cumulants: Irreducible $n$-particle correlations that remove Poisson contribution and probe genuine correlations

Ordinary cumulants: mix correlations of different order

$\hat{C}_n \sim \langle N(N-1)(N-2)\ldots\rangle_c$

$$\hat{C}_1 = C_1$$
$$\hat{C}_2 = C_2 - C_1$$
$$\hat{C}_3 = C_3 - 3C_2 + 2C_1$$
$$\hat{C}_4 = C_4 - 6C_3 + 11C_2 - 6C_1$$

Bzdak, Koch, Strodthoff, PRC 95, 054906 (2017)

# More fluctuations! $p_T$ fluctuations

**Mean $p_T$ fluctuations:**

$$\langle \Delta p_{T,i} \Delta p_{T,i} \rangle \sim \langle \Delta \langle p_T \rangle^2 \rangle$$

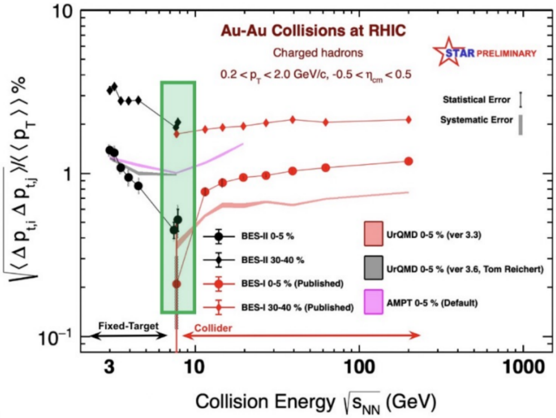
Mean $p_T$ probes the temperature

Gardim et al, Nature Phys. (2020)

$$\langle p_T \rangle \propto T_{eff}$$

$$\langle \Delta p_{T,i} \Delta p_{T,i} \rangle \sim \langle \Delta T^2 \rangle$$

**In equilibrium:** $\langle \Delta T^2 \rangle = \frac{T^2}{V c_V}$




R. Manikandhan (STAR), QM2025

At the critical point $c_V \to \infty$ → Minimum in $\sqrt{s_{NN}}$ dependence?

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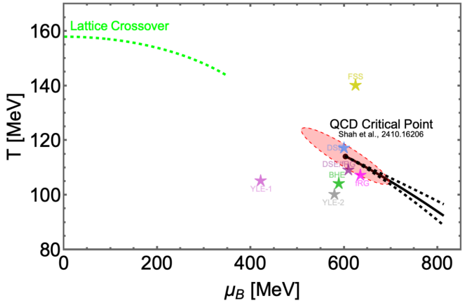
# Summary




Critical point estimate at $O(\mu_B^2)$:

$T_c = 114 \pm 7$ MeV, $\mu_B = 602 \pm 62$ MeV

**Estimates from recent literature:**

YLE-1: D.A. Clarke et al. (Bielefeld-Parma), arXiv:2405.10196

YLE-2: G. Basar, PRC 110, 015203 (2024)

BHE: M. Hippert et al., arXiv:2309.00579

fRG: W-J. Fu et al., PRD 101, 054032 (2020)

DSE/fRG: Gao, Pawlowski., PLB 820, 136584 (2021)

DSE: P.J. Gunkel et al., PRD 104, 052022 (2021)

FSS: A. Sorensen et al., arXiv:2405.10278

- No indication of critical behavior from lattice QCD for $\mu_B < 450$ MeV.
- Several effective theories predict the location of the critical point to $T \sim 90 - 120$ MeV and $\mu_B \sim 500 - 650$ MeV.
- No critical behavior describe proton cumulants at $\sqrt{s_{NN}} \geq 20$ GeV.
- This trend changes around $\sqrt{s_{NN}} \sim 10$ GeV; in particular, for factorial cumulants and the presence of the CP could be a reasonable explanation.

# Appendix

# Ordinary vs. Factorial Cumulants in Heavy-Ion Collisions

**Ordinary Cumulants**

$C_n \sim \langle \delta N^n \rangle$

- Built from moments of distribution (mean, variance, etc).
- Sensitive to critical fluctuations.
- Connected to thermodynamic susceptibilities.
- Higher orders diverge near critical point.

**Factorial Cumulants**

$\hat{C}_n \sim \langle N(N-1)(N-2)\ldots \rangle_c$

- Built from factorial moments.
- Vanish for Poisson baseline $\Rightarrow$ better contrast.
- More robust under detection inefficiencies.
- Useful in experimental fluctuation analyses.

# Which Cumulants Are Better for the QCD Critical Point?

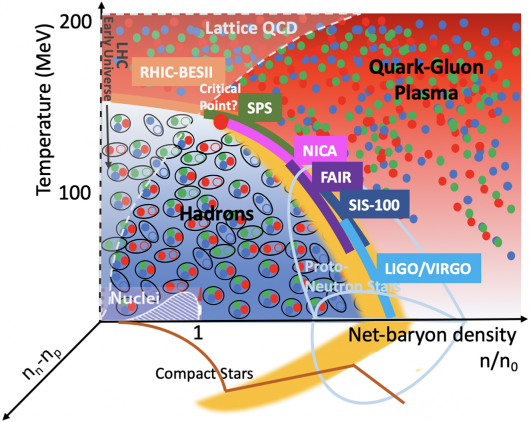
**Both are useful, but serve different purposes:**

- **Ordinary cumulants:** $C_n \sim \langle \delta N^n \rangle$
  - Theoretically well-defined.
  - Connected to QCD susceptibilities.
- **Factorial cumulants:** $\hat{C}_n \sim \langle N(N-1)(N-2)\ldots \rangle_c$
  - Cleaner signals under real-world detector conditions.
  - Efficient background suppression (e.g., Poisson noise).
- *Best approach: use both and compare.*

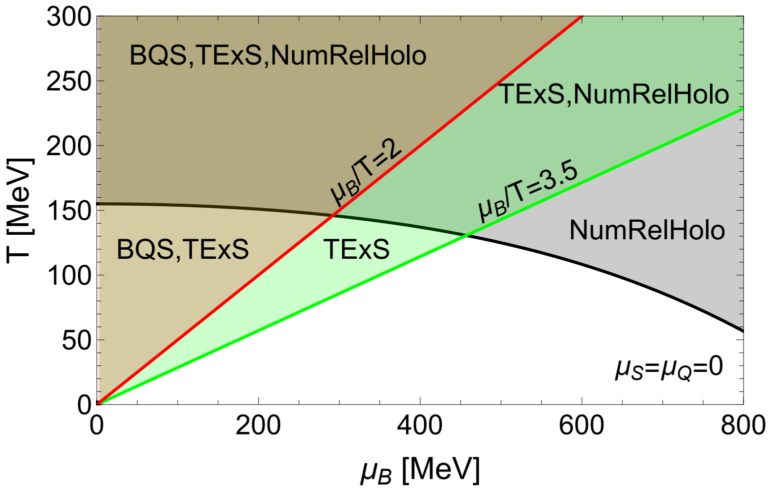
See: Bzdak et al. Phys. Rept. 853 (2020), Kitazawa Asakawa PRC 85 (2012), STAR Collaboration (2022)

# What happens at finite/large densities?

- We need to merge the lattice QCD EoS with other effective theories.
- Study the regime of validity of each effective model.
- Constrained internal parameters to adhere know experimental and theoretical limits.
- Test models to validate/exclude them.
- EoS to guide/interpret experimental data.



# Range of validity for the MUSES Heavy Ion EoS'




M. Reinke Pelicer et al. *in preparation*

## BQS EoS

## $10 < T < 600$ MeV; $\mu_B < 450$ MeV

J. Noronha-Hostler, et al., PRC (2019)

$$\frac{p(T,\mu_B,\mu_Q,\mu_S)}{T^4} = \sum_{i,j,k} \frac{1}{i!j!k!} \chi_{ijk}^{BQS} \left(\frac{\mu_B}{T}\right)^i \left(\frac{\mu_Q}{T}\right)^j \left(\frac{\mu_S}{T}\right)^k$$

$$\chi_{ijk}^{BQS} = \left.\frac{\partial^{i+j+k}(p/T^4)}{\partial\left(\frac{\mu_B}{T}\right)^i \partial\left(\frac{\mu_Q}{T}\right)^j \partial\left(\frac{\mu_S}{T}\right)^k}\right|_{\mu_B,\mu_Q,\mu_S=0}$$

## 2D Ising T.Ex.S

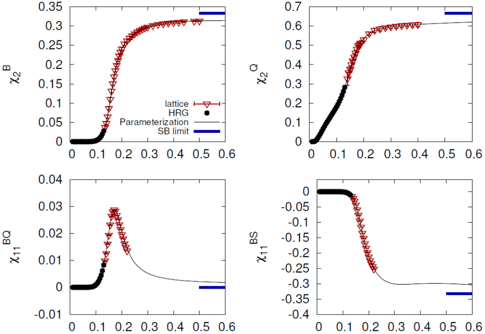
## $10 < T < 800$ MeV; $\mu_B < 700$ MeV

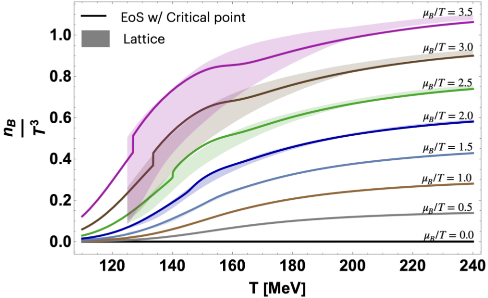
M. Kahangirwe, et al., PRD 109 (2024)

$$T\frac{\chi_1^B(T,\mu_B)}{\mu_B} = \chi_2^B(T',0)$$

$$T'(T,\mu_B) = T\left[1 + \kappa_2^{BB}(T)\left(\frac{\mu_B}{T}\right)^2 + \kappa_4^{BB}(T)\left(\frac{\mu_B}{T}\right)^4 + \dots\right]$$

- Includes a 3D Ising model critical behavior into a lattice alternative expansion EoS.



## HRG Model

$0 < T < 160$ MeV; $\mu_B < 1000$ MeV

V. Vovchenko, CPC (2019)

- Provides a realistic hadronic EoS at low $T$
- Interacting hadrons can be modeled by an ideal gas of resonances.
- For a realistic EoS at higher densities, Van der Waals interactions are added.
- Describes the liquid-gas phase transition.

## Holography (NumRelHolo)

$40 < T < 400$ MeV; $\mu_B < 1200$ MeV

J. G., et al., PRD (2021), PRD (2022)

- Based on the gauge/gravity duality and constrained to reproduce lattice-QCD thermodynamics
- Large coverage of the EoS in the strongly-interacting regime.
- Predicts the location of the QCD CP.