# Dynamics of Fluctuations

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# Why dynamic modeling?

We do not expect that we can directly compare experimental data with equilibrium expectations for suceptibilities:

- Finite size effects, rapid expansion.
- Non-equilibrium effects (critical slowing, memory).
- Consistent treatment of fluctuations and dissipation.
- Freezeout, resonances, global charge conservation, etc.

# Hydrodynamics of fluctuations

**Stochastic**

Random hydro variables: $\breve{\psi}$

$$\partial_t \breve{\psi} = -\nabla \cdot \Big( \text{Flux}[\breve{\psi}] + \text{ Noise} \Big)$$

$+$ fewer variables and eqs.

$-$ cutoff dependence

**Deterministic**

$\psi \equiv \langle \breve{\psi} \rangle$, $G \equiv \langle \breve{\psi}\breve{\psi} \rangle$, etc.

$$\partial_t \psi = -\nabla \cdot \text{Flux}[\psi, G];$$

$$\partial_t G = \mathsf{L}[G; \psi].$$

$-$ more variables and eqs.

$+$ no cutoff dependence
(renormalization)

# Deterministic approach to non-Gaussian fluctuations

- *Infinite* hierarchy of coupled equations *An et al 2009.10742 PRL*

  for hydrodynamic correlators $H_n \equiv \langle\underbrace{\delta\breve{\psi}\ldots\delta\breve{\psi}}_{n}\rangle^{\text{connected}}$:

$$\partial_t H_n = -n\Gamma(H_n - \bar{H}_n[\psi, H, \ldots, H^{n-1}]) + \mathcal{O}(\varepsilon^n);$$

  To leading order, the equations are iterative and "linear".

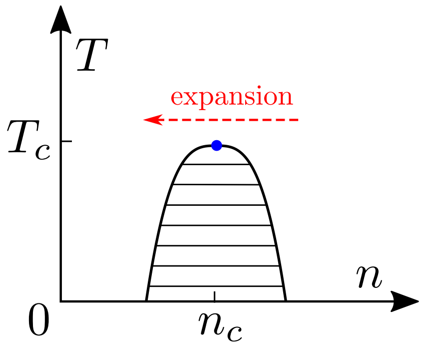
- Generalized Wigner transform:

$$H(x_1, \ldots, x_n) \to W(x; \boldsymbol{q}_1, \ldots, \boldsymbol{q}_n)$$

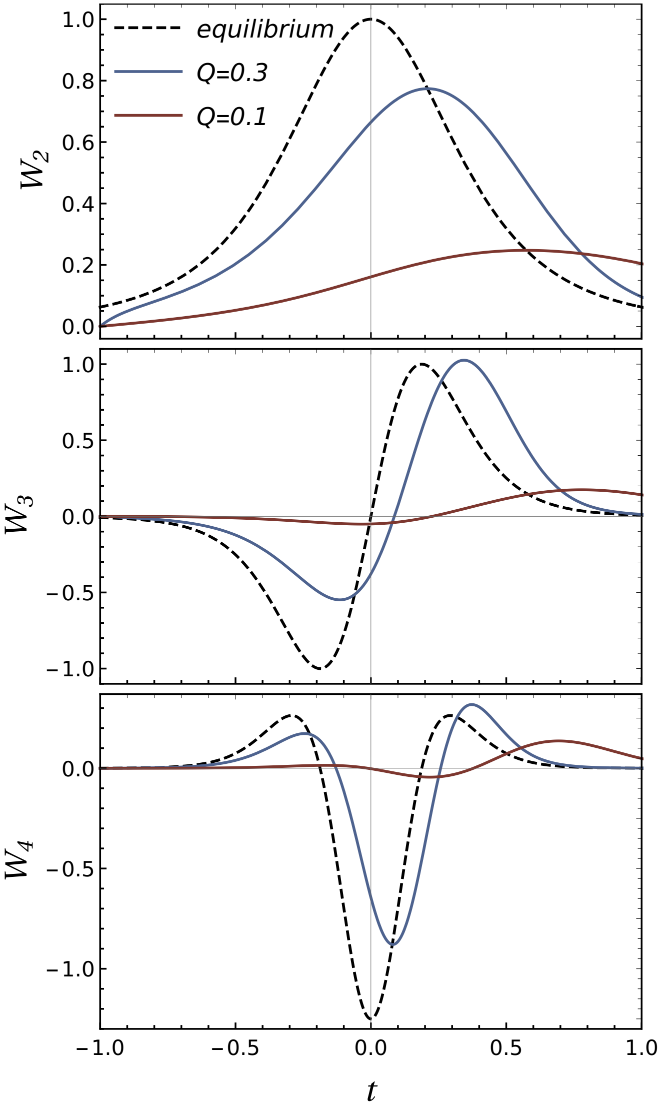
  $W_n$'s quantify magnitude and non-gaussianity of fluctuation harmonics with wave-vectors $\boldsymbol{q}_i$.

# Example: expansion through a critical region

*An et al 2009.10742, PRL*




- Two main features:
  - Lag, ”memory”.
  - Smaller $Q$ – slower evolution. Conservation laws.



# Stochastic Hydrodynamics

A. De, C. Shen, J. Kapusta, PRC 106 (2022) 5, 054903
A. Sakai, K. Murase, T. Hirano, PRC 102 (2020) 6, 064903
M. Nahrgang, M. Bluhm, T. Schäfer, S. Bass, APPS 10 (2017) 687
M. Nahrgang, M. Bluhm, T. Schäfer, S. Bass, PRD 99 (2019) 11, 116015

## Ordinary Hydro + Stochastic Noise

$$\partial_\mu T^{\mu\nu} = 0, \quad T^{\mu\nu} = T^{\mu\nu}_{\text{ideal}} + T^{\mu\nu}_{\text{viscous}} + S^{\mu\nu}_{\text{noise}}\,,$$
$$\partial_\mu J^{\mu} = 0, \quad J^{\mu} = J^{\mu}_{\text{ideal}} + J^{\mu}_{\text{viscous}} + I^{\mu}_{\text{noise}}\,.$$

- Fluctuation-Dissipation Relation (FDR)

$$\langle I^\mu(x_1) I^\nu(x_2)\rangle = 2T\sigma\Delta^{\mu\nu}\delta^{(4)}(x_1 - x_2)$$

J. Kapusta, B. Muller, M. Stephanov, PRC 85 (2012) 054906

## Advantages (comparing with hydro-kinetics)

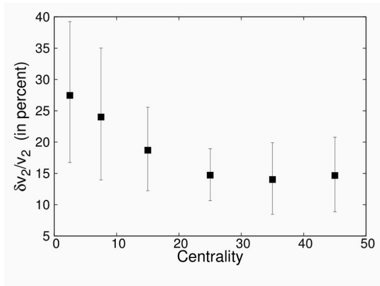
- Directly relates to final observables
- Easily incorporated with various IC models or mechanisms

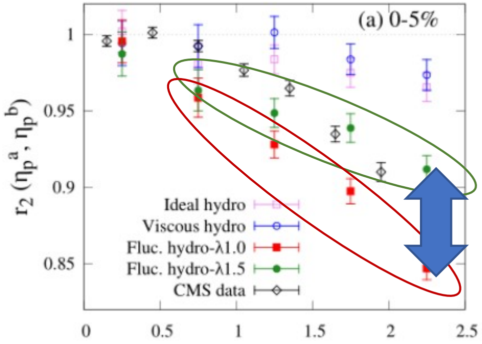
## Challenges

- Instabilities (large gradients, negative density, multiplicative problem, … )
- Results are sensitive on grid-size or cut-off
- Requires large statistics (with IC and afterburner fluctuations)

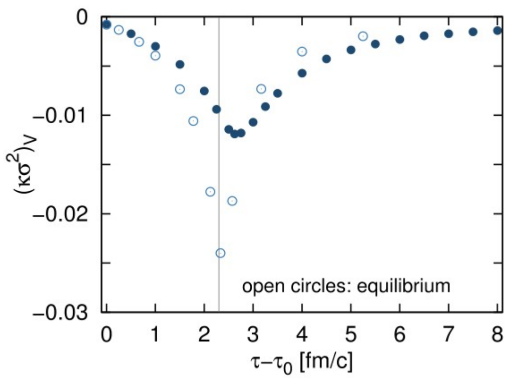
## Recent progress

- Non-critical flow (Murase, De, Chun, etc.)




- Critical observables (Nahrgang, etc.)



# Linearized Stochastic Charge Transport

**For LO, the evolution of $T^{\mu\nu}$ and $N^{\mu}$ can be calculated separately**

## Charge-neutral Background

Energy-momentum conservation

$$\boxed{\partial_\mu T^{\mu\nu} = 0}$$

$$T^{\mu\nu} = e u^\mu u^\nu - P\Delta^{\mu\nu} + \Pi^{\mu\nu}$$

## Baryon Fluctuation + Transportation

$$\boxed{\partial_\mu N_i^\mu = 0}$$

Charge conservation

$$N_i^\mu = n_i u^\mu + q_i^\mu$$

Diffusion with noise

$$\boxed{\Delta^{\mu\nu} D q_\nu = -\frac{1}{\tau_q}(q^\mu - q^\mu_{NS} - \xi^\mu)}$$

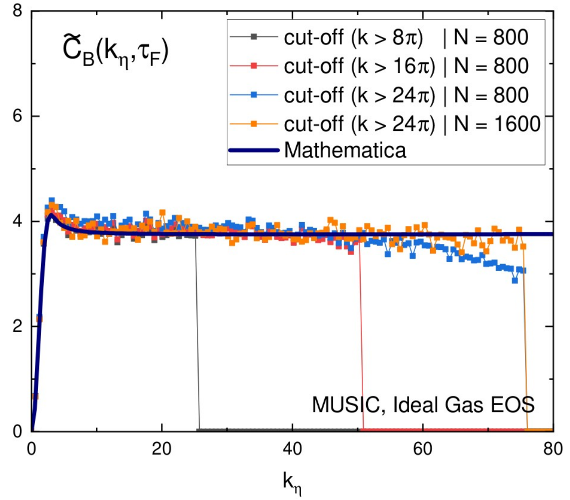
$$\langle \xi^\mu(x)\xi^\nu(x')\rangle = 2\kappa_B \Delta^{\mu\nu}\delta^{(4)}(x - x')$$

N. Borghini, BF, S. Schlichting, Phys. Rev. D 111 (2025) 11, 116019

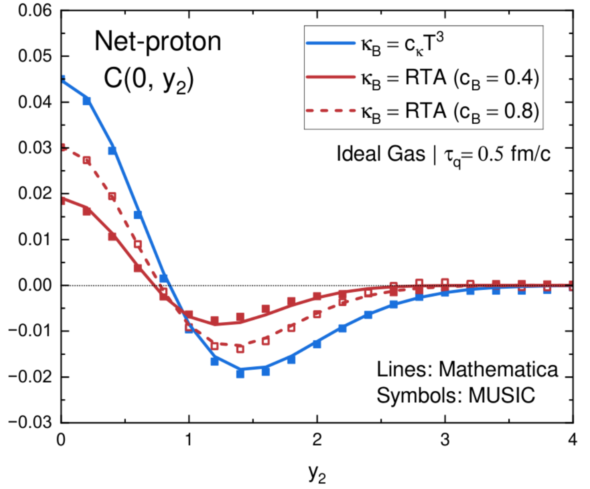
### Application in 3+1-d MUSIC

- **Stable:** based on hydro K-T algorithm
- **Accurate:** independent of cut-off momentum or grid size
- **Fast:** saves ~98% computation time comparing with traditional fluctuating hydro.

MUSIC 2PC independent of cut-off or grid size




Net-proton correlation sensitive on $\kappa_B$

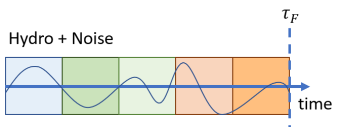


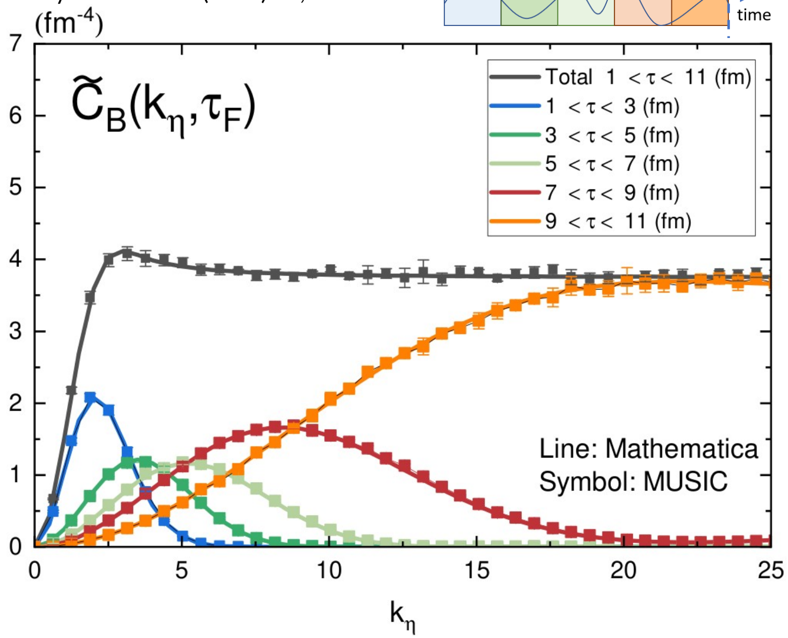
# 2 Point Correlation Results in Fourier Space

## (Noise emerge) time ordering of 2PC

- Long range correlation – early noise (diffusion)
- Short range correlation – late noise (equilibrium)
- Experimental measurements mostly reflects eq. fluctuations

N. Borghini, BF, S. Schlichting
Phys. Rev. D 111 (2025) 11, 116019

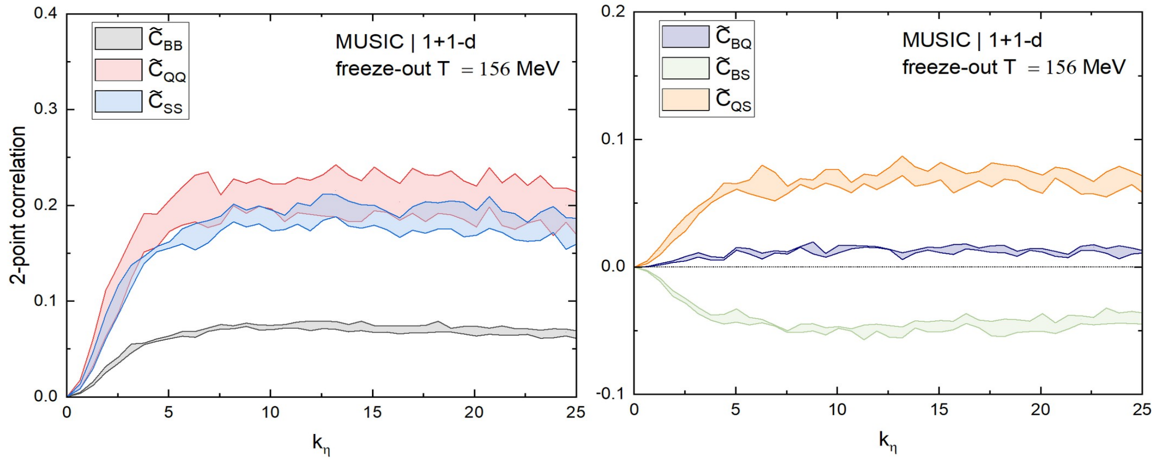



## Diagonal and off-diagonal correlations of {B,Q,S}

- Significant QQ and SS correlations
- Sizeable off-diagonal correlations: mixed charge transport
- Model uncertainties in quantitative level
- Future: n-point correlation and charge conserved frz-out

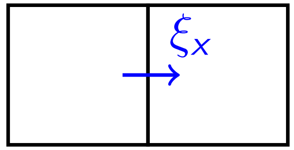
N. Borghini, BF, S. Schlichting, In progress



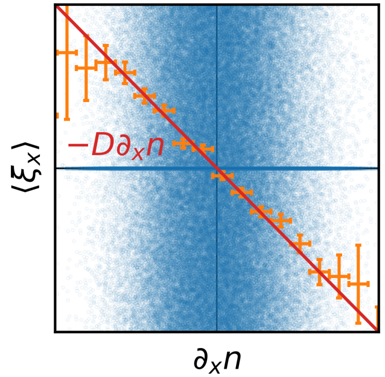
# Stochastic Hydrodynamics via Metropolis algorithm

## Simple Stochastic Diffusion Example

- $$\partial_t n = -\vec{\nabla}(\underbrace{-D\vec{\nabla}n}_{\text{Diffusion}} + \underbrace{\vec{v}n}_{\text{Advection}})$$




- Thermal fluctuations $\propto \exp[-\beta H]$ with $H = \int_{\vec{x}} \frac{n^2}{2\chi}$
  - Replace $-D\vec{\nabla}n \to \vec{\xi}$ with $\langle \xi_k \rangle = -D\partial_k n$
  - Fluctuation-dissipation theorem $\langle \xi_k(t,x)\xi_l(t',x')\rangle - \langle \xi \rangle^2 = 2TD\chi\delta_{kl}\delta(x-x')\delta(t-t')$




- Numerically: Ideal adv. dynamics + Metropolis update
  - Propose Gaussian $\xi_k$
  - Accept/Reject step $p_{acc} = \min(1, \exp(-\beta\Delta H))$
  - Automatically $\langle \vec{\xi} \rangle = \int D[\vec{\xi}]\, \vec{\xi}\, p_{\text{prop}}\, p_{\text{acc}} = -D\vec{\nabla}n + \mathcal{O}(\Delta t^3)$

# Stochastic Hydrodynamics via Metropolis algorithm

Advantages: Metropolis regulates $\delta^{(4)}(x-x') \sim 1/(\Delta t \Delta x^3)$, ensures equilibration under all conditions, avoids discretization ambiguities

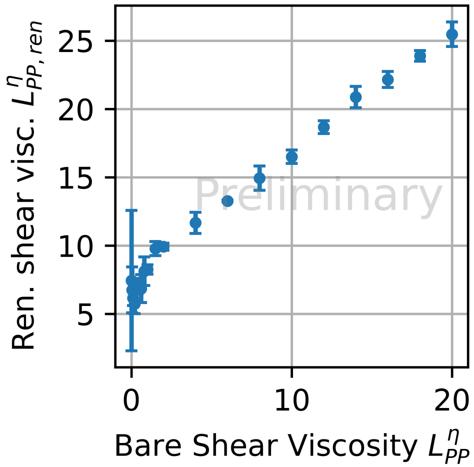
Problem: Noisy configurations difficult for numerical PDE solvers (ideal step)
$\rightsquigarrow$ No exact entropy conservation in ideal step

Fluctuations and non-linear interactions:

$$\rightsquigarrow \langle T^{\mu\nu}[\phi]\rangle = T^{\mu\nu}\left[\langle\phi\rangle\right] + \frac{\delta^2 T^{\mu\nu}}{\delta\phi_a \delta\phi_b}\langle\delta\phi_a(x)\delta\phi_b(x)\rangle + \ldots$$

Renormalization of transport coefficients, equation of state

Right: Renormalization of shear viscosity in full two-dimensional non-relativistic hydrodynamics



# Hydrodynamic equation for critical mode

Equation of motion for critical mode $\phi$ ("model H")

$$\frac{\partial\phi}{\partial t} = \kappa\,\nabla^2\frac{\delta\mathcal{F}}{\delta\phi} - g\left(\vec{\nabla}\phi\right)\cdot\frac{\delta\mathcal{F}}{\delta\vec{\pi}^T} + \zeta \qquad (g=1)$$

Diffusion Advection Noise

Equation of motion for momentum density $\pi$

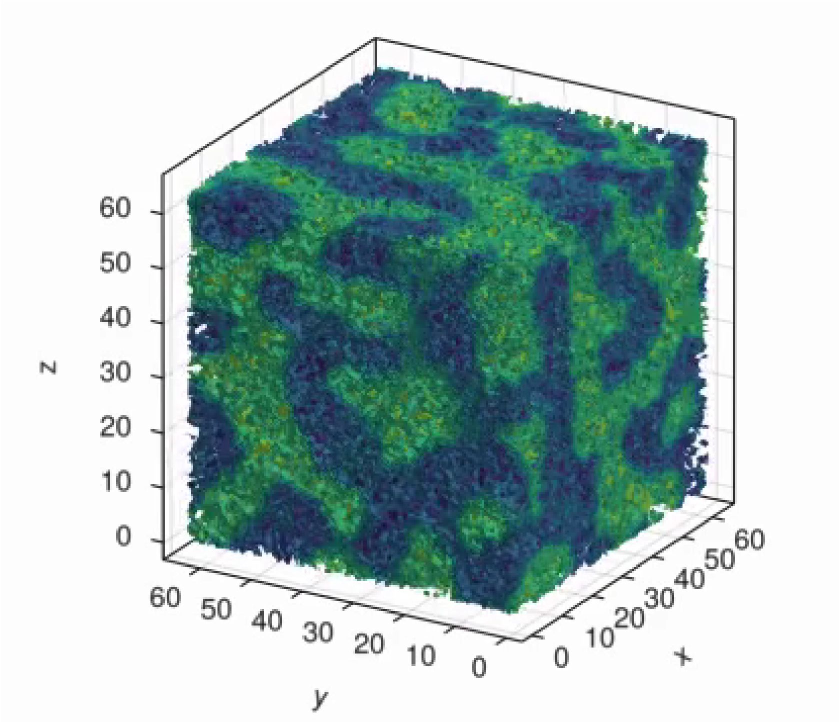
$$\frac{\partial\vec{\pi}^T}{\partial t} = \eta\,\nabla^2\frac{\delta\mathcal{F}}{\delta\vec{\pi}^T} + g\left(\vec{\nabla}\phi\right)\cdot\frac{\delta\mathcal{F}}{\delta\phi} - g\left(\frac{\delta\mathcal{F}}{\delta\vec{\pi}^T}\cdot\vec{\nabla}\right)\vec{\pi}^T + \vec{\xi}$$

Free energy functional: Order parameter $\phi$, momentum density $\vec{\pi} = w\vec{v}$

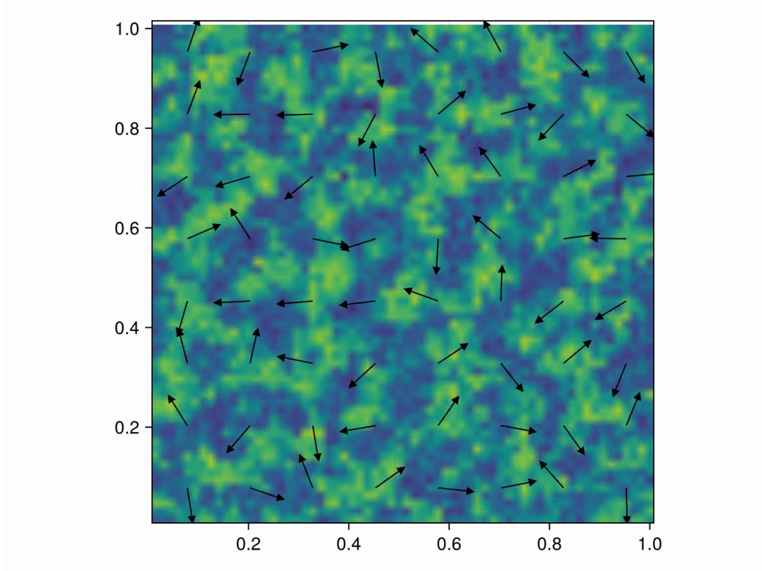
$$\mathcal{F} = \int d^3x\left[\frac{1}{2w}\vec{\pi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{m^2}{2}\phi^2 + \lambda\phi^4\right] \qquad D = m^2\kappa$$

# Numerical results (critical Navier-Stokes)

Order parameter (3d)



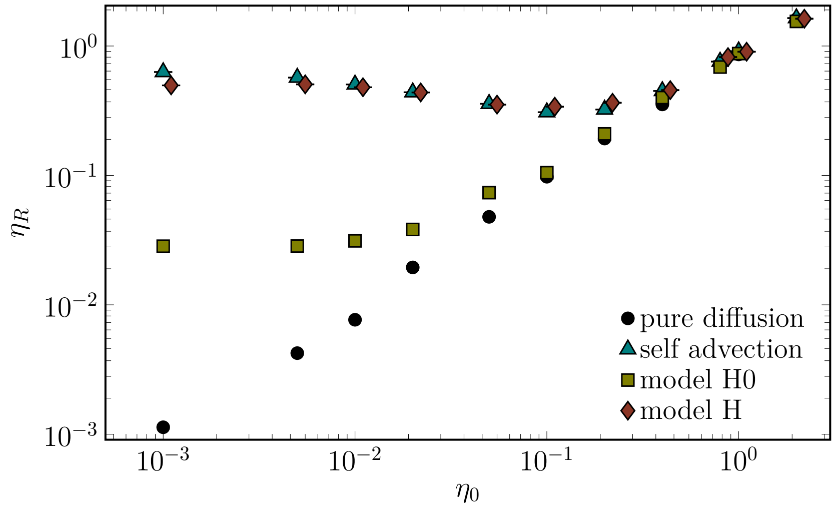

Order parameter/velocity field (2d)



# Renormalized viscosity and dynamic scaling

Renormalization of $\eta$

"Stickiness of shear waves"



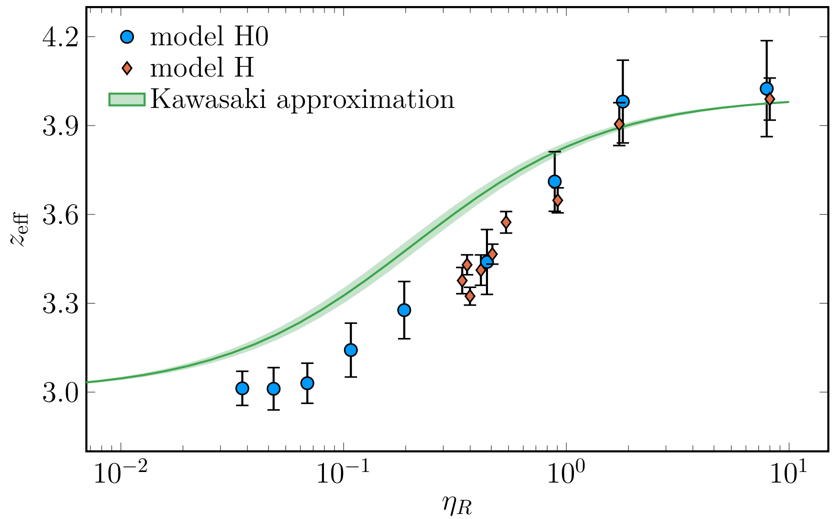

Top: Model H

Middle: No self-advection

Bottom: No advection

Crossover from $\tau_R \sim \xi^4$ at large $\eta_R$

to $\tau_R \sim \xi^3$ for small $\eta_R$




Blue: Model H0

Orange: Model H

Band: Mode-coupling theory

## Summary

- Viscous hydro reads with $T^{0\mu} = (\mathcal{E}, M^x)$:

$$\partial_t \mathcal{E} \quad + \quad \partial_x M = 0$$
$$\partial_t M + \partial_x \mathcal{T}^{xx}(\beta) = \partial_x \left( T\kappa^{xxxx} \partial_x \beta_x \right)$$

- The equations are strictly first order in time and stable.
  - Solve using an ideal step and Crank-Nicholson like step
  - The only variables are $(\mathcal{E}, M)$
  - Only parameters are $\eta$ and $\zeta$ and equation of state
- Each Lorentz observer defines a hydro-frame by setting up spatial volume, measuring the charge, and defining $\beta_\mu$ from the EOS

$$\mathrm{d}S^0 = -\beta_0 \, \mathrm{d}\mathcal{E} - \beta_x \, \mathrm{d}M^x$$

# Freezeout of fluctuations

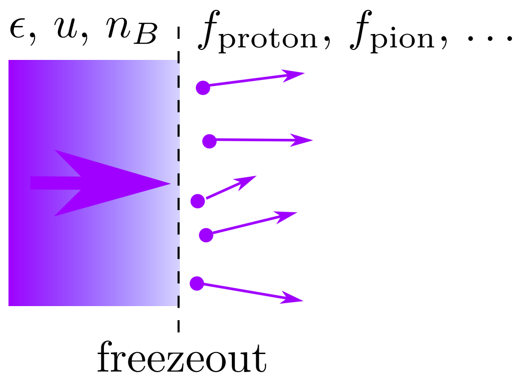
*Pradeep, MS, 2211.09142, PRL*

- Freezeout: mapping of correlators of hydrodynamic fluctuations ($\psi = \epsilon, n_B, u$)

  $\langle \delta\psi \ldots \delta\psi \rangle = H_n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$

  to particle correlators

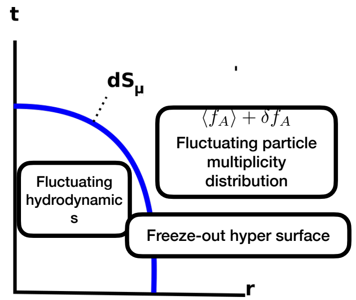
  $\langle \delta f \ldots \delta f \rangle = G_n(\boldsymbol{x}_1, \boldsymbol{p}_1, \ldots, \boldsymbol{x}_n, \boldsymbol{p}_n)$.




- There is a mapping from $H_n$ to $G_n$ which:
  - Satisfies *all* conservation laws, and
  - Maximizes entropy.

# Freezing out event-by-event fluctuations

## Maximum Entropy Freeze-out




- Dual descriptions at freeze-out - Hydrodynamics + fluctuations & Particle multiplicity distributions
- **Matching conditions** to ensure that **conserved quantities** are preserved across freeze-out.
- There exist **infinitely many particle distributions** that reproduce the same macroscopic (hydrodynamic) averages.
- The freeze-out prescription amounts to choosing one such distribution — the **Maximum Entropy** principle provides a natural and consistent choice.

**Cooper & Frye , 74 — Freeze-out of mean yields**
**Everett, Chattopadhyay, Heinz, 21 — Matching conditions for viscous hydrodynamics**
**Pradeep & Stephanov, 22 — Generalization to freeze-out of fluctuations**

# Hope : Data ➔ EoS of QCD near CP+ Dynamics

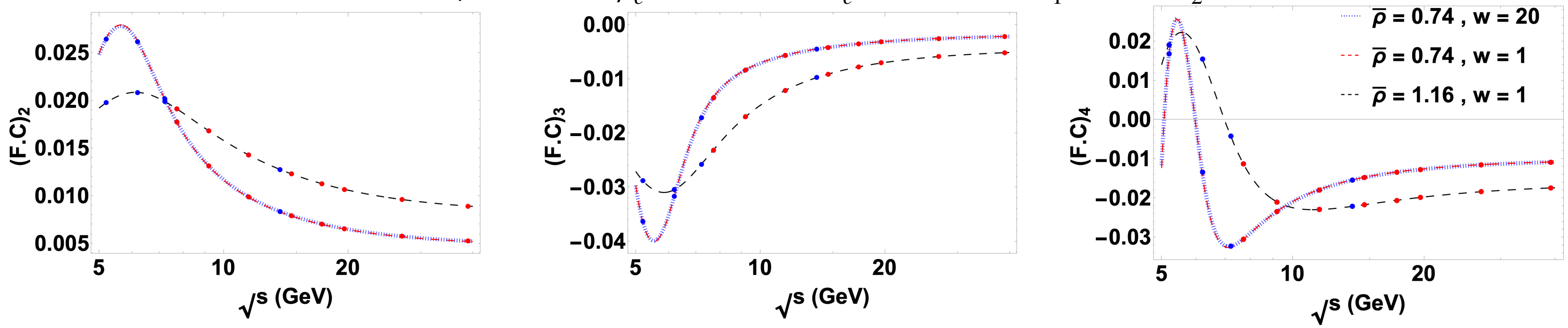
## Lessons about equilibrium properties learnt so far

- $\hat{\Delta}\omega_p^k \sim w^{-1.2}\Delta T_f^{1.2-k} \sim \xi^{k(5-\eta)/2-3}$ - Magnitude sensitive to w and the closeness of the freeze-out curve to cross-over line

**arXiv : 2508.19237, Karthein, Pradeep, Rajagopal, Stephanov, Yin**

- $\bar{\rho} = \rho\, w^{0.36}$ controls the **position of peaks/dips** along a freeze-out curve

*Illustrative choices for non-universal parameters* **:** $\mu_c = 500$ **MeV** , $T_c = 115$ **MeV** , $\alpha_1 = 10^\circ$ , $\alpha_2 = -6^\circ$




$w = 1$ plots has been rescaled by the factor $20^{-1.2}$

- The trajectory of the **Lee-Yang singularities** of QCD near the CP, can be used to **constrain** $\bar{\rho}$ , setting bounds on the equilibrium position of peaks/dips once other non-universal parameters are fixed.

**Basar, Pradeep, Stephanov(in preparation)**

## Summary

What exists: Deterministic equations for Gaussian and non-Gaussian fluctuations (in non-trivial background, but not coupled to velocity fluctuations).

Direct simulation of linearized stochastic hydro, full non-linear model H (in a box), stochastic algorithm for relativistic hydro in density frame. Also: Freezeout prescriptions.

## Goals and Questions

How does the stochastic fluid baseline compare to the RQMD baseline?

How to proceed: Stochastic simulation on top of standard hydro backgound? Or aim for complete simulation?

Stochastic fluids vs stochastic kinetics vs molecular dynamics.