Finite element code for QED+QCD factorization

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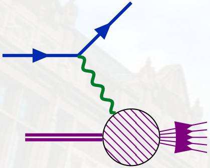
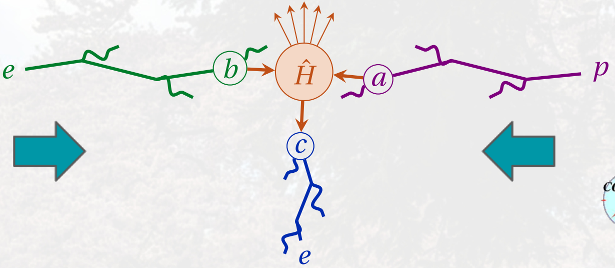
Radiative corrections: the standard way

- Theorists produce Born-level cross section
- Nature has QED radiation—present in measurements
- Experimentalists "correct" data to (try to) remove radiative effects



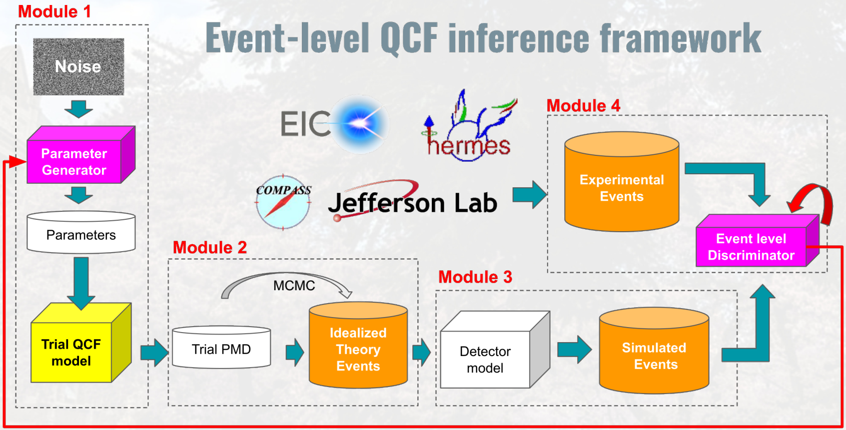
Radiative corrections: the joint factorization way

- Theorists incorporate QED radiation induced by collisions
- Theoretical predictions compared to actual data
- Necessary for a "folding" paradigm & event-level analysis



Event-level inference

- **Event-level inference:** combine theory & detector simulation to predict realistic events
- Tune parton distributions, fragmentation functions, etc., by minimizing distance from actual events
  - Use AI/ML in the pipeline
- Philosophy of **QuantOm collaboration**
  - Team of physicists, mathematicians & physicists at ANL/JLab/VT



Outline

1. Joint QED+QCD factorization for DIS
   - Formalism: Cammarota, Qiu, Watanabe & Zhang, 2505.23487
   - Caveat: I'm not an expert—my role is creating code for this
   - See Jian-Wei Qiu's cake seminar (slides / video) for in-depth talk about formalism
2. Finite element methods
   - Will be most of the talk
   - Focused on (combined QED+QCD) evolution equations
3. Numerical demonstration
   - Just for QED+QCD evolution so far


Joint QED+QCD formalism

- **Quantum interference**: a practical reason for theory to do QED effects
- Interference between diagrams can't be removed at cross section level
- All diagrams of fixed order needed to cancel IR singularities
  - Cancellations occur between real and virtual diagrams!

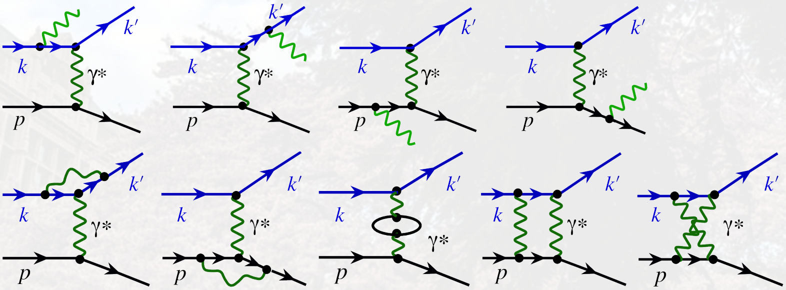


Figure: Cammarota, Qiu, Watanabe & Zhang, 2505.23487

Pinched singularities and photon distributions

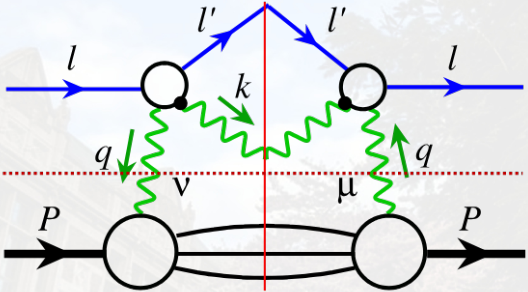


Figure: Jian-Wei Qiu, cake seminar (video)

- Cut diagram with α_{QED} corrections
- Observed $Q^2 = -(q+k)^2$
  - k is integrated
  - q is not observed momentum
  - *Squared* amplitude; note the propagators:

$$\frac{1}{q^2 + i\epsilon} \qquad \frac{1}{q^2 - i\epsilon}$$

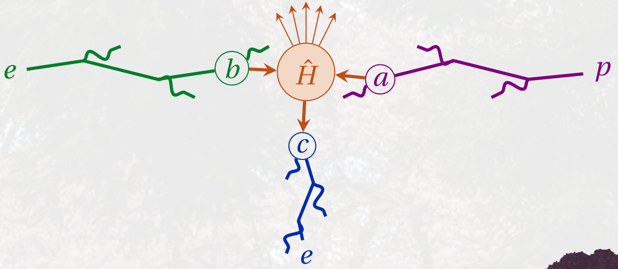
- $q^2 = 0$ is pinched—**real photon**

- Need to include photon distribution inside hadron!
  - Requires modifying DGLAP evolution

What's being factorized

$$2E\frac{d\sigma_{ep\to eX}}{d^3 l'} \approx \frac{1}{s}\int\frac{d\zeta}{\zeta^2}\overbrace{D_{e/c}(\zeta,\mu^2)}^{\text{lepton fragmentation function}}\int\frac{d\xi}{\xi}\underbrace{f_{b/e}(\xi,\mu^2)}_{\text{lepton distribution function}}\int\frac{dx}{x}\overbrace{f_{a/p}(x,\mu^2)}^{\text{parton distribution function}}\underbrace{\hat{H}_{ab\to cX}\left(\xi l, xp, \frac{l'}{z}, \mu^2\right)}_{\text{hard coefficient}}$$

- Three non-perturbative functions
- Each needs to be fit empirically
- a, b, c can *each* be q, g, e, or γ
- Evolution equations mix QED & QCD!
- Hard coefficients in 2505.23487



Joint QED+QCD evolution

$$\frac{\mathrm{d}}{\mathrm{d}\log(\mu^2)}\begin{bmatrix} f_e(x,\mu^2) \\ f_{\bar e}(x,\mu^2) \\ f_\gamma(x,\mu^2) \\ f_q(x,\mu^2) \\ f_{\bar q}(x,\mu^2) \\ f_g(x,\mu^2) \end{bmatrix} = \begin{bmatrix} K_{ee} & K_{e\bar e} & K_{e\gamma} & K_{eq} & K_{e\bar q} & K_{eg} \\ K_{\bar e e} & K_{\bar e\bar e} & K_{\bar e\gamma} & K_{\bar e q} & K_{\bar e\bar q} & K_{\bar e g} \\ K_{\gamma e} & K_{\gamma\bar e} & K_{\gamma\gamma} & K_{\gamma q} & K_{\gamma\bar q} & K_{\gamma g} \\ K_{qe} & K_{q\bar e} & K_{q\gamma} & K_{qq} & K_{q\bar q} & K_{qg} \\ K_{\bar q e} & K_{\bar q\bar e} & K_{\bar q\gamma} & K_{\bar q q} & K_{\bar q\bar q} & K_{\bar q g} \\ K_{ge} & K_{g\bar e} & K_{g\gamma} & K_{gq} & K_{g\bar q} & K_{gg} \end{bmatrix} \otimes \begin{bmatrix} f_e(x,\mu^2) \\ f_{\bar e}(x,\mu^2) \\ f_\gamma(x,\mu^2) \\ f_q(x,\mu^2) \\ f_{\bar q}(x,\mu^2) \\ f_g(x,\mu^2) \end{bmatrix}$$

- Same evolution equation for PDFs of hadron and lepton
- Combine **pure QED evolution**, **pure QCD evolution**, and **mixed evolution**

Finite elements

Needs for numerical code

- Need numerical package for evolution & factorization formulas:

$$2E\frac{d\sigma_{ep\to eX}}{d^3 l'} \approx \frac{1}{s}\int \frac{d\zeta}{\zeta^2} D_{e/c}(\zeta,\mu^2) \int \frac{d\xi}{\xi} f_{b/e}(\xi,\mu^2) \int \frac{dx}{x} f_{a/p}(x,\mu^2) \hat{H}_{ab\to cX}\left(\xi l, xp, \frac{l'}{z}, \mu^2\right)$$

$$\frac{d f_{a/X}(x,\mu^2)}{d\log(\mu^2)} = \sum_b \int_x^1 \frac{dy}{y} K_{a/b}\left(\frac{x}{y},\mu^2\right) f_{b/X}(y,\mu^2)$$

- Options: Mellin or x-space. Let's do x-space.
- Requirements for x-space codes:
  - **Fast**: for use in global analysis.
  - **Differentiable**: for machine learning applications.
  - **Standalone**: to be easily usable by anyone (for model calculations, lattice QCD, ...)
- **Finite elements** satisfy all the needs!
  - Basically a fancy way of discretizing x-space.
- **This talk is about finite elements for evolution.**

(I haven't implemented the cross section formula yet...)

Discretizing the integral

- Right-hand size of evolution equation is an integral:

$$\frac{\mathrm{d}f(x,Q^2)}{\mathrm{d}\log(Q^2)} = \int_x^1 \frac{\mathrm{d}y}{y} K\left(\frac{x}{y}, Q^2\right) f(y,Q^2)$$

- Integral in evolution equation approximated using **Gauss-Kronrod quadrature**.

$$\int_x^1 \frac{\mathrm{d}y}{y} K\left(\frac{x}{y}, Q^2\right) f(y,Q^2) \approx \sum_{g=1}^{N_g} \frac{w_g}{y_g} K\left(\frac{x}{y_g}, Q^2\right) f(y_g,Q^2)$$

  - Discretized grid $\{x_i\}$ and quadrature grid $\{y_g\}$ are not the same.
  - x_i-dependent interpolation must be done.
  - **Interpixels** are used for interpolation.

- **Interpixels** (**interp**olated **pixel**): interpolation basis functions.
  - Exploit linearity of polynomial interpolation:

$$P[y_1 + y_2](x) = P[y_1](x) + P[y_2](x)$$

  - PDF pixelation is a sum of pixels:

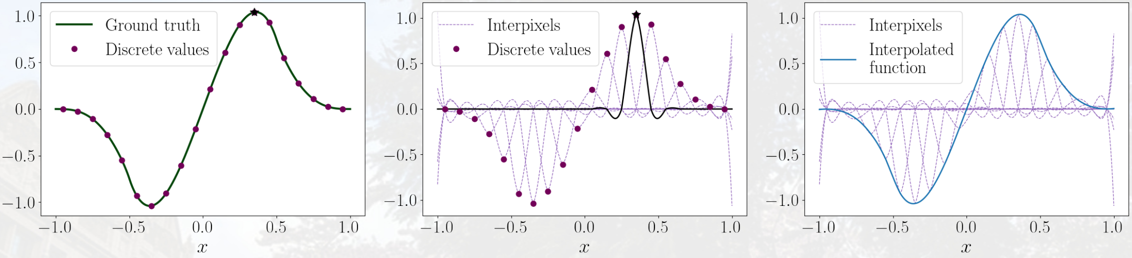
$$\boldsymbol{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} = f_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + f_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \ldots + f_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \equiv f_1 \hat{e}_1 + f_2 \hat{e}_2 + \ldots + f_n \hat{e}_n$$

  - Interpolated pixelation is a sum of interpixels!

$$P[\boldsymbol{f}](x) = f_1 P[\hat{e}_1](x) + f_2 P[\hat{e}_2](x) + \ldots + f_n P[\hat{e}_n](x)$$

- Interpixels are an example of a **finite element**.
  - Used previously in some PDF evolution codes, e.g., HOPPET and APFEL.

Interpixel demo



- Interpixel is a *piecewise* polynomial of fixed order.
  - Increase N_x *without* increasing interpolation order (avoids Runge phenomenon).
  - I'm using fifth-order Lagrange interpolation.
  - Knots at the discrete x_i grid points.
- Each interpixel has oscillations.
  - Oscillations cancel in sum.

Integral discretization: now with interpixels!

- PDF at Gaussian weight points from piecewise polynomial interpolation:

$$f(y_g,\xi,Q^2) \approx \sum_{j=1}^{N_x} f_j(Q^2) P[\hat{e}_j](y_g)$$

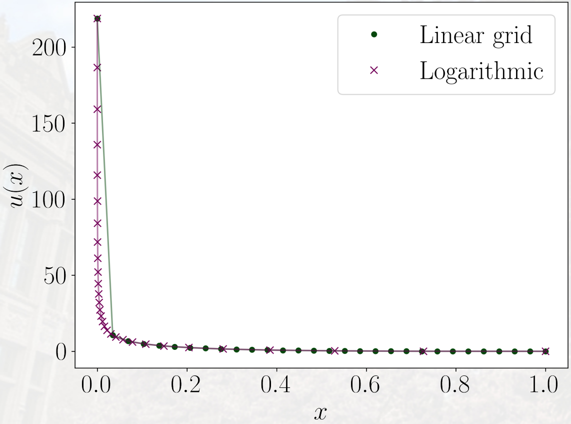
  - Interpolation decomposed into basis functions (**interpixels**).

- Integral is only over interpixels:

$$\int_{x_i}^{1} \frac{\mathrm{d}y}{y} K\left(\frac{x_i}{y}, Q^2\right) f(y,Q^2) \approx \sum_{j=1}^{N_x} \underbrace{\left(\sum_{g=1}^{N_g} \frac{w_g}{y_g} K\left(\frac{x_i}{y_g}, Q^2\right) P[\hat{e}_j](y_g) \right)}_{\equiv K_{ij}(Q^2)} f_j(Q^2)$$

  - Absorb interpixel into kernel matrix.
  - Integral over interpixel **independent of specific PDF**.
  - Method can be generalized to distributions (plus prescription etc.)

Need for non-linear grids



- Hadron PDFs change rapidly at small x
  - Linear x grid leads to poor numerics
  - Typically use linear spacing in $\log(x)$
- Define a map:

$$\phi : [a, b] \to [0, 1]\,, \qquad \phi(\eta) = x$$

  - Build interpixels in η
  - Define ϕ so that x is spaced as needed
  - e.g., $\phi(\eta) = \exp(\eta)$ for logarithmic grid

Non-linear grids in evolution

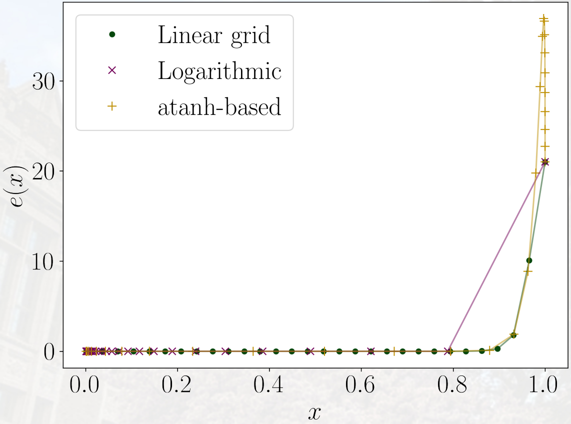
- Variable transform in integral:

$$\int_x^1 \frac{\mathrm{d}y}{y} K\left(\frac{x_i}{y}, Q^2\right) f(y, Q^2) = \int_{\phi^{-1}(x_i)}^1 \frac{\mathrm{d}\eta}{\phi(\eta)} \frac{\mathrm{d}\phi(\eta)}{\mathrm{d}\eta} K\left(\frac{x_i}{\phi(\eta)}, Q^2\right) f(\phi(\eta), Q^2)$$

$$\approx \sum_{j=1}^{N_x} \underbrace{\left(\sum_{g=1}^{N_g} \frac{w_g}{\phi(\eta_g)} \left(\frac{\mathrm{d}\phi(\eta)}{\mathrm{d}\eta}\right)\bigg|_{\eta=\eta_g} K\left(\frac{x_i}{\phi(\eta_g)}, Q^2\right) P[\hat{e}_j](\eta_g)\right)}_{\equiv K_{ij}(Q^2)} f_j(Q^2)$$

  - The interpixels interpolate the linear η space
  - Discretization points $y(\eta_j)$ are non-linear
  - Jacobian incorporated in kernel matrix K_{ij}

Logarithmic grids are not enough



- Lepton PDFs sharply peaked at small x
  - Electron-in-electron is delta-ish
- $\log(x)$ grids even worse than linear
- atanh-based grid takes care of both ends!

$$x_i = \phi(\eta_i) = \frac{1}{2}\big(1 + \tanh(\eta_i)\big)$$

$$\eta_i = \phi^{-1}(x_i) = \operatorname{atanh}\big(2x_i - 1\big)$$

- $x = 0$ and $x = 1$ forbidden—must set min & max values
  - $x_{\min} \sim 10^{-5}$ and $x_{\max} \sim 1 - 10^{-5}$ seem reasonable.

Differential matrix equation

- Discretization+interpixels turns the evolution equation into a **matrix differential equation**:

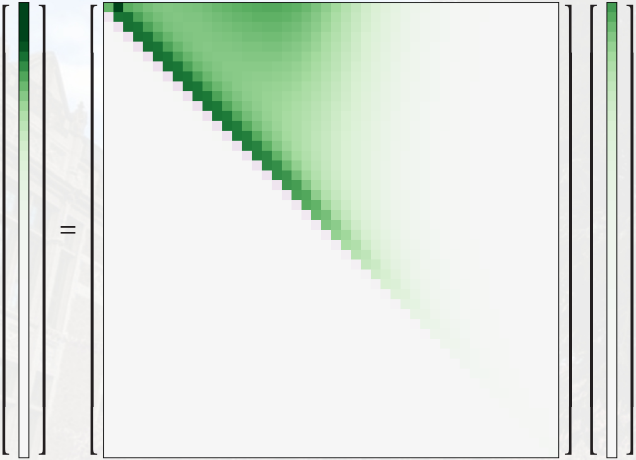
$$\frac{\mathrm{d} f_i(Q^2)}{\mathrm{d}\log(Q^2)} = \sum_{j=1}^{N_x} K_{ij}(Q^2) f_j(Q^2)$$

- Can be solved with standard techniques, like Runge-Kutta.

$$f_i(Q^2_{\mathrm{fin}}) = \sum_{j=1}^{N_x} M_{ij}(Q^2_{\mathrm{ini}} \to Q^2_{\mathrm{fin}}) f_j(Q^2_{\mathrm{ini}})$$

  - Only K_{ij}—not f_j itself—is needed to build M_{ij}.

Evolution matrices: what they do



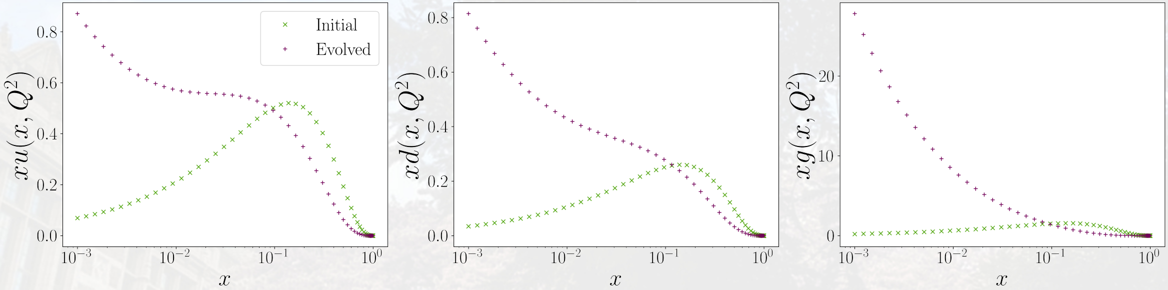
$$f_i(Q^2_{\text{fin}}) = \sum_{j=1}^{N_x} M_{ij}(Q^2_{\text{ini}} \to Q^2_{\text{fin}})\, f_j(Q^2_{\text{ini}})$$

- Evolution matrix is **transfer matrix**
- Says how evolution maps pixels to pixels
- Depends only on the interpixels
- Independent of specific PDF

Numerical demonstration

Standard PDF evolution

Inside proton



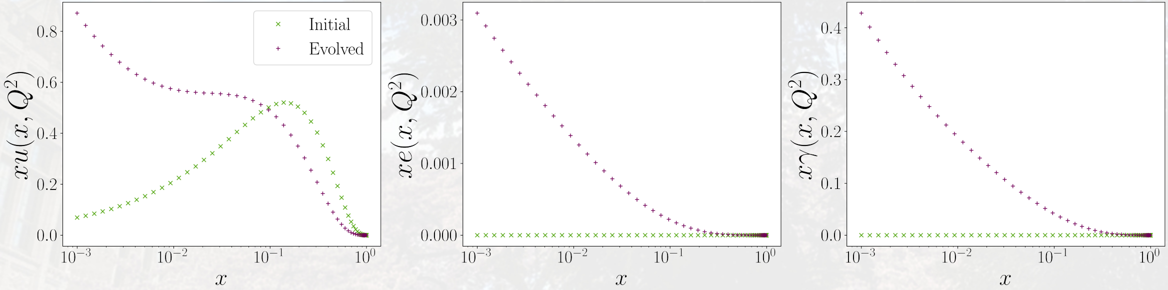
- The atanh-based grid works fine for standard PDF evolution.
- For demo, using initial scale of $\mu^2 = m_c^2$ and parametric model:

$$u(x)\,,\, d(x)\,,\, g(x) \sim x^{-1/2}(1-x)^3$$

  - Normalized to satisfy quark & momentum sum rules

But there are photons!

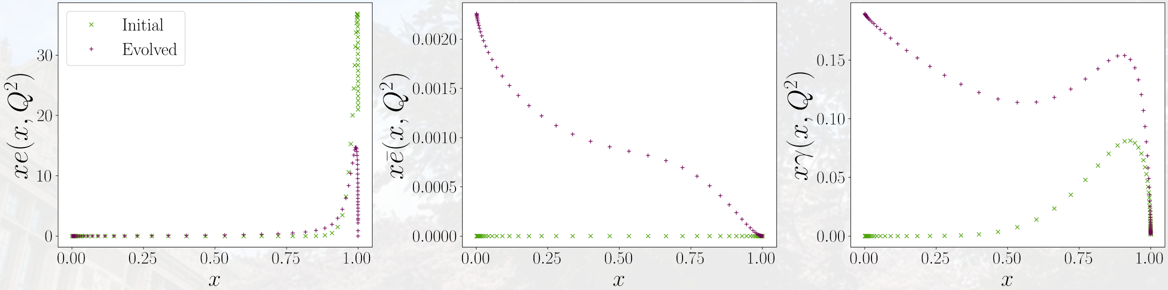
Inside proton



- There are now electrons, positrons & photons inside hadrons.
- Small contribution to momentum sum rule, $< 1\%$.

Electron evolves too

Inside electron



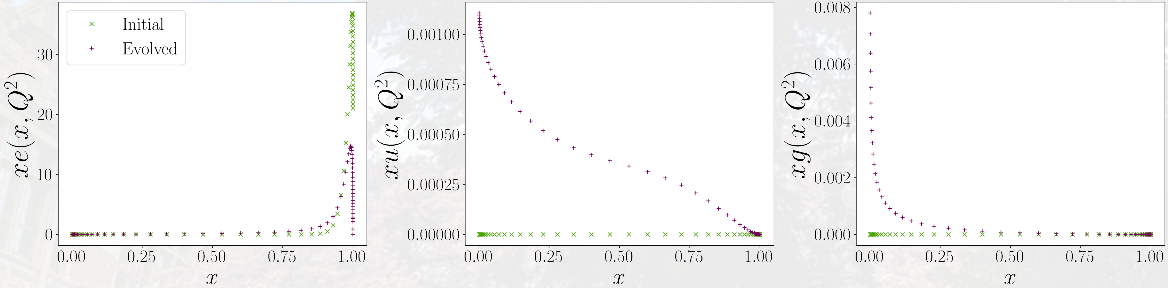
- The QED corner of the evolution matrix
- At $\mu^2 = m_c^2$, using

$$e(x) \sim x^{50}(1-x)^{1/8} \qquad \gamma(x) \sim x^5(1-x)^{1/2}$$

  - Normalized to satisfy electron & momentum sum rules
  - As suggested by Cammarota, Qiu, Watanabe & Zhang, 2505.23487

Electron as a hadron?

Inside electron



- There are quarks and gluons inside electron too
- There are very few though, $< 1\%$ to momentum sum rule

Outlook

Summary

- Cammarota, Qiu, Watanabe & Zhang (2505.23487) worked out joint QED+QCD factorization
  - Applicable to DIS & other reactions
- I've started on a finite element code of their formalism
  - Code will be fast and AI/ML-friendly
  - We plan to use in global analysis
  - When ready, the code will be public & open-source
- There's a lot left to do
  - So far only incorporated evolution
  - Still need to incorporate cross section formulas
  - Benchmarking against CQWZ's Mellin space method needs to be done

QuantOm Collaboration

Jefferson Lab

SciDAC award: *Femtoscale Imaging of Nuclei using Exascale Platforms*

DOE contract No. DE-AC05-06OR23177

Thank you for your time!