Mass decomposition on the light front

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Intended takeaways

1. Mass decomposition depends on coordinate system employed.
   - Differs for light front and instant form.

$$M_{\text{IF}} = \sum_{q,g} M\Big(A_{q,g}(0) + \bar{c}_{q,g}(0)\Big)$$

$$M_{\text{LF}} = \sum_{q,g} M\Big(A_{q,g}(0) + 2\bar{c}_{q,g}(0)\Big)$$

2. Light front coordinates **are not** the infinite momentum frame.
   - Conceptually differ
   - **Mathematically differ**: P^0 is not P^- in any frame.

These results are not new; see Sec. IV.C of:

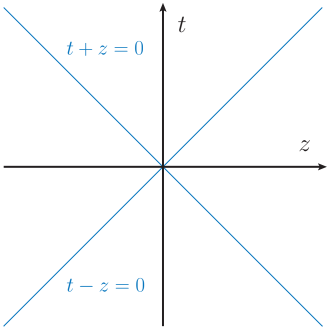
- Lorcé, Metz, Pasquini, & Rodini,
  JHEP 11 (2021) 121 [2109.11785]

...I just want to make sure they're fully understood and appreciated by everyone.

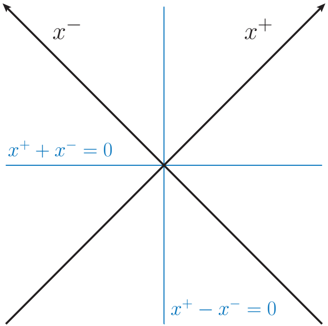
Light front coordinates

Light front coordinates are a different foliation of spacetime.

$$x^{\pm} = \frac{t \pm z}{\sqrt{2}} \qquad \mathbf{x}_\perp = (x, y) \qquad \tau = x^+ = \text{time}$$



Minkowski coordinates



Light front coordinates

Light front: Myths and Facts

- **Myth:** Light front coordinates are a reference frame.
- **Fact:** Light front coordinates can be employed in any reference frame.

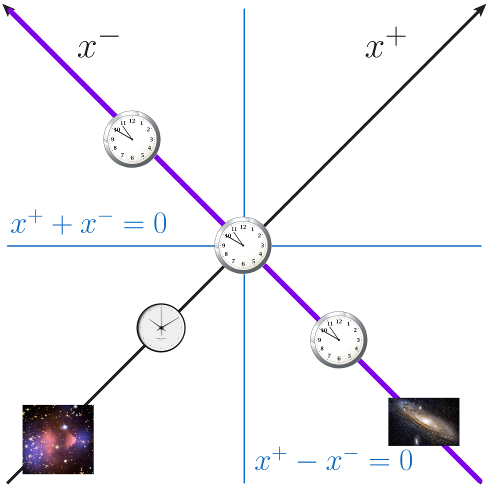
- **Myth:** Light front coordinates describe the perspective of light.
- **Fact:** Light front coordinates describe *our* perspective.
  - . . . but only in the z direction!
  - Light has no perspective.

- **Myth:** Light front coordinates come from boosting to infinite momentum.
- **Fact:** Light front coordinates from from redefining:
  1. *Simultaneity*
  2. What we mean by *boosting*
  3. How we break the Poincaré group into generators (*Galilean subgroup*)

Light front & time synchronization

Light front **redefines simultaneity**

- Fixed $x^+ = \frac{t+z}{\sqrt{2}}$ means *simultaneous*
- Look in the $+\hat{z}$ direction...
  - Whatever you **see** *right now*, is **happening** *right now*.
  - *Only* true for $+\hat{z}$ direction though.
- Light front coordinates are what *we* see.
  - ...at least in one fixed direction.
  - (great for small systems—hadrons!)
  - *Not* what light "sees."



Terrell rotations

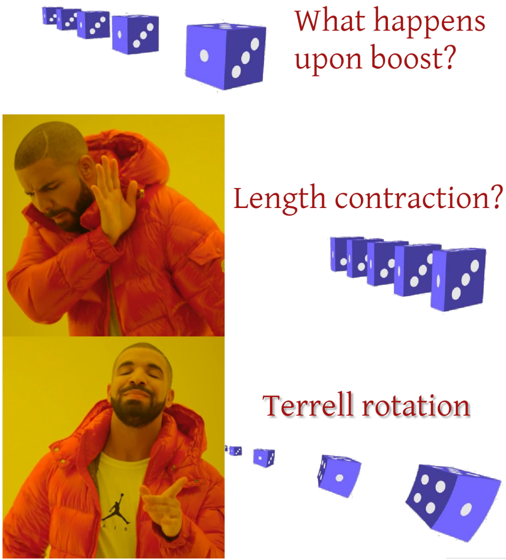
- Lorentz-boosted objects appear *rotated*.
  - **Terrell rotation**
  - Optical effect: contraction + delay
- **Light front transverse boost** *undoes* Terrell rotation:

$$B_x^{(\mathrm{LF})} = \frac{1}{\sqrt{2}}\Big(K_x + J_y\Big)$$

  - Combination of ordinary boost + rotation!
  - Leaves x^+ (time) invariant!
  - Changes p_z, but leaves p^+ invariant:

$$P^+ = \frac{E + p_z}{\sqrt{2}} = \frac{\sqrt{p_z^2 + \mathbf{p}_\perp^2 + M^2} + p_z}{\sqrt{2}}$$

- Dice images by Ute Kraus, `https://www.spacetimetravel.org/`



Galilean subgroup

- Poincaré group has a $(2+1)$D **Galilean subgroup**.
  - x^+ is time and $\mathbf{x}_\perp$ is space under this subgroup.
  - x^- can be integrated out.
  - $P^+ = \frac{1}{\sqrt{2}}(E_{\mathbf{p}} + p_z)$ is the central charge.
  - x^+ and P^+ are invariant under this subgroup!
- Basically, light front coordinates should give a **fully relativistic** 2D picture that looks like *non-relativistic* physics.
  - But with P^+ in place of M.

$$\frac{\mathrm{d}\mathbf{P}_\perp}{\mathrm{d}x^+} = P^+ \frac{\mathrm{d}^2\mathbf{x}_\perp}{\mathrm{d}x^{+2}}$$

$$H = P^- = H_{\text{rest}} + \frac{\mathbf{P}_\perp^2}{2P^+}$$

$$\mathbf{v}_\perp = \frac{\mathbf{P}_\perp}{P^+}$$

See Matthias Burkardt, Int. J. Mod. Phys. A18 (2003) 173

The EMT and momentum densities

- Continuity equation:

$$\partial_\mu \hat{T}^{\mu\nu}(x) = 0$$

- For any choice τ of time ($\bar{x}$ for space):

$$\frac{\partial}{\partial \tau}\left[\hat{T}^{\tau\nu}(\tau,\bar{x})\right] + \sum_{\alpha=1}^{3} \frac{\partial}{\partial \bar{x}^\alpha}\left[\hat{T}^{\alpha\nu}(\tau,\bar{x})\right] = 0$$

  - e.g., $\tau = x^+$ for light front, $\tau = x^0$ for instant form.
- Integrate over infinite spatial ($\bar{x}$) volume:

$$\frac{\partial}{\partial \tau}\left[\int \mathrm{d}^3\bar{x}\, \hat{T}^{\tau\nu}(\tau,\bar{x})\right] = \frac{\partial}{\partial \tau}\left[\hat{P}^{\nu}(\bar{x},\tau)\right] = 0\,.$$

Energy density operator

- Four-momentum density for any coordinate system:

$$\hat{\mathscr{P}}^\nu(\bar{x},\tau) = \hat{T}^{\tau\nu}(\tau,\bar{x})$$

- Which ν to use for *energy* density depends on coordinate system.
  - Instant form: $\nu = 0$ because $\hat{H} = \hat{P}^0$
  - Light front: $\nu = -$ because $\hat{H} = \hat{P}^-$
- Two energy densities:

$$\hat{\mathscr{H}}_{\mathrm{IF}}(\mathbf{x},t) = \hat{T}^{00}(\mathbf{x},t)$$

$$\hat{\mathscr{H}}_{\mathrm{LF}}(\mathbf{x}_\perp,x^-,x^+) = \hat{T}^{+-}(\mathbf{x}_\perp,x^-,x^+)$$

- I have *not* chosen a frame or boosted to infinite momentum!

Translation dictionary

- Coordinate definitions:

$$x^+ = \frac{1}{\sqrt{2}}\left(x^0 + x^3\right) \qquad x^- = \frac{1}{\sqrt{2}}\left(x^0 - x^3\right)$$

$$x^0 = \frac{1}{\sqrt{2}}\left(x^+ + x^-\right) \qquad x^3 = \frac{1}{\sqrt{2}}\left(x^+ - x^-\right)$$

- Tensor relationships:

$$\hat{T}^{+-}(x) = \frac{1}{2}\left(\hat{T}^{00}(x) + \hat{T}^{30}(x) - \hat{T}^{03}(x) - \hat{T}^{33}(x)\right)$$

$$\hat{T}^{00}(x) = \frac{1}{2}\left(\hat{T}^{++}(x) + \hat{T}^{+-}(x) - \hat{T}^{-+}(x) - \hat{T}^{--}(x)\right)$$

  - The *same* x is used as arguments in both.
  - **Cannot** set $x^+ = 0$ and $x^0 = 0$ at the same time while leaving x^3 and x^- free.
  - Red terms cancel for symmetric EMT.

Boosting the EMT

- Suppose we apply a Lorentz boost in the z direction (*active* boost):

$$\hat{T}'^{\mu\nu}(x) = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\hat{T}^{\alpha\beta}(\Lambda^{-1}x) \qquad \Lambda^{\mu}{}_{\alpha} = \begin{bmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{bmatrix}$$

- EMT components transform as (spacetime argument suppressed):

$$\hat{T}'^{00} = \gamma^2\hat{T}^{00} + \beta\gamma^2\hat{T}^{03} + \beta\gamma^2\hat{T}^{30} + \beta^2\gamma^2\hat{T}^{33}$$

$$\hat{T}'^{03} = \beta\gamma^2\hat{T}^{00} + \gamma^2\hat{T}^{03} + \beta^2\gamma^2\hat{T}^{30} + \beta\gamma^2\hat{T}^{33}$$

$$\hat{T}'^{30} = \beta\gamma^2\hat{T}^{00} + \beta^2\gamma^2\hat{T}^{03} + \gamma^2\hat{T}^{30} + \beta\gamma^2\hat{T}^{33}$$

$$\hat{T}'^{33} = \beta^2\gamma^2\hat{T}^{00} + \beta\gamma^2\hat{T}^{03} + \beta\gamma^2\hat{T}^{30} + \gamma^2\hat{T}^{33}$$

$$\hat{T}'^{+-}(x) = \hat{T}^{+-}(\Lambda^{-1}x)$$

Boosting to infinite momentum?

- Let's take the $\beta \to 1$ ($\gamma \to \infty$) limit.
  - Instant form energy density:

$$\hat{T}^{00} \xrightarrow[\text{boost}]{} \gamma^2 \hat{T}^{00} - \beta\gamma^2 \hat{T}^{03} - \beta\gamma^2 \hat{T}^{30} + \beta^2\gamma^2 \hat{T}^{33} \xrightarrow[\beta\to 1]{} \infty$$

  - Light front energy density:

$$\hat{T}^{+-} \xrightarrow[\text{boost}]{} \hat{T}^{+-} = \text{finite}$$

  - Instant form **does not** become light front at infinite momentum!
- For a direct light front perspective:

$$\begin{matrix} V^+ \\ V^- \\ \mathbf{v}_\perp \end{matrix} \xrightarrow[\text{boost}]{} \begin{matrix} e^{+\eta} V^+ \\ e^{-\eta} V^- \\ \mathbf{v}_\perp \end{matrix}$$

$$\eta = \tanh^{-1} \beta$$

$$V^2 = 2V^+V^- - \mathbf{V}_\perp^2$$

$\hat{T}^{+-}$ transforms like V^+V^-

What about total energy?

- Light front energy vanishes at infinite momentum!

$$P^- \xrightarrow[\text{boost}]{} e^{-\eta} P^- \xrightarrow[\eta\to\infty]{} 0$$

$$P^- = \int \mathrm{d}x^- \mathrm{d}^2\mathbf{x}_\perp \, \hat{T}^{+-}(x) \xrightarrow[\text{boost}]{} \int \mathrm{d}(e^{-\eta}x^-)\mathrm{d}^2\mathbf{x}_\perp \, \hat{T}^{+-}(x) = e^{-\eta}P^- \xrightarrow[\eta\to\infty]{} 0$$

$$P^- = \frac{M^2 + \mathbf{P}_\perp^2}{2P^+} \xrightarrow[P^+\to\infty]{} 0$$

- Instant form energy becomes infinite:

$$P^0 \xrightarrow[\text{boost}]{} \gamma P^0 + \beta\gamma P^3 \xrightarrow[\beta\to 1]{} \infty$$

$$P^0 = \sqrt{M^2 + \mathbf{P}^2} \xrightarrow[P^3\to\infty]{} \infty$$

- These quantities differ in the infinite momentum limit.
- Light front **is not** instant form at infinite momentum!

Regarding decompositions

- Density given by expectation value with **physical state** $|\Psi\rangle$.

$$T^{\mu\nu}(x) = \langle\Psi|\hat{T}^{\mu\nu}(x)|\Psi\rangle$$

- Break down into quark/gluon pieces:

$$\hat{T}^{\mu\nu}(x) = \hat{T}^{\mu\nu}_q(x) + \hat{T}^{\mu\nu}_g(x)$$

- Total momentum/energy given by spatial integral:

$$P^\nu_{q,g} = \int \mathrm{d}^3\bar{x}\,\langle\Psi|\hat{T}^{\tau\nu}_{q,g}(x)|\Psi\rangle$$

- Consider **spin-zero** example:

$$\langle p|\hat{T}^{\mu\nu}_{q,g}(0)|p\rangle = 2p^\mu p^\nu A_{q,g}(0) + 2M^2\bar{c}_{q,g}(0)$$

Quick calculation sketch

I won't flesh out the calculation; but if you use:

- **Completeness relations:**

$$1 = \int \frac{\mathrm{d}^3\mathbf{p}}{2E_{\mathbf{p}}(2\pi)^3} |\mathbf{p}\rangle\langle\mathbf{p}| \qquad 1 = \int \frac{\mathrm{d}p^+\mathrm{d}^2\mathbf{p}_\perp}{2p^+(2\pi)^3} |p^+,\mathbf{p}_\perp\rangle\langle p^+,\mathbf{p}_\perp|$$

- **Wave functions:**

$$\psi_{\mathrm{IF}}(\mathbf{p}) = \langle\mathbf{p}|\Psi\rangle \qquad \psi_{\mathrm{LF}}(p^+,\mathbf{p}_\perp) = \langle p^+,\mathbf{p}_\perp|\Psi\rangle$$

$$\int \frac{\mathrm{d}^3\mathbf{p}}{2E_{\mathbf{p}}(2\pi)^3} \Big|\psi_{\mathrm{IF}}(\mathbf{p})\Big|^2 = 1 \qquad \int \frac{\mathrm{d}p^+\mathrm{d}^2\mathbf{p}_\perp}{2p^+(2\pi)^3} \Big|\psi_{\mathrm{LF}}(p^+,\mathbf{p}_\perp)\Big|^2 = 1$$

- **Translation operators:**

$$\hat{T}^{\mu\nu}(x) = e^{i\hat{P}\cdot x}\hat{T}^{\mu\nu}_{q,g}(0)e^{-\hat{P}\cdot x}$$

Then you get ...

Arbitrary frame energy decompositions

- **Instant form decomposition**:

$$P^0_{q,g} = \langle E_{\mathbf{p}} \rangle A_{q,g}(0) + M \left\langle \frac{1}{E_{\mathbf{p}}} \right\rangle \bar{c}_{q,g}(0)$$

- **Light front decomposition**:

$$P^-_{q,g} = M^2 \left\langle \frac{1}{2p^+} \right\rangle \Big(A_{q,g}(0) + 2\bar{c}_{q,g}(0) \Big) + \left\langle \frac{\mathbf{p}_\perp^2}{2p^+} \right\rangle A_{q,g}(0)$$

- $\langle \hat{O} \rangle$ is **expectation value** with physical state $|\Psi\rangle$.
  - Could use momentum eigenstates to remove angled brackets.

Infinite momentum frame energy decompositions

- **Instant form decomposition:**

$$P^0_{q,g} = \langle E_{\mathbf{p}} \rangle A_{q,g}(0) + M \left\langle \frac{1}{E_{\mathbf{p}}} \right\rangle \bar{c}_{q,g}(0) \xrightarrow[p_z \to \infty]{} \langle p_z \rangle A_{q,g}(0) + \mathcal{O}\left(\frac{1}{p_z}\right)$$

- **Light front decomposition:**

$$P^-_{q,g} = M^2 \left\langle \frac{1}{2p^+} \right\rangle \Big(A_{q,g}(0) + 2\bar{c}_{q,g}(0) \Big) + \left\langle \frac{\mathbf{p}_\perp^2}{2p^+} \right\rangle A_{q,g}(0) \xrightarrow[p^+ \to \infty]{} e^{-\eta} P^-_{q,g}$$

- These don't match in infinite momentum limit.
  - Actually, $P^0_{q,g}$ tends to $\sqrt{2} P^+_{q,g}$.

Rest frame energy decompositions

- **Instant form decomposition**:

$$P^0_{q,g} = \langle E_{\mathbf{p}} \rangle A_{q,g}(0) + M \left\langle \frac{1}{E_{\mathbf{p}}} \right\rangle \bar{c}_{q,g}(0) \xrightarrow[\text{rest}]{} M\Big(A_{q,g}(0) + \bar{c}_{q,g}(0)\Big)$$

- **Light front decomposition**:

$$P^-_{q,g} = M^2 \left\langle \frac{1}{2p^+} \right\rangle \Big(A_{q,g}(0) + 2\bar{c}_{q,g}(0)\Big) + \left\langle \frac{\mathbf{p}_\perp^2}{2p^+} \right\rangle A_{q,g}(0) \xrightarrow[\text{rest}]{} \frac{M}{\sqrt{2}}\Big(A_{q,g}(0) + 2\bar{c}_{q,g}(0)\Big)$$

- These results give **mass decompositions**
  - Interpreting mass as **rest frame energy**.
- Light front and instant form give **different decompositions**!

Why does it matter?

1. I want to set the record straight regarding light front vs. IMF.
2. Proton mass decomposition entails ambiguities.
   - Renormalization scheme
   - Coordinate system
   - What we mean by mass (rest frame energy, Galilean charge, Lorentz scalar, ...)
3. Need to think about **what physics** the decomposition should encode.
   - Are we interested in a rest frame energy budget?
   - Does a particle picture work in the framework we're using? (Light front, Fock space)
   - Can **total hadron motion** and **intrinsic structure** be cleanly separated?

$$P^-_{q,g} = M^2 \left\langle \frac{1}{2p^+} \right\rangle \Big(A_{q,g}(0) + 2\bar{c}_{q,g}(0) \Big) + \left\langle \frac{\mathbf{p}_\perp^2}{2p^+} \right\rangle A_{q,g}(0)$$

   - Can the mass decomposition be studied in simple, intuitive models?
     (See Jerry's talk on Thursday!)
   - Is an energy budget for the **pion in the chiral limit** worth investigating?

Regarding DIS and the EMT

- The gluon momentum fraciton found in DIS is:

$$\langle x_g \rangle = \frac{\langle p|\hat{T}_g^{++}(0)|p\rangle}{\langle p|\hat{T}^{++}(0)|p\rangle} = A_g(0)$$

- Classically?

$$T_g^{++} = F^{+\rho}F_\rho{}^{+} = \frac{1}{4}(\mathbf{E}^2 + \mathbf{B}^2) + (\mathbf{E}\times\mathbf{B})_3 + \frac{1}{2}(E_3^2 - B_1^2 - B_2^2) \neq \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)$$

- In renormalized QFT?

$$[T_\gamma^{++}]_R = \frac{1}{4}\Big[\mathbf{E}^2 + \mathbf{B}^2\Big]_R - [(\mathbf{BE})_3]_R - \frac{1}{2}[E_3^2] + \frac{1}{2}\left[\sum_{k=1}^{\infty} B_{3k}B_{3k}\right]_R \neq \frac{1}{2}\Big[\mathbf{E}^2 + \mathbf{B}^2\Big]_R$$

  - Recall space is ∞-dimensional in dimreg (see Collins)
  - Magnetic field is 2-form if space isn't 3D.
  - Can't distribute $[]_R$ over infinite sum.

The End

Thank you for your time!