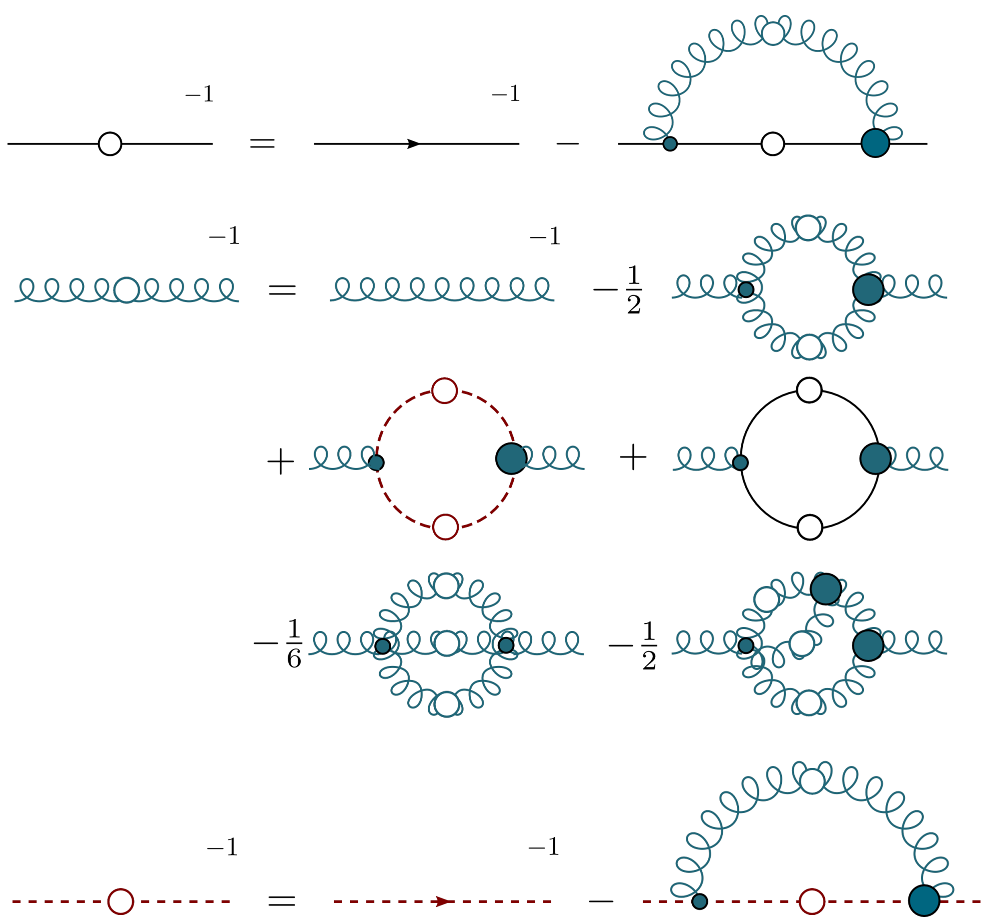
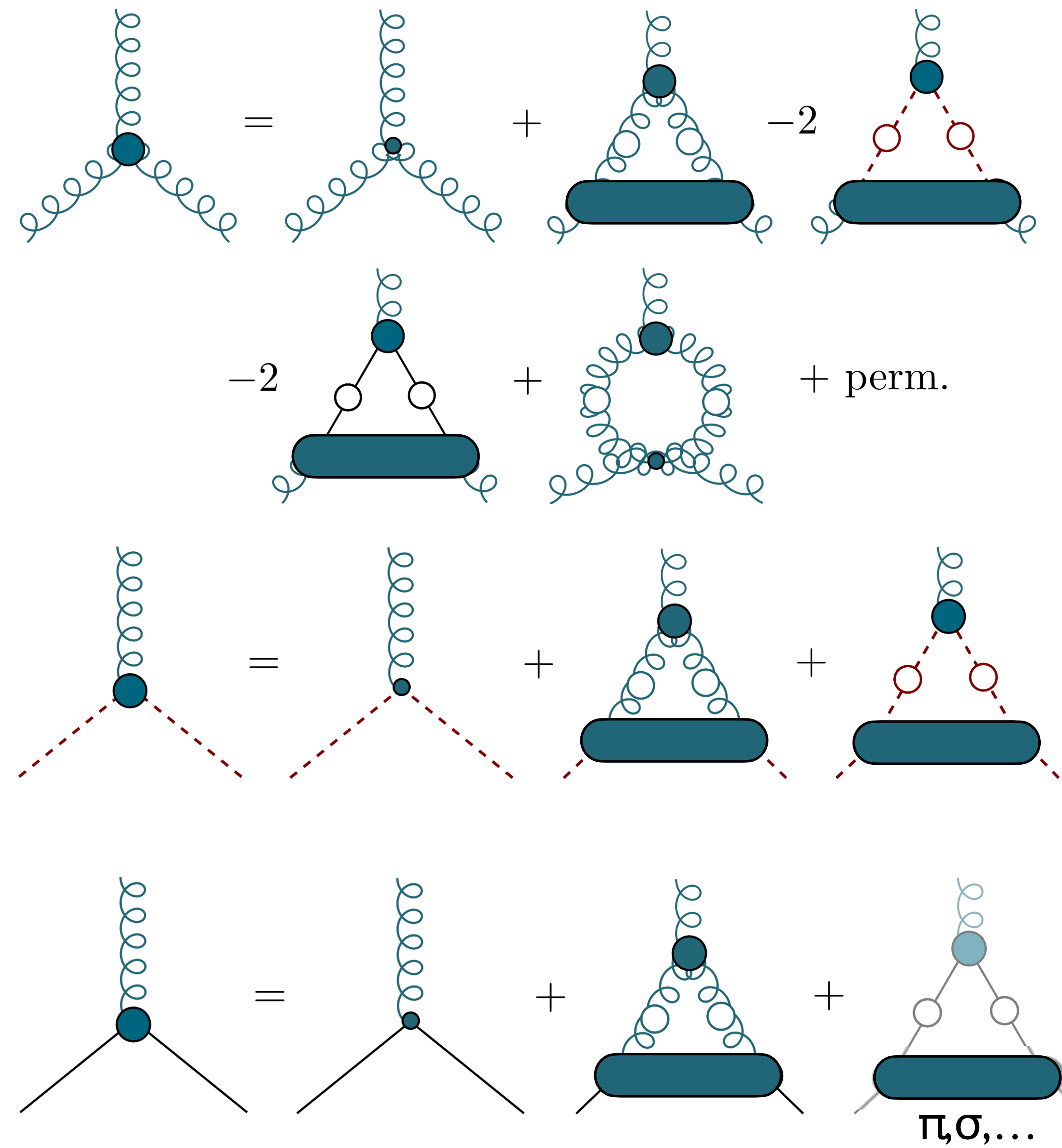
# Columbia plot and critical point

## propagators




## vertices

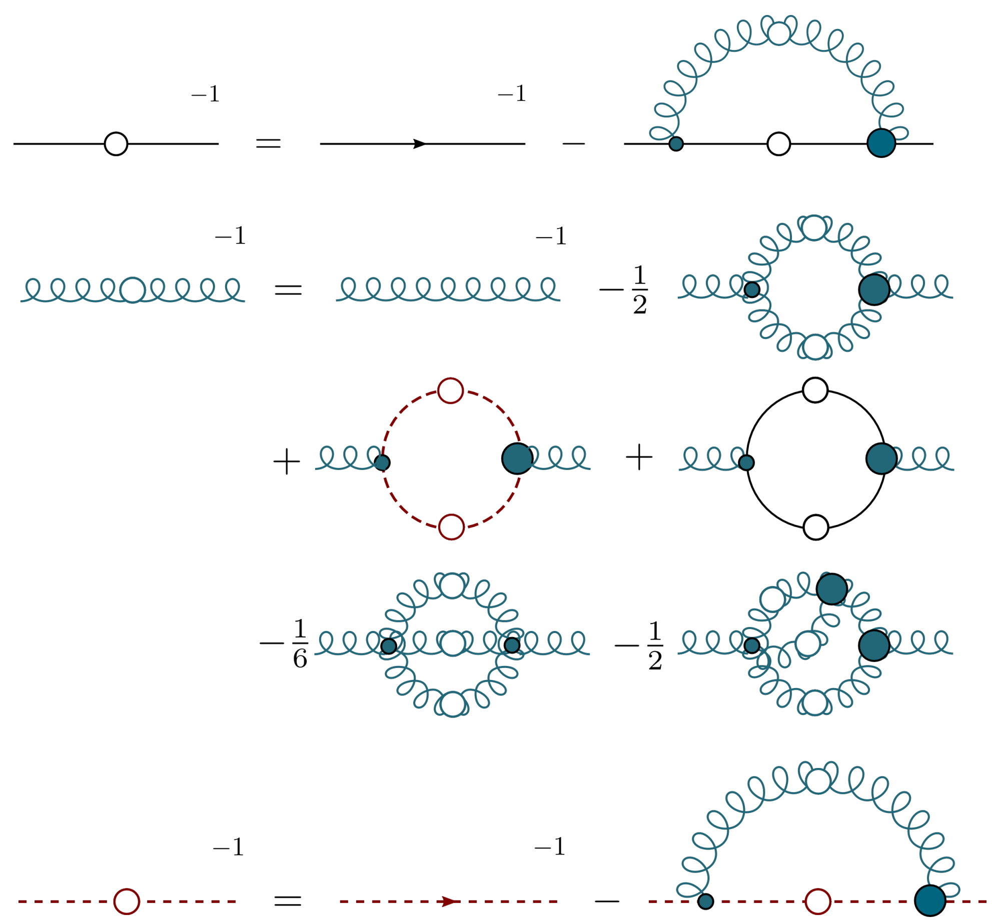



Review: Eichmann, Sanchis-Alepuz, Williams, Alkofer, CF, PPNP 91, 1-100 [1606.09602]

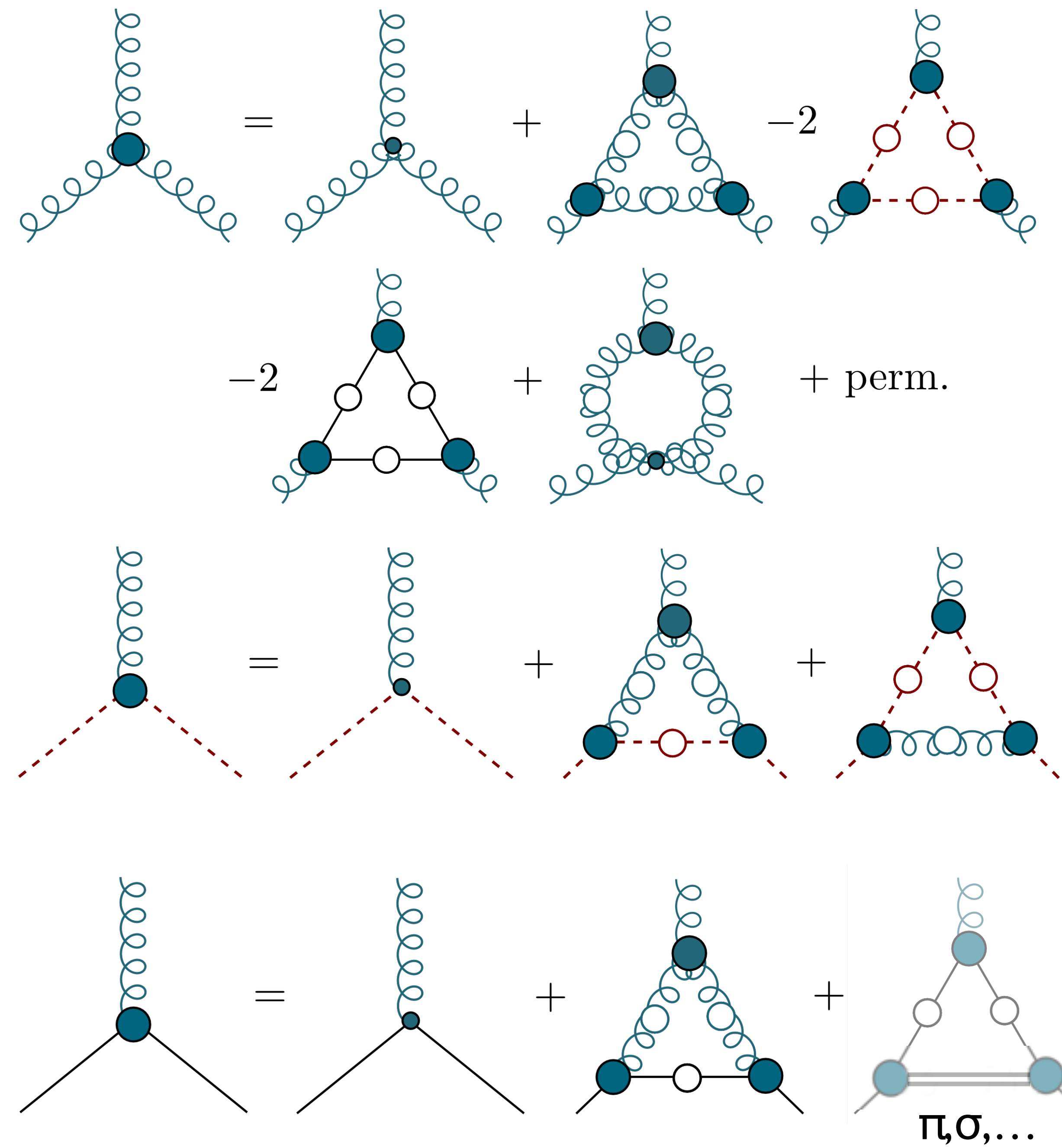
CF, Alkofer, PRD67 (2003) 094020
Williams, CF, Heupel, PRD93 (2016) 034026
Huber, PRD 101 (2020) 114009

# QCD with DSE

## propagators




## vertices




Review: Eichmann, Sanchis-Alepuz, Williams, Alkofer, CF, PPNP 91, 1-100 [1606.09602]

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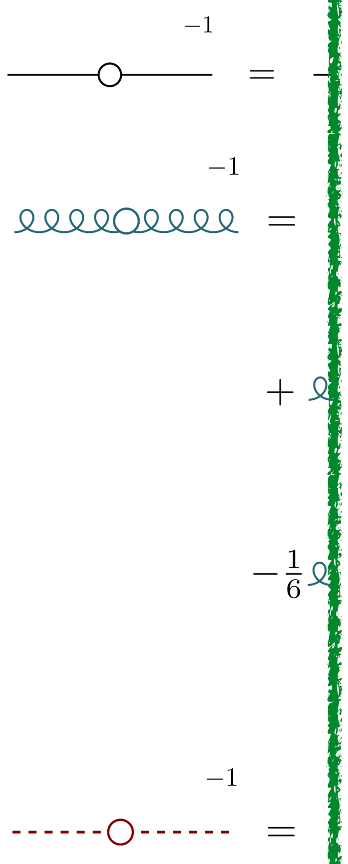
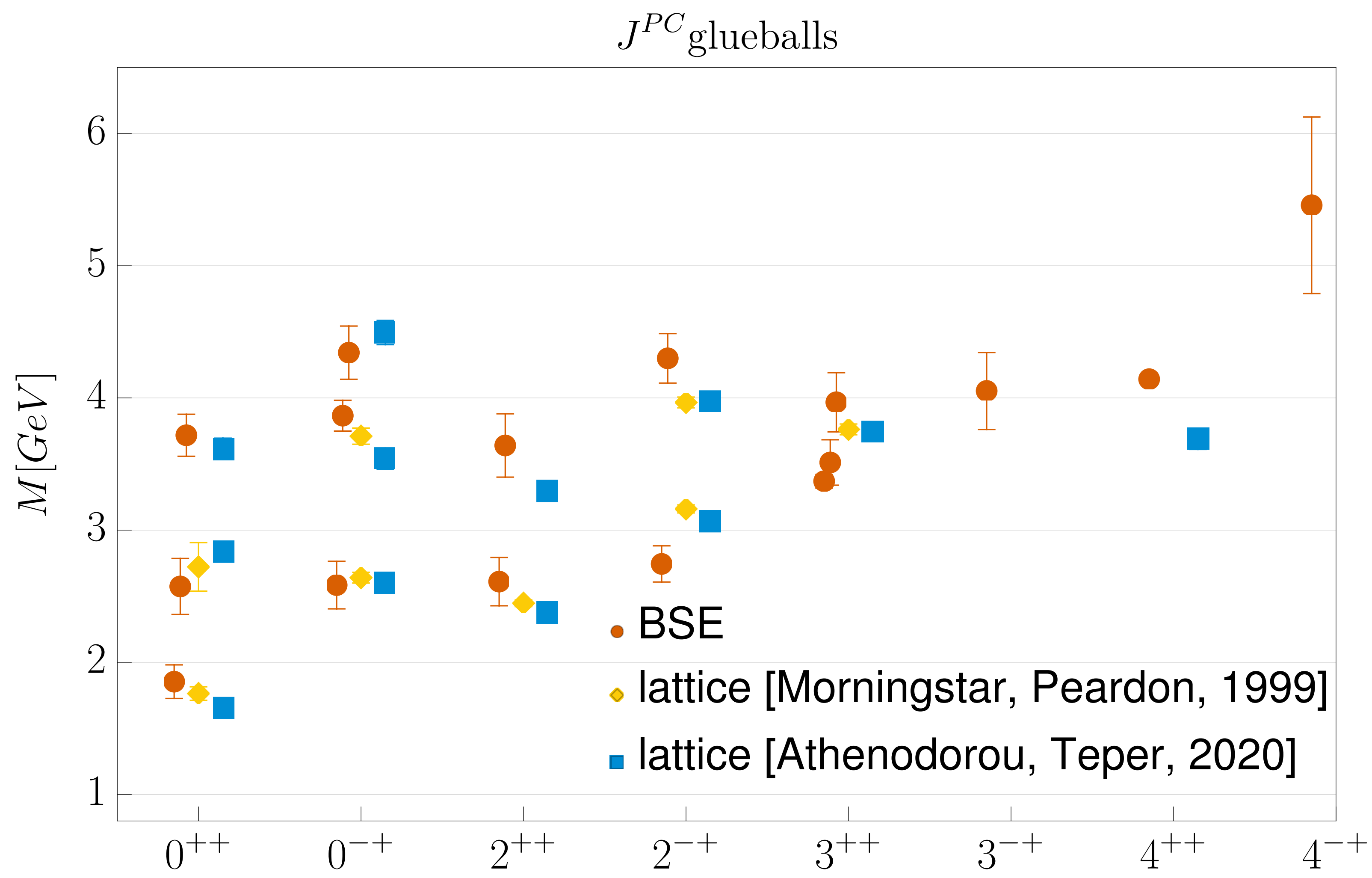
Williams, CF, Heupel, PRD93 (2016) 034026
Huber, PRD 101 (2020) 114009

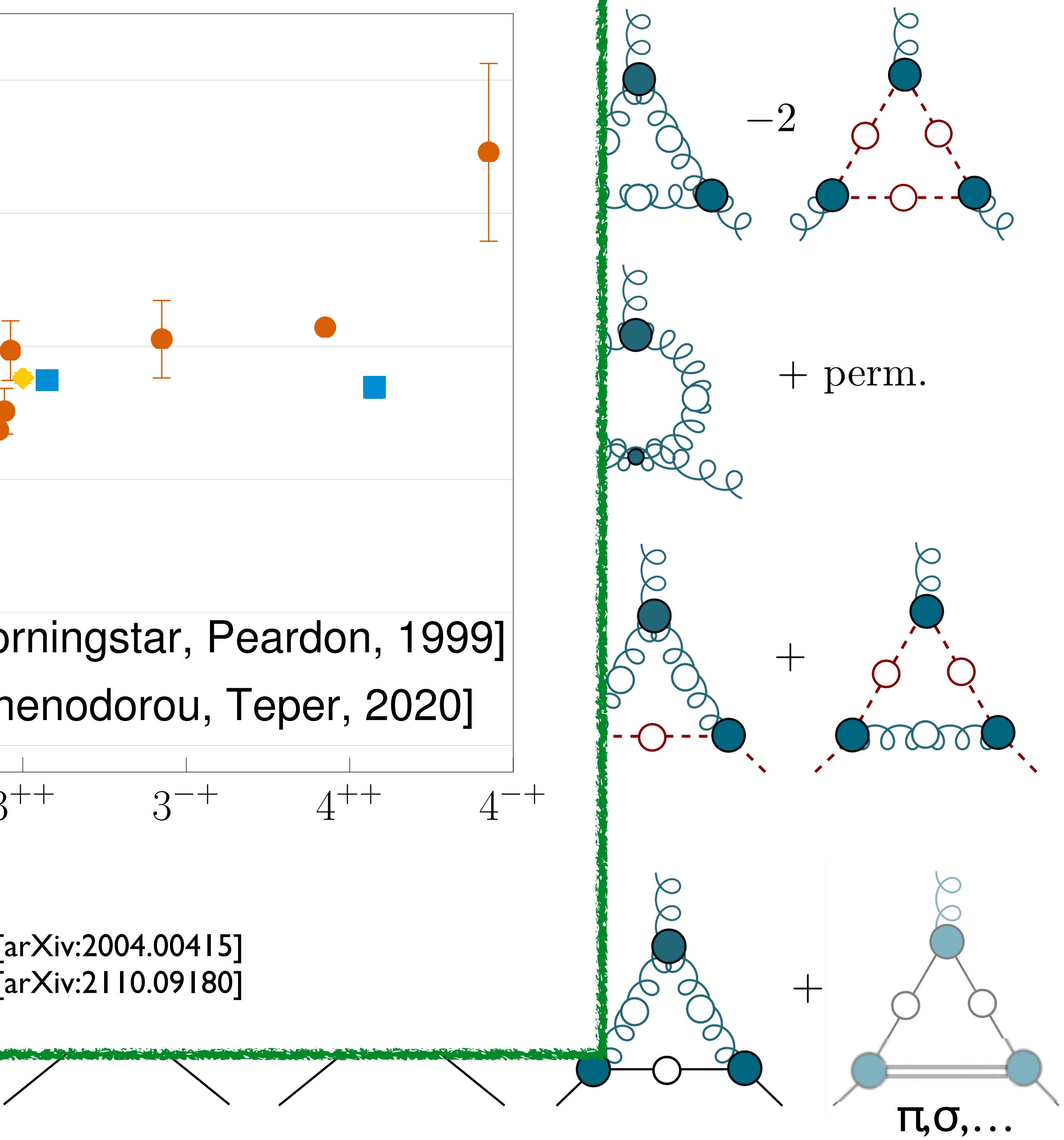
# QCD with DSE

pr






CF, Huber, Sanchis-Alepuz, EPJC 80 (2020) [arXiv:2004.00415]
Huber, CF, Sanchis-Alepuz, EPJC 81 (2021) [arXiv:2110.09180]

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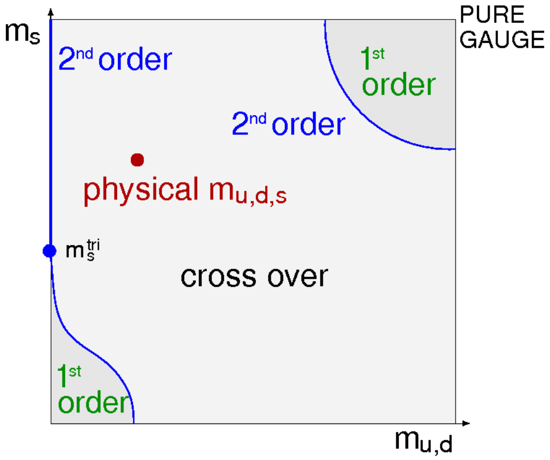
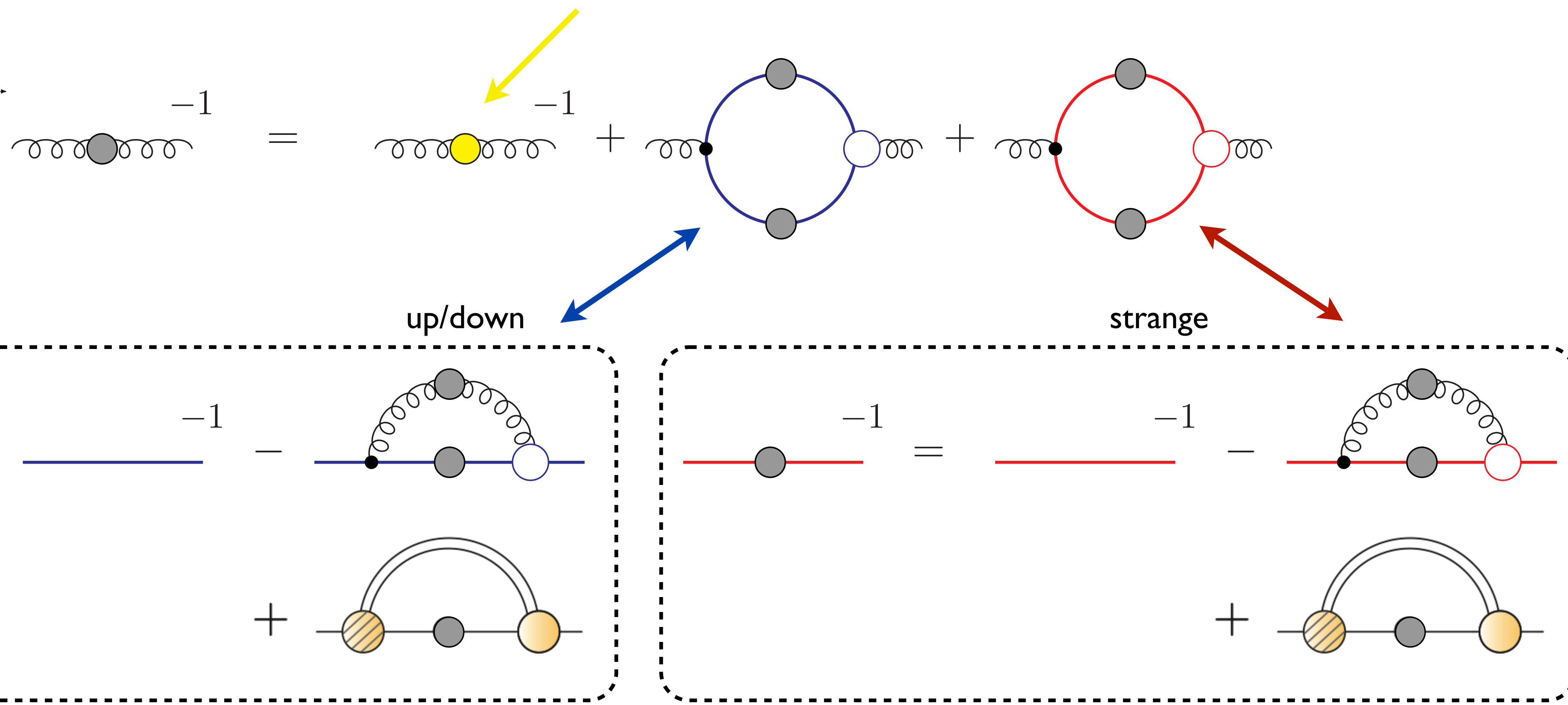
CF, Alkofer, PRD67 (2003) 094020
Williams, CF, Heupel, PRD93 (2016) 034026
Huber, PRD 101 (2020) 114009

# $N_f$=2+1-QCD with DSEs and meson backcoupling




T-dep. propagator from lattice (pure glue; $N_f$=0)



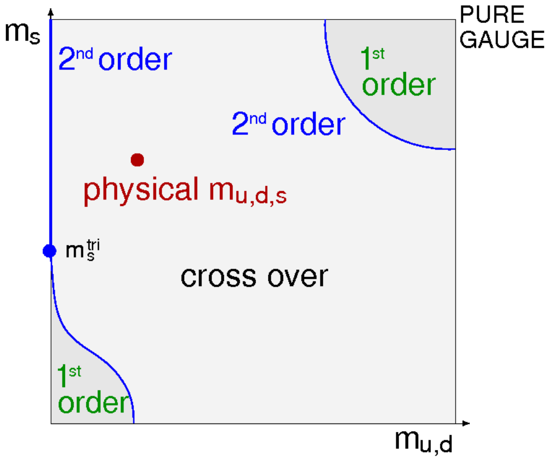

light mesons

strange mesons

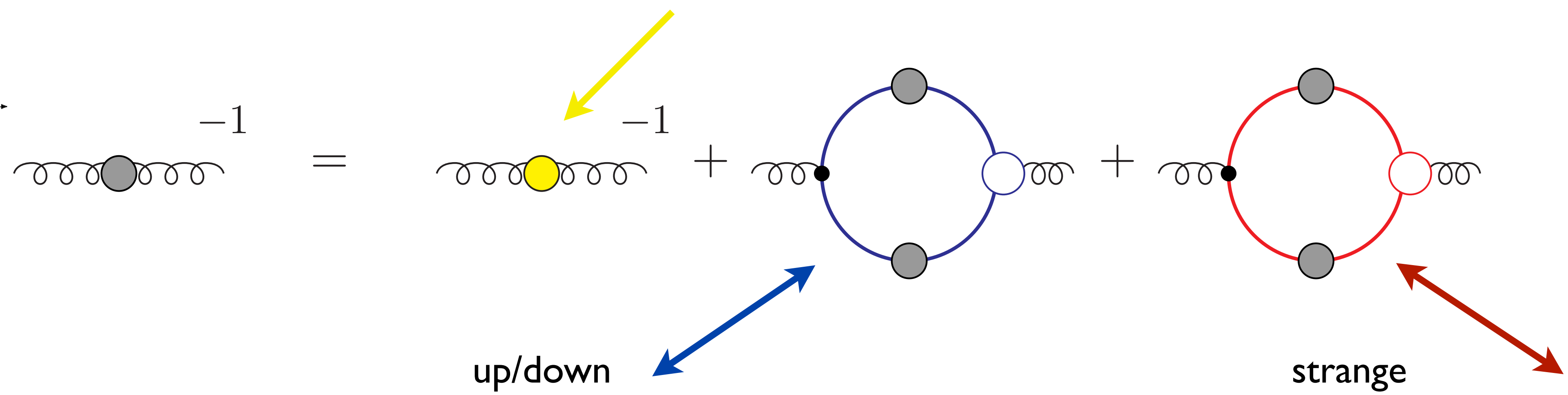
CF, Luecker, Welzbacher, PRD 90 (2014) 034022

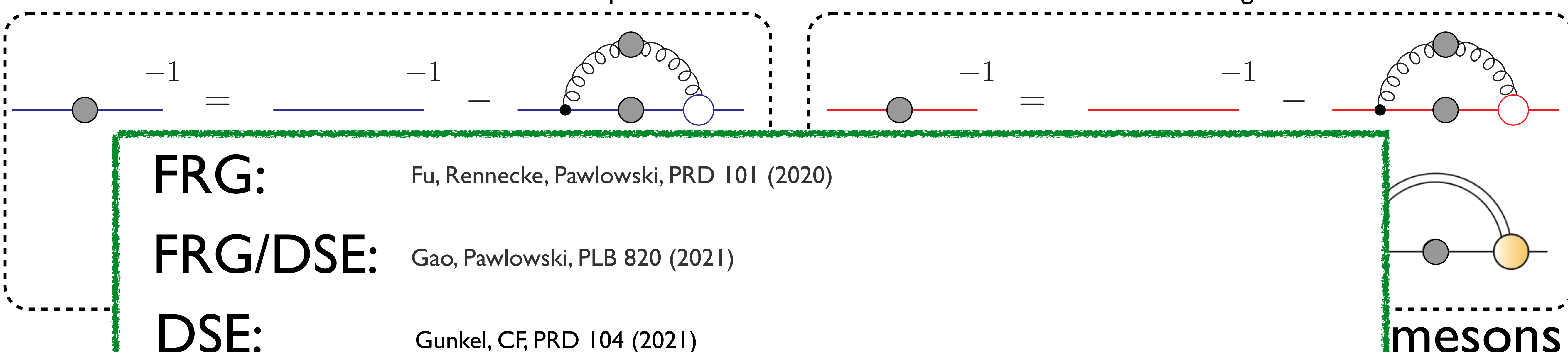
Gunkel, CF, PRD 104 (2021) [2106.08356]

# $N_f$=2+1-QCD with DSEs and meson backcoupling




T-dep. propagator from lattice (pure glue; $N_f$=0)




FRG: Fu, Rennecke, Pawlowski, PRD 101 (2020)

FRG/DSE: Gao, Pawlowski, PLB 820 (2021)

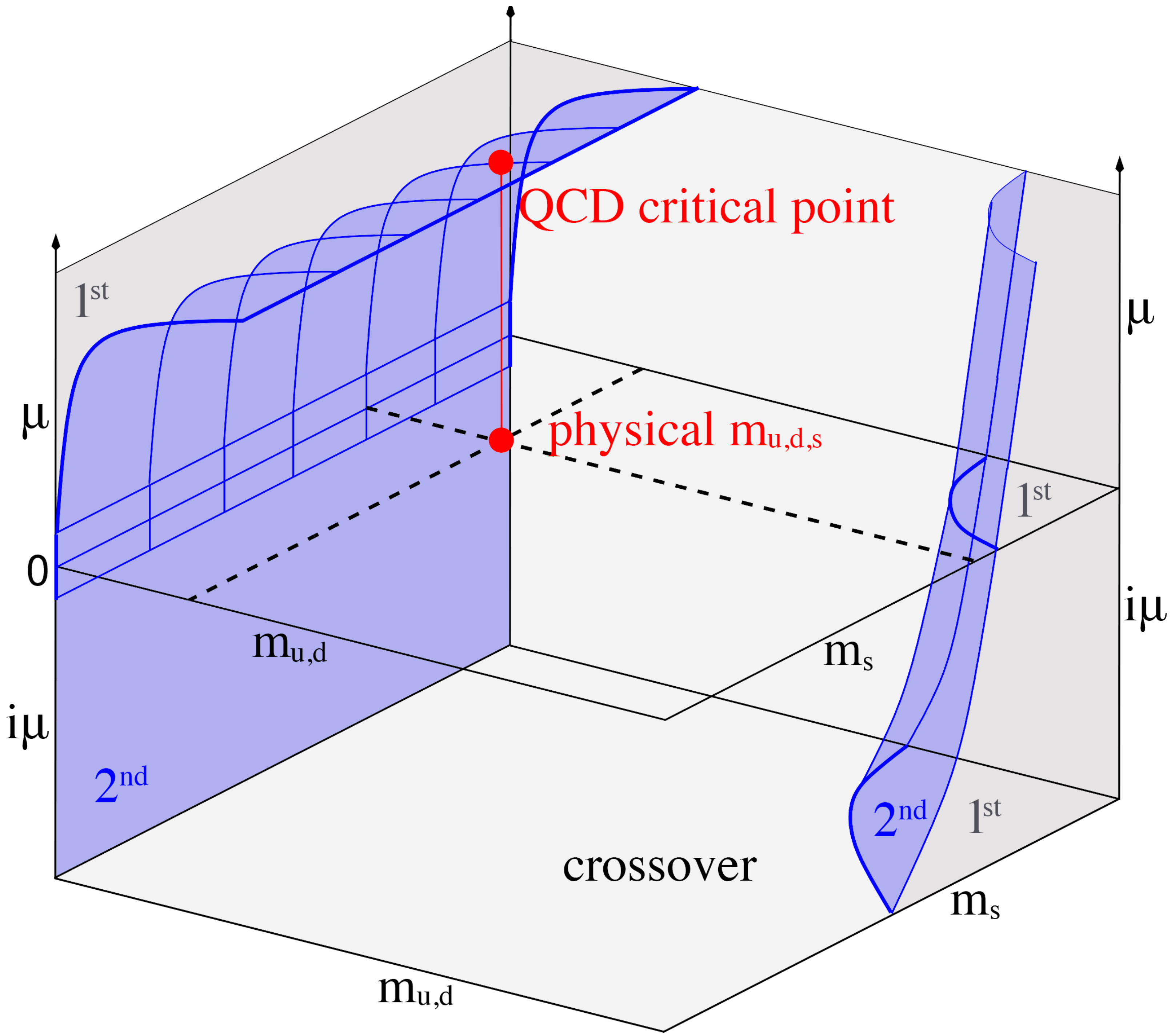
DSE: Gunkel, CF, PRD 104 (2021)

quality of truncations on same level


CF, Luecker, Welzbacher, PRD 90 (2014) 034022

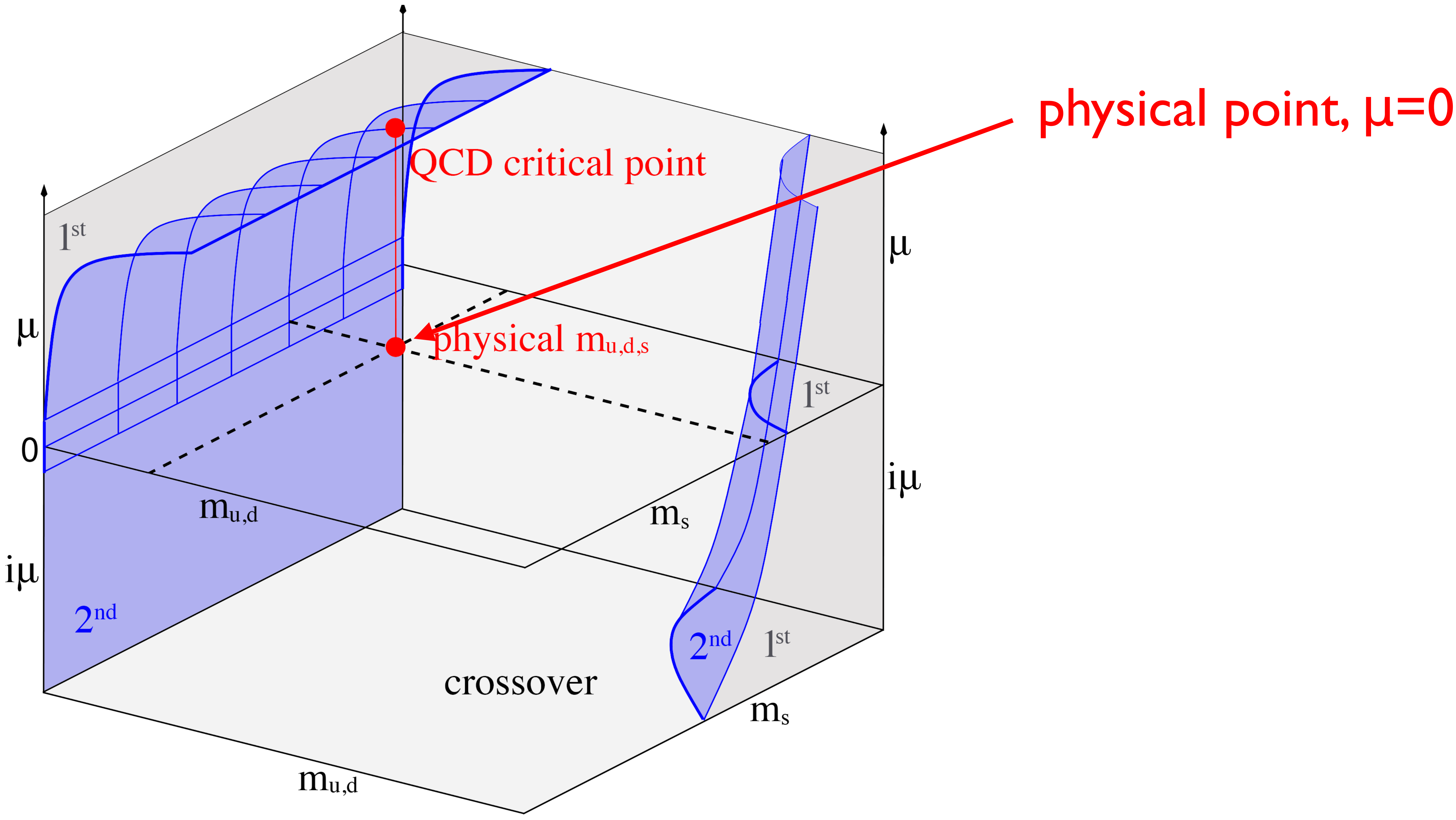
Gunkel, CF, PRD 104 (2021) [2106.08356]

# Roadmap



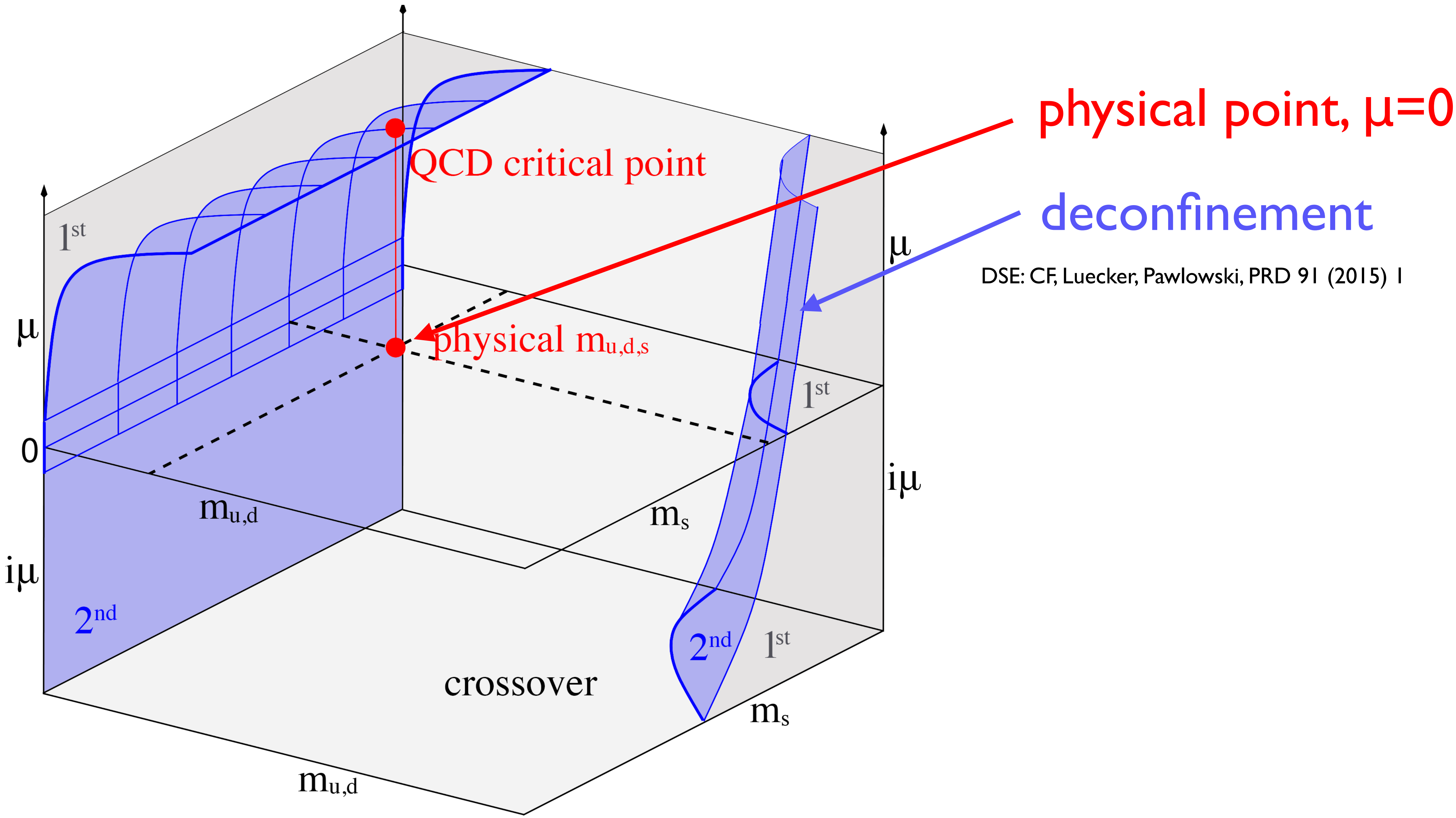
1st
μ
0
iμ
2nd
QCD critical point
physical $m_{u,d,s}$
$m_{u,d}$
$m_s$
μ
1st
iμ
2nd
1st
crossover
$m_s$
$m_{u,d}$

# Roadmap



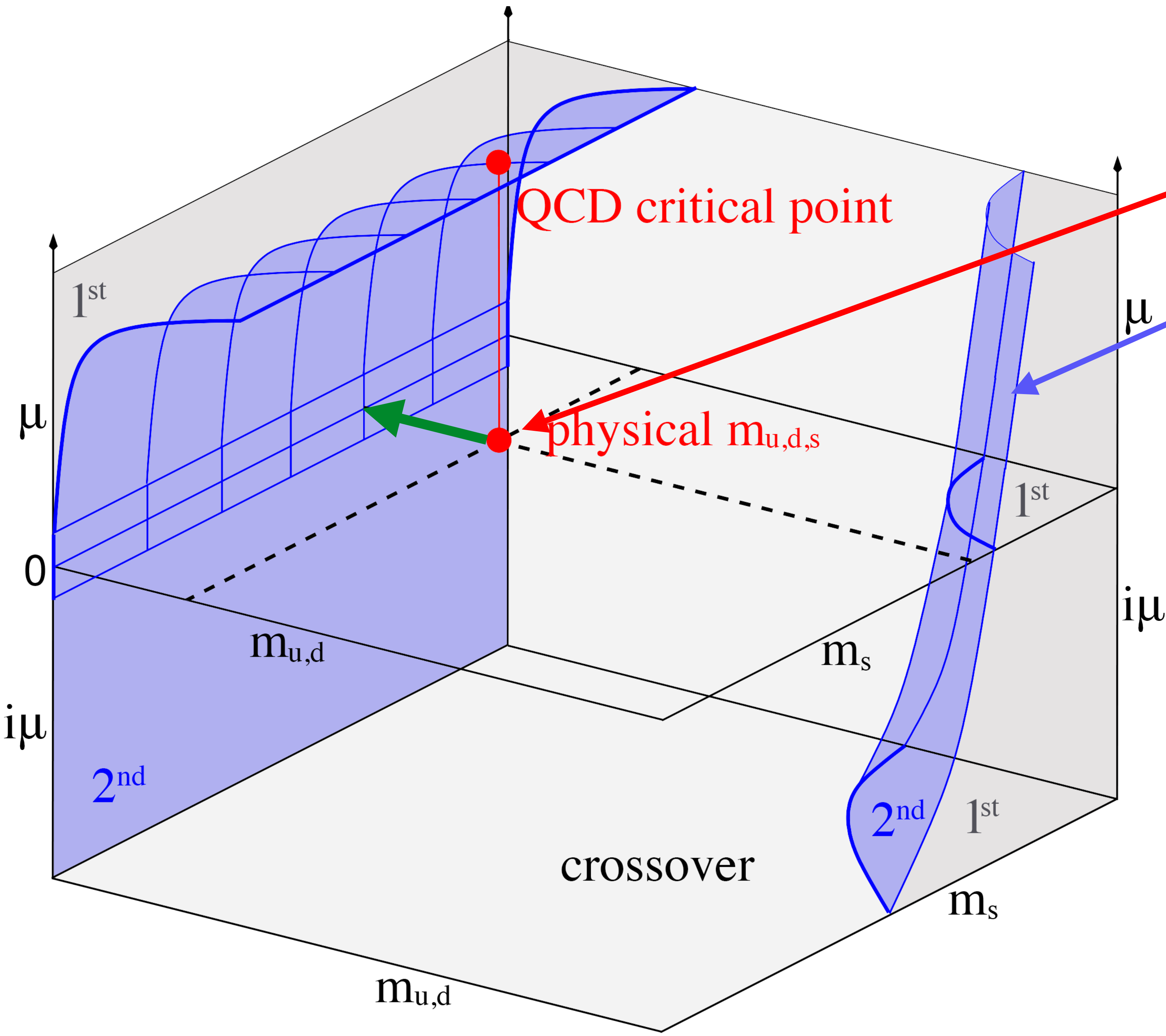
physical point, μ=0
QCD critical point
physical m_{u,d,s}
1st
μ
0
iμ
2nd
m_{u,d}
m_s
crossover
1st
iμ
μ
2nd
1st
m_s
m_{u,d}

# Roadmap




DSE: CF, Luecker, Pawlowski, PRD 91 (2015) 1

# Roadmap




physical point, μ=0

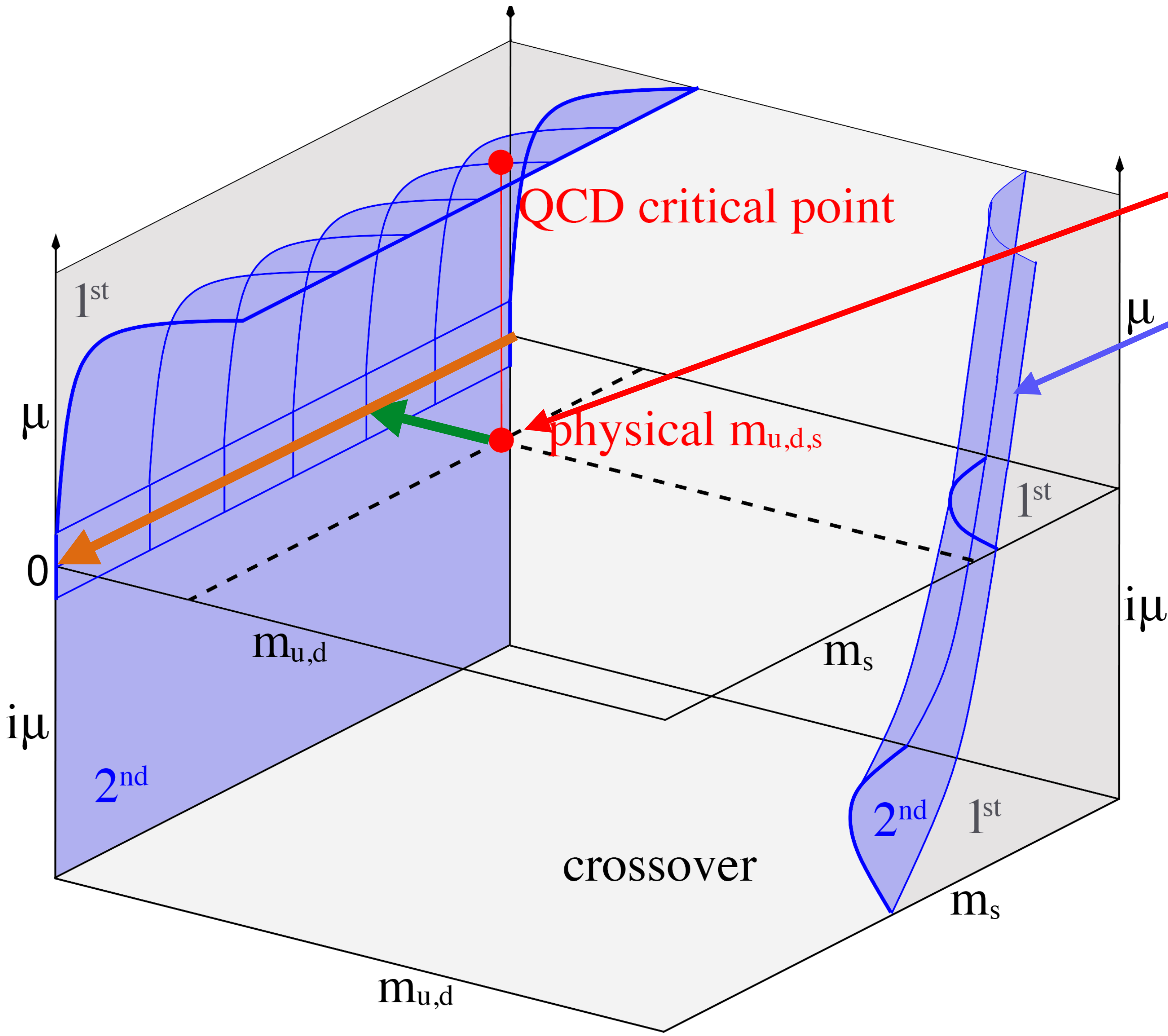
deconfinement

DSE: CF, Luecker, Pawlowski, PRD 91 (2015) 1

towards chiral limit

FRG: Braun et al, PRD 102 (2020) 5, 056010
FRG/DSE: Gao and Pawlowski PRD 105(2022) 094020
DSE: Bernhardt and CF, PRD 108 (2023) 114018

# Roadmap




DSE: CF, Luecker, Pawlowski, PRD 91 (2015) 1

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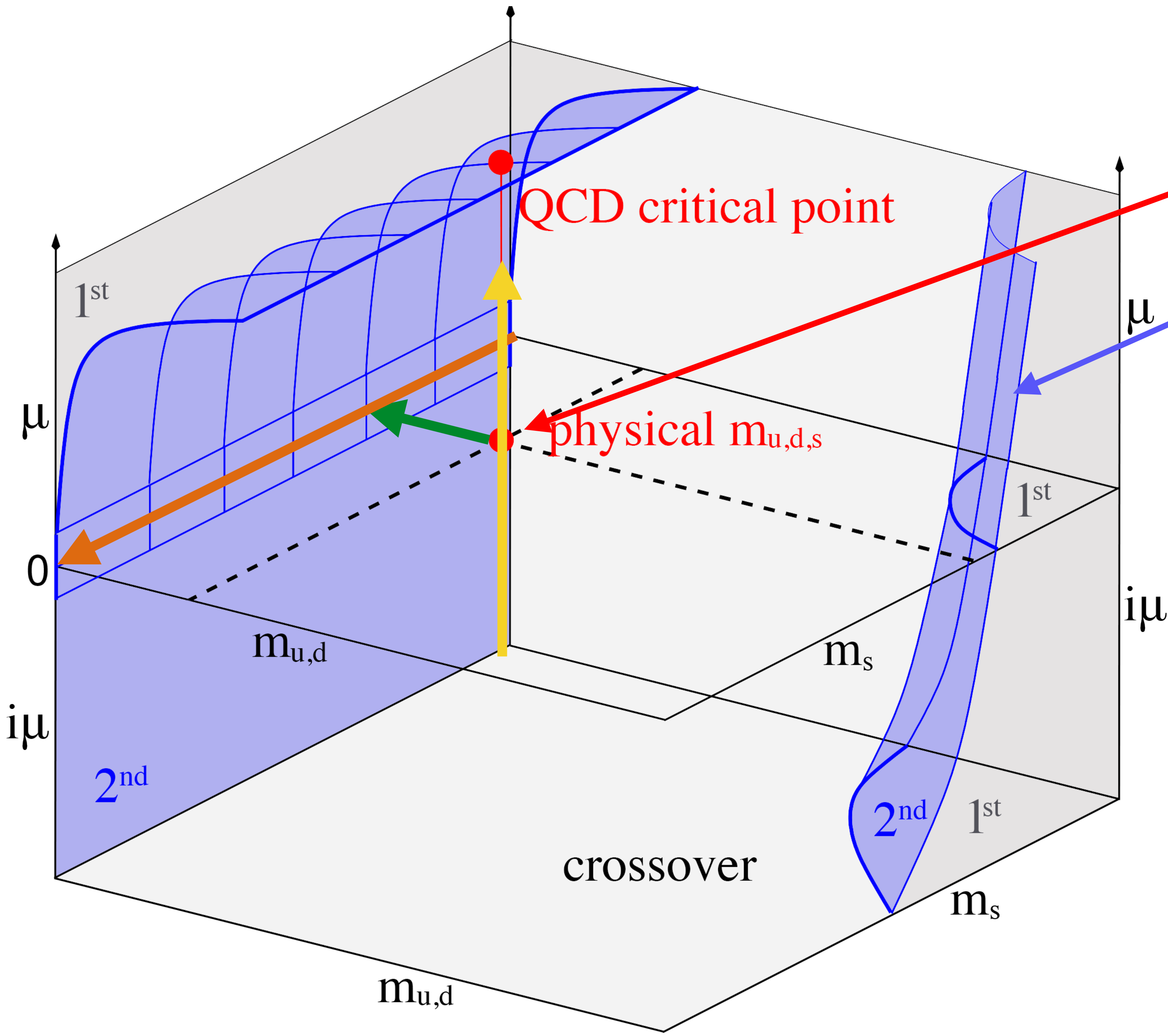


physical point, $\mu$=0

deconfinement

DSE: CF, Luecker, Pawlowski, PRD 91 (2015) 1

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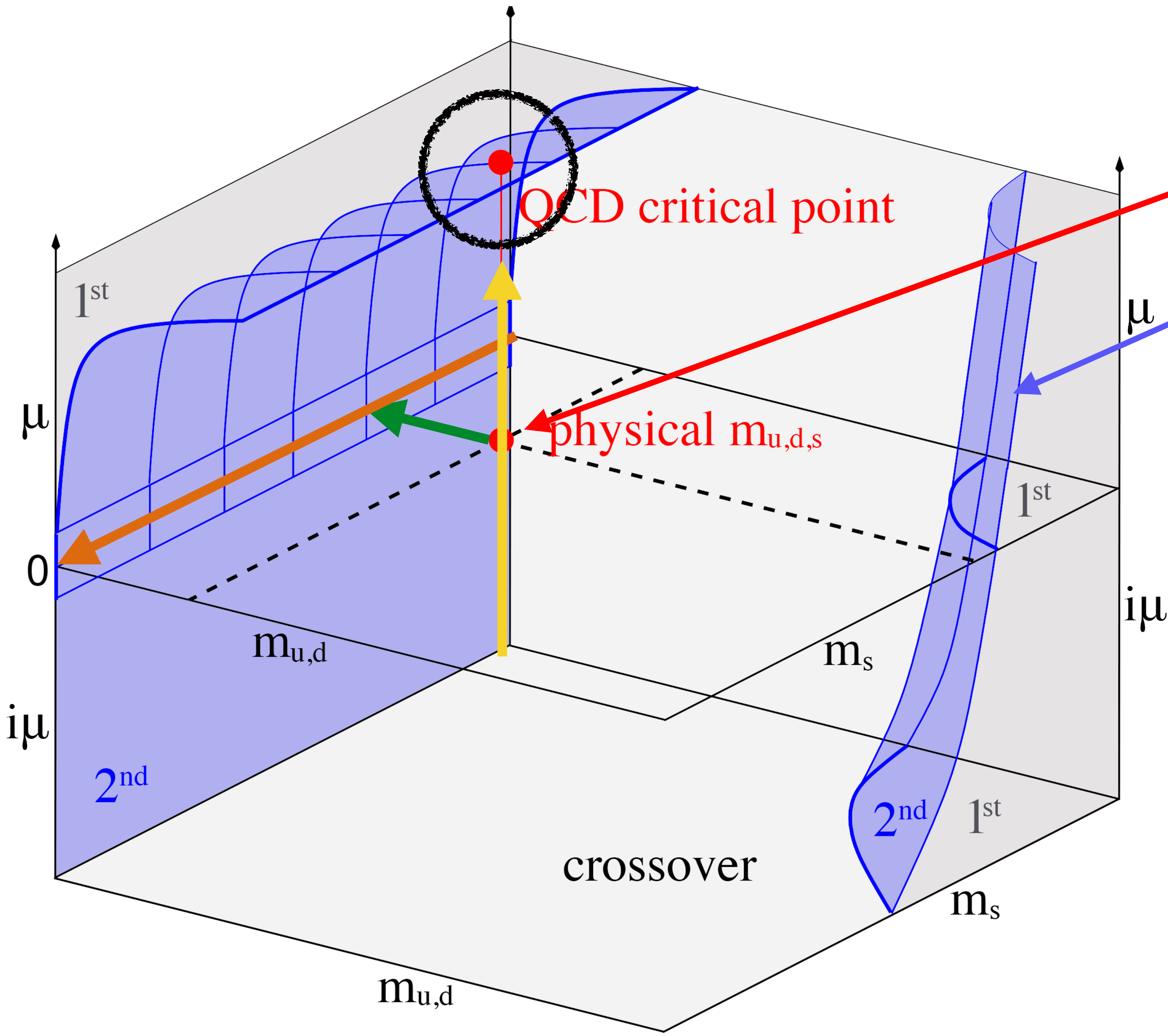
chiral limit

DSE: Bernhardt and CF, PRD 108 (2023) 114018

imaginary $\mu$

FRG (Nf=2): Braun et. al. PRL 106 (2011)
DSE: Bernhardt, CF, EPJA 59 (2023) 8, 181

# Roadmap




physical point, μ=0

deconfinement

DSE: CF, Luecker, Pawlowski, PRD 91 (2015) 1

towards chiral limit

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chiral limit

DSE: Bernhardt and CF, PRD 108 (2023) 114018

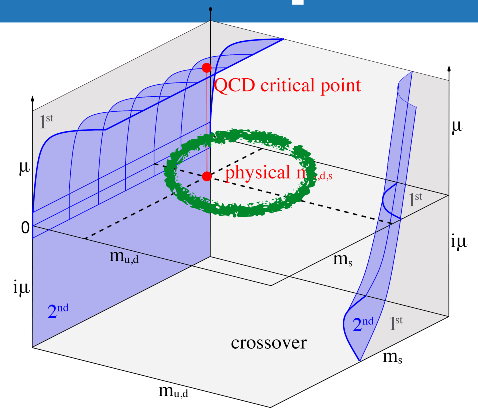
imaginary μ

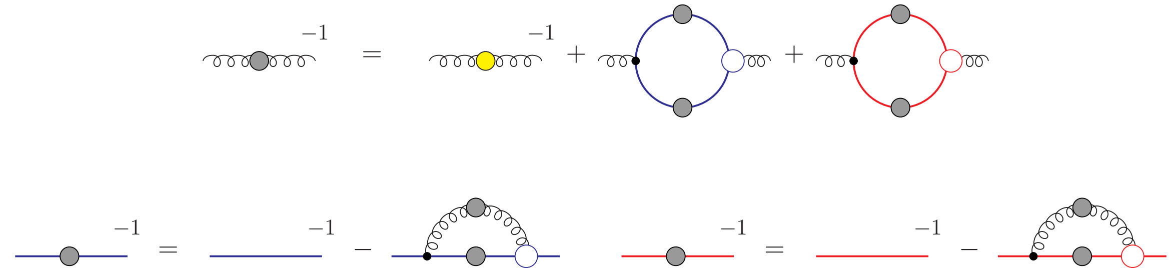
FRG (Nf=2): Braun et. al. PRL 106 (2011)
DSE: Bernhardt, CF, EPJA 59 (2023) 8, 181

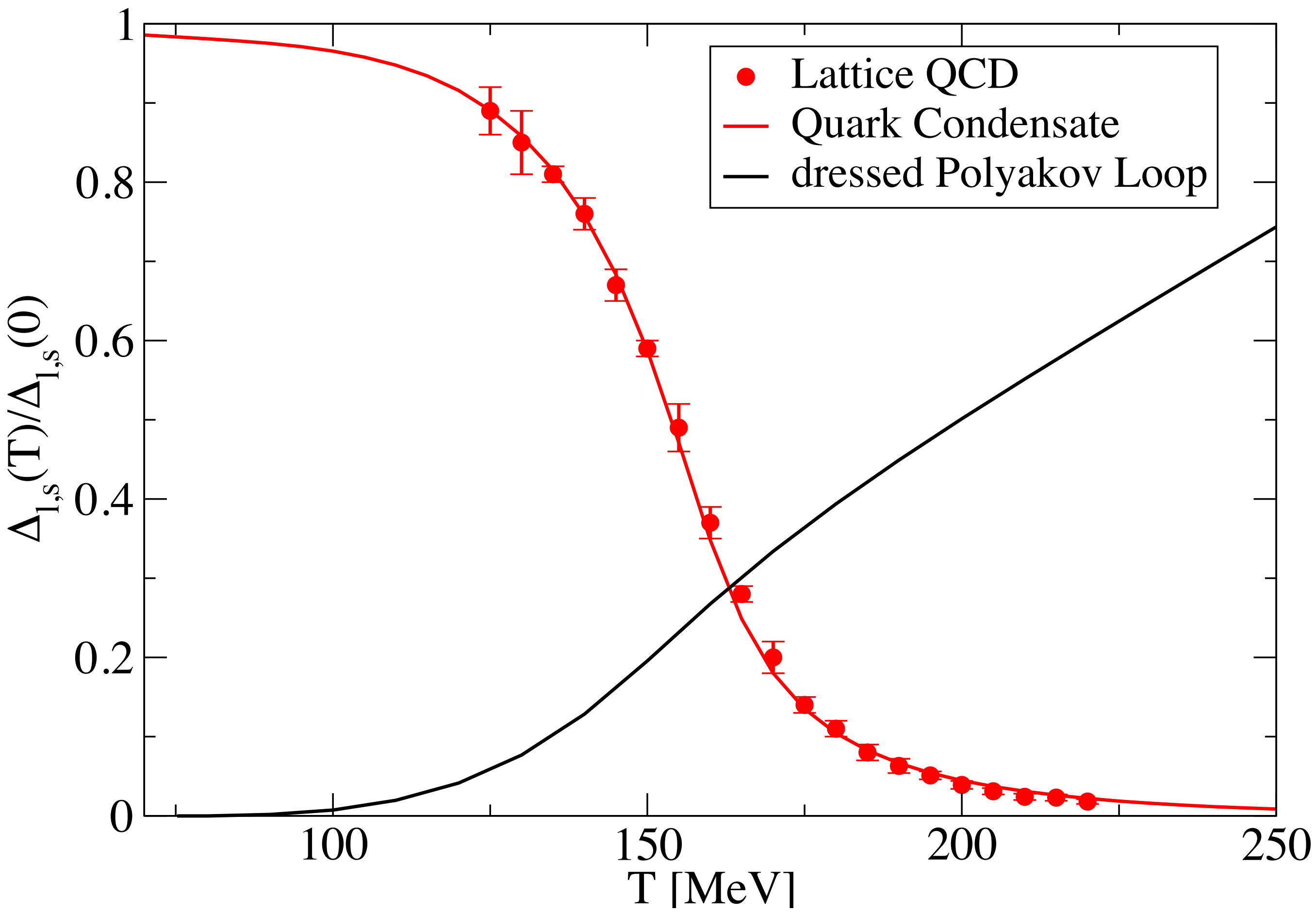
location of CEP

FRG: Fu, Rennecke, Pawlowski, PRD 101 (2020)
FRG/DSE: Gao, Pawlowski, PLB 820 (2021)
DSE: Gunkel, CF, PRD 104 (2021)

# $N_f$=2+1, μ=0, physical point

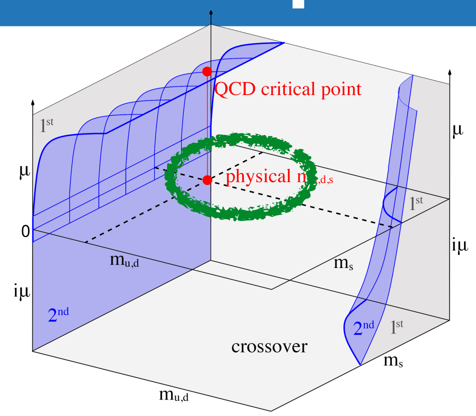


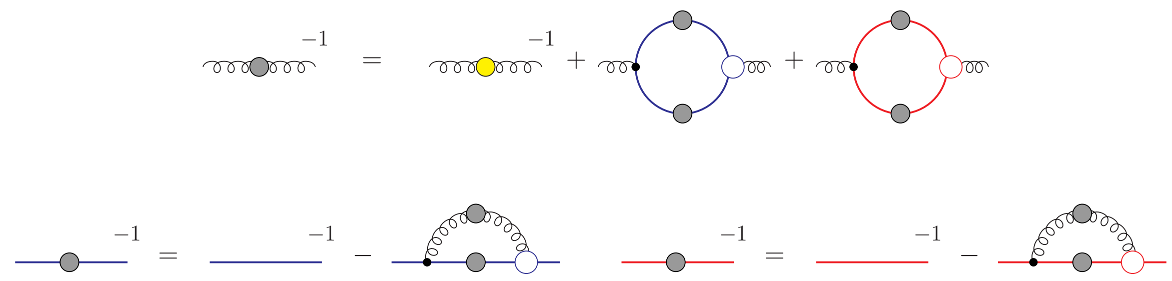



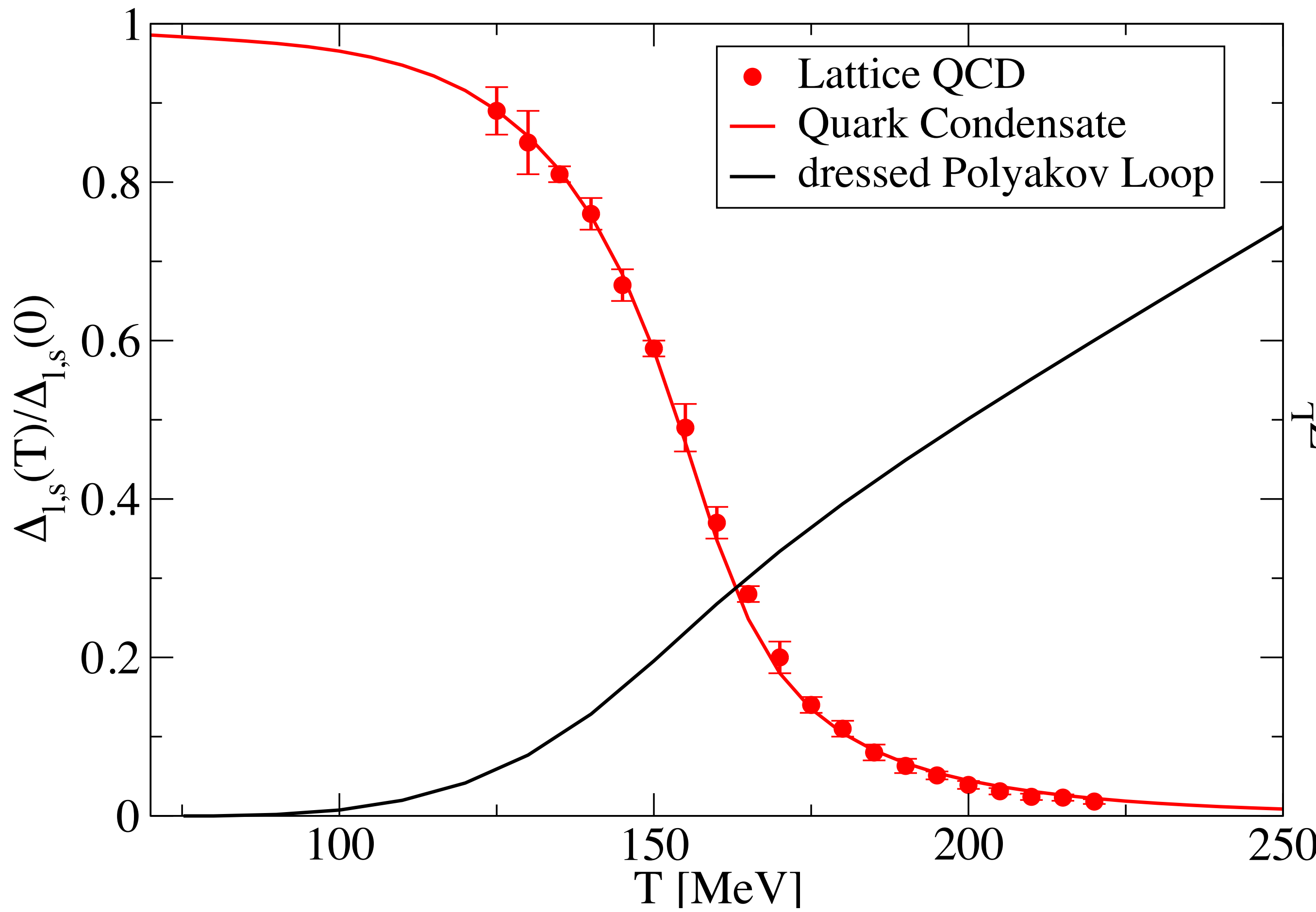


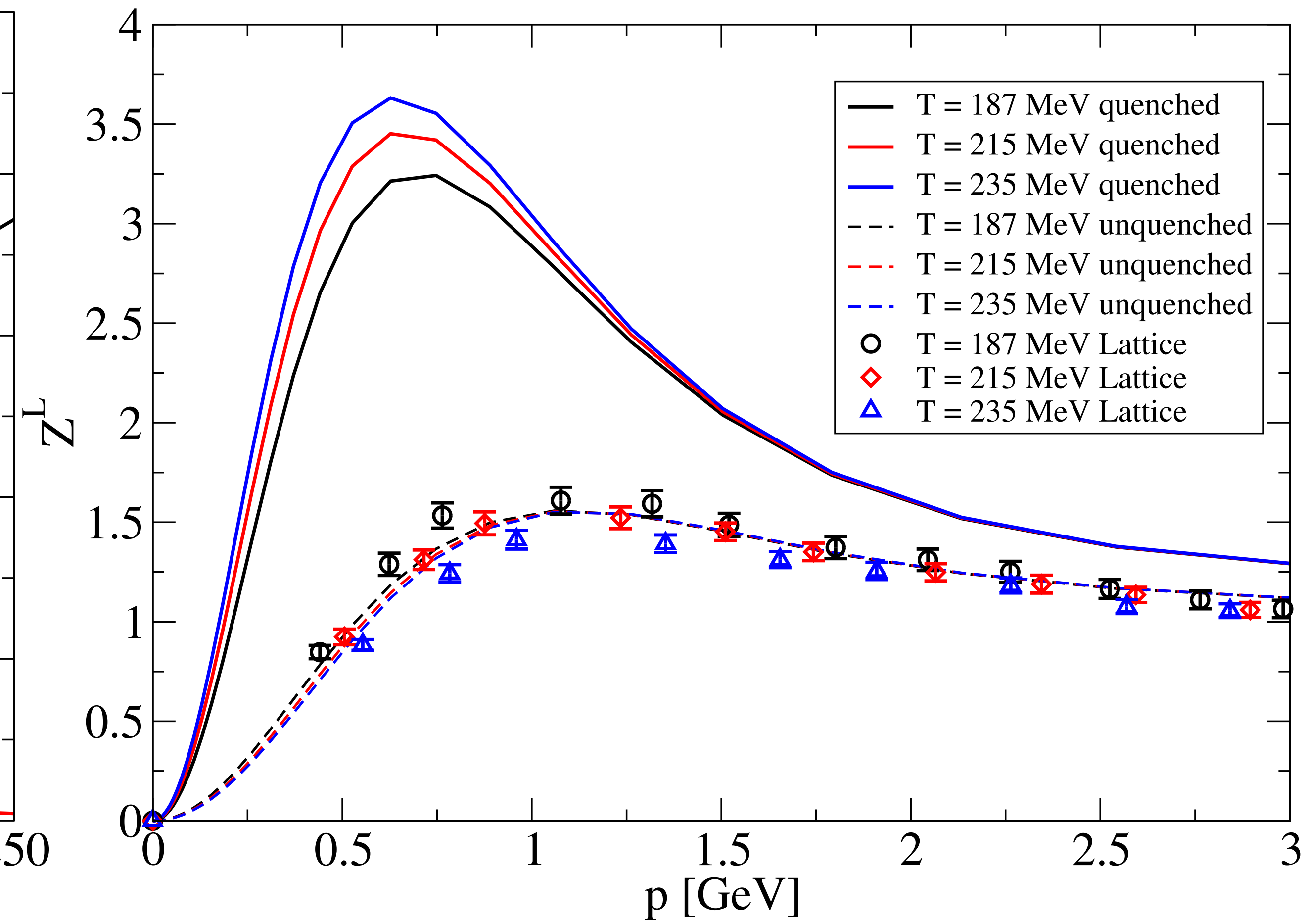

Lattice: Borsanyi *et al.* [Wuppertal-Budapest], JHEP 1009(2010) 073
DSE: CF, Luecker, PLB 718 (2013) 1036,
CF, Luecker, Welzbacher, PRD 90 (2014) 034022

# $N_f$=2+1, μ=0, physical point





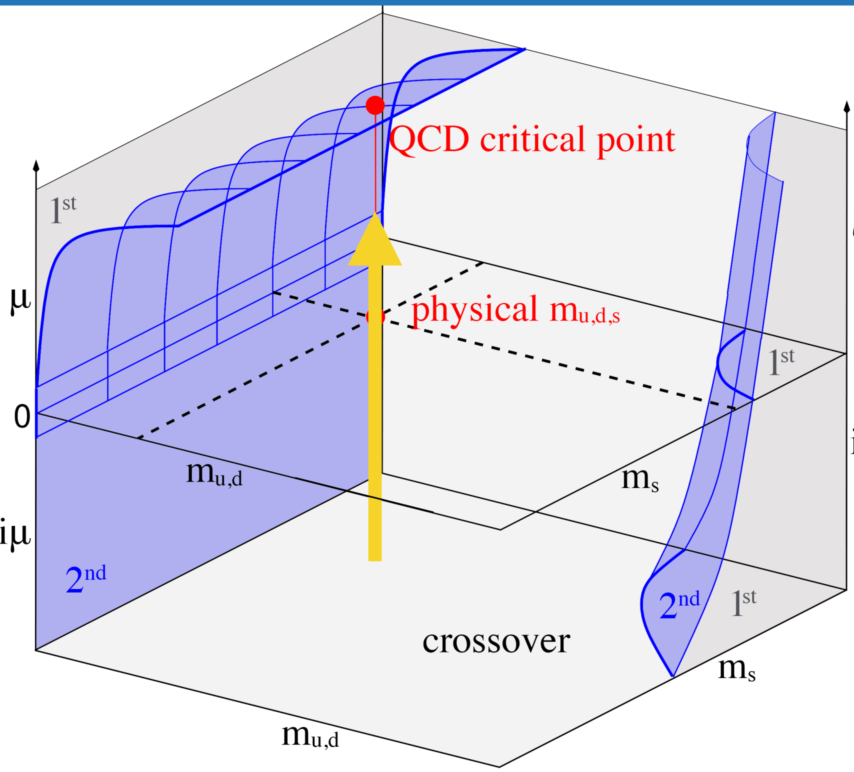

Lattice: Borsanyi *et al.* [Wuppertal-Budapest], JHEP 1009(2010) 073
DSE: CF, Luecker, PLB 718 (2013) 1036,
CF, Luecker, Welzbacher, PRD 90 (2014) 034022

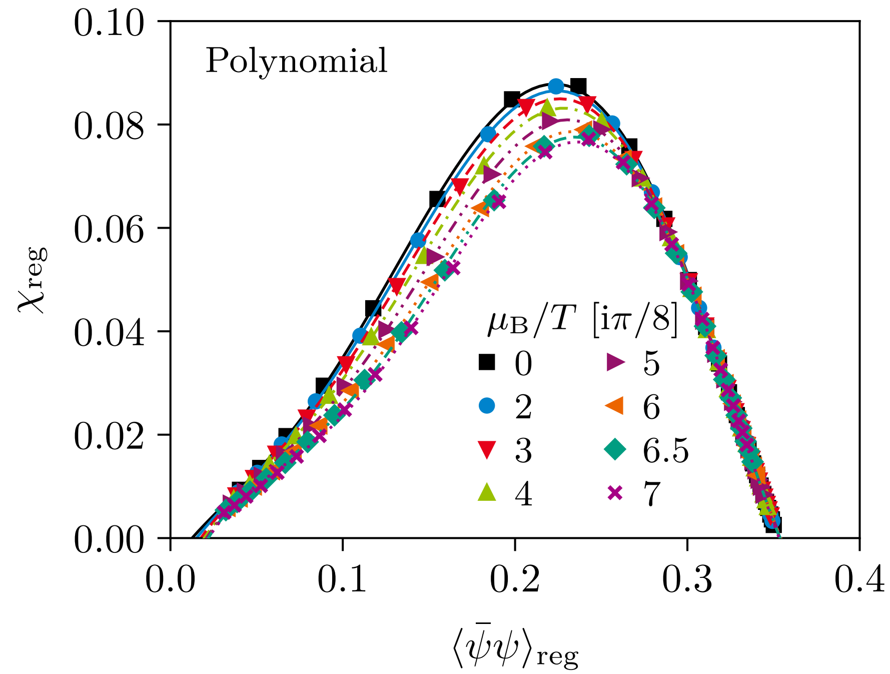
Lattice: Aouane, et al. PRD D87 (2013), [arXiv:1212.1102]
DSE: CF, Luecker, PLB 718 (2013) 1036, [arXiv:1206.5191]
CF, Luecker, Welzbacher, PRD 90 (2014) 034022

- quantitative agreement: DSE prediction verified by lattice
  FRG: similar results

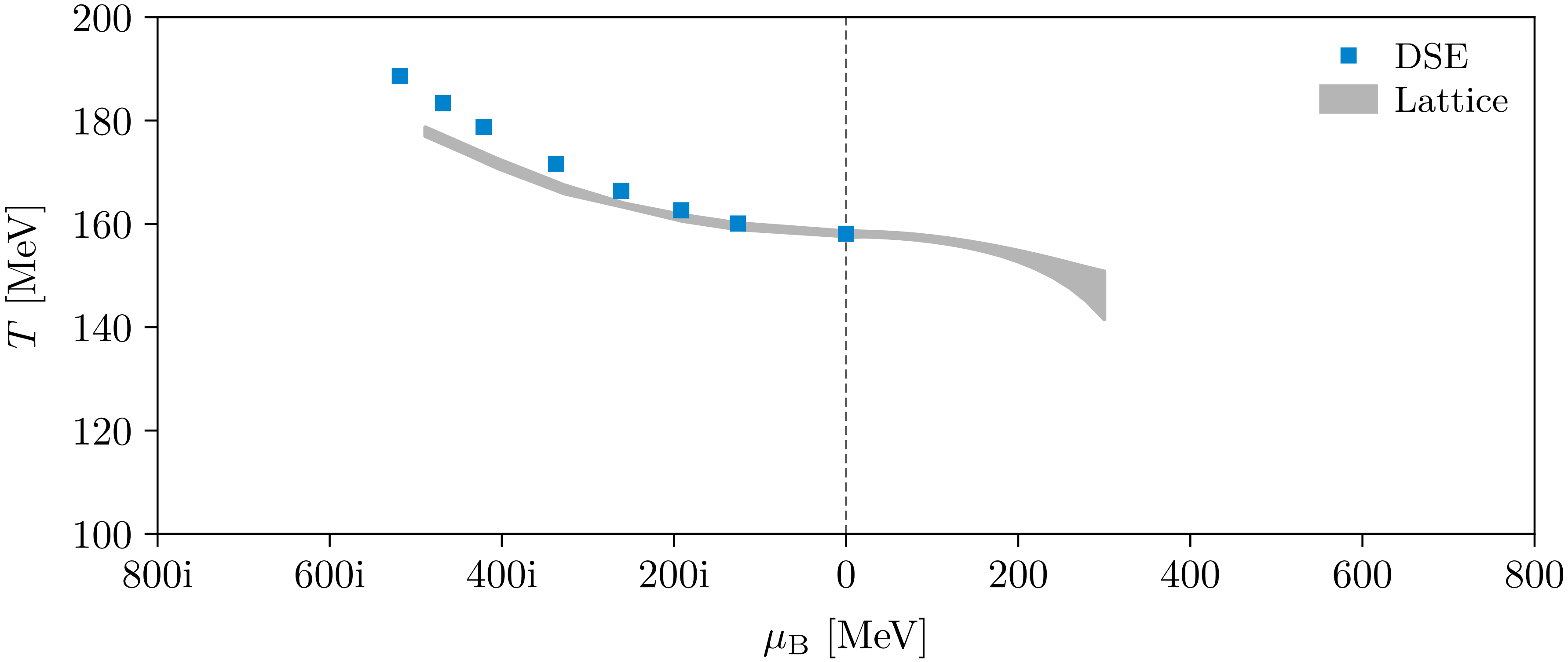
Fu, Rennecke, Pawlowski, PRD 101 (2020)

# Extrapolation from imaginary chemical potential




$$\chi(T) = \frac{\partial\langle\bar\psi\psi\rangle(T)}{\partial m_{\mathbf{u}}}$$



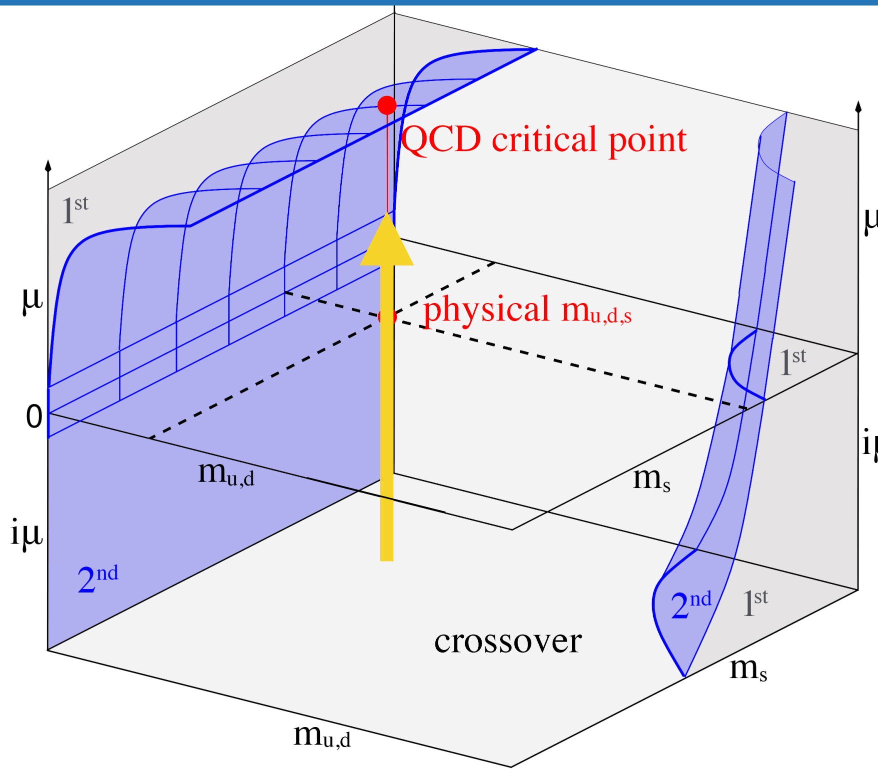

Lattice: Borsanyi et al. PRL 125 052001 (2020)
DSE: Bernhardt, CF, EPJA 59 (2023) 8, 181

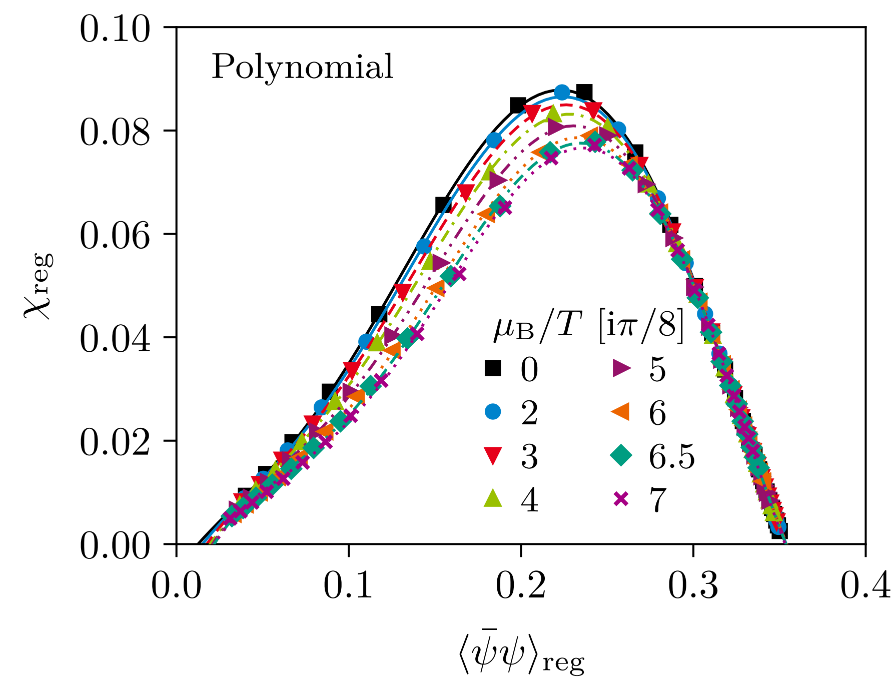
$$\frac{T_c(\mu_B)}{T_c} = 1 - \kappa_2\left(\frac{\mu_B}{T_c}\right)^2 - \kappa_4\left(\frac{\mu_B}{T_c}\right)^4$$

$$\kappa_2^{\rm poly} = 0.0196\,, \quad \kappa_4^{\rm poly} = 0.00015\,,$$

see also FRG (Nf=2): Braun et. al. PRL 106 (2011)

# Extrapolation from imaginary chemical potential

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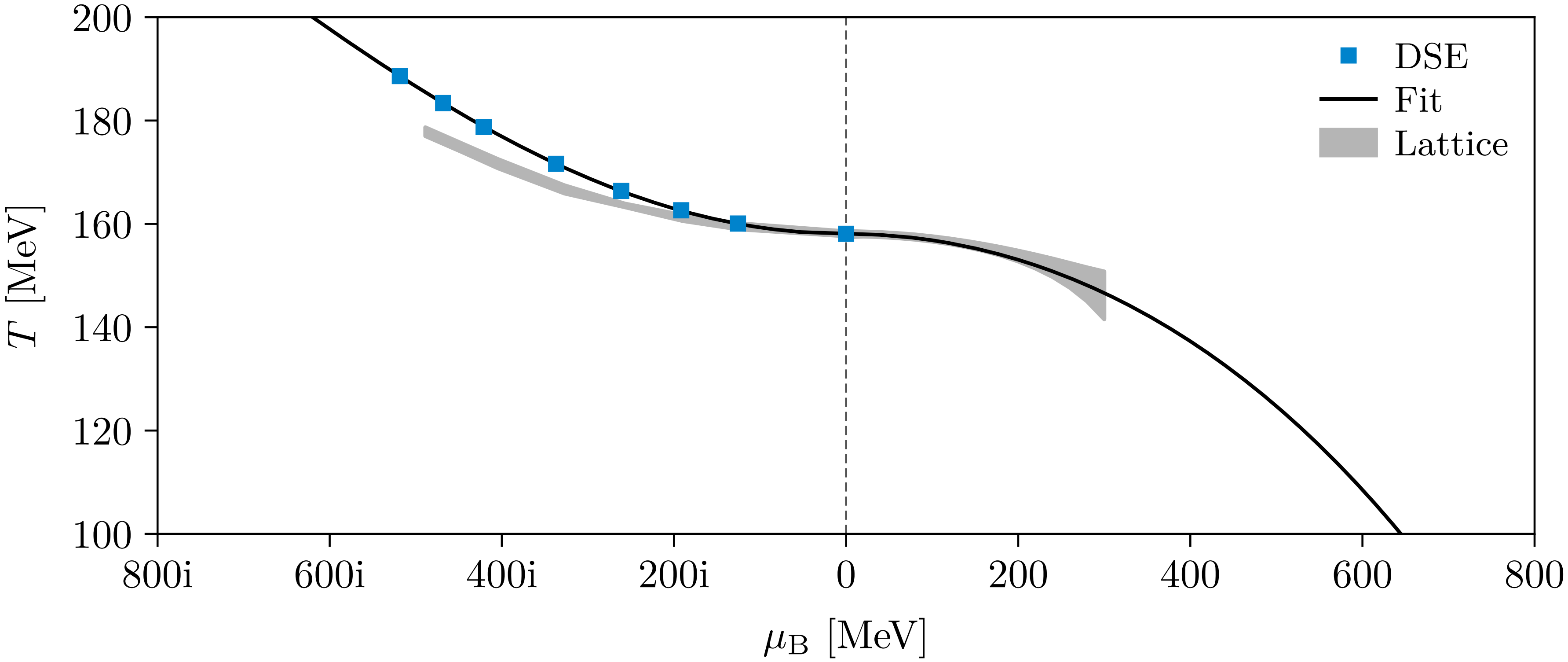


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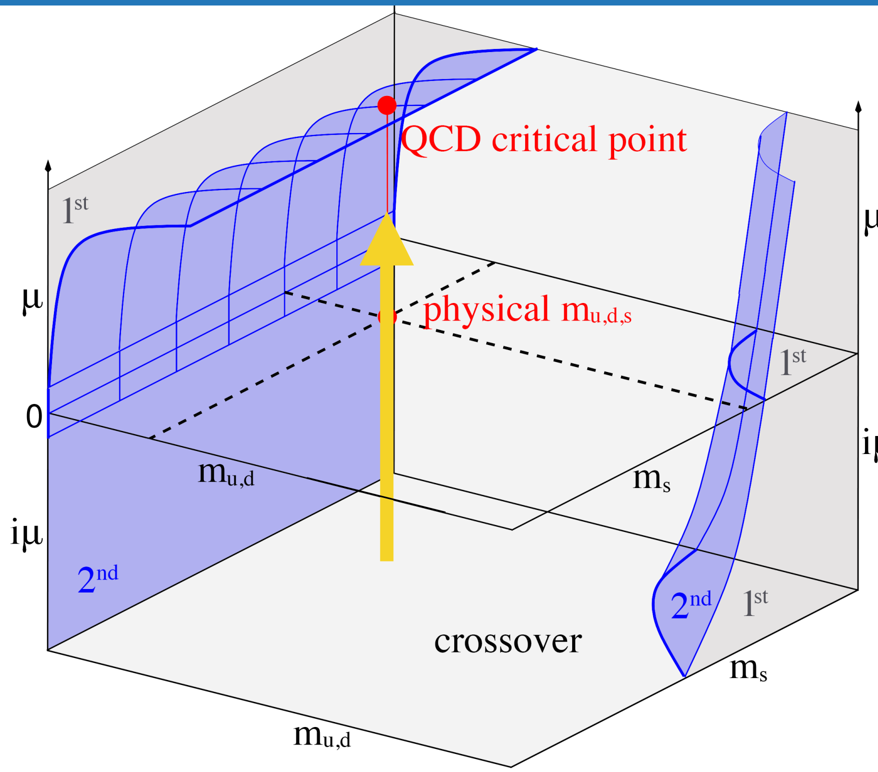


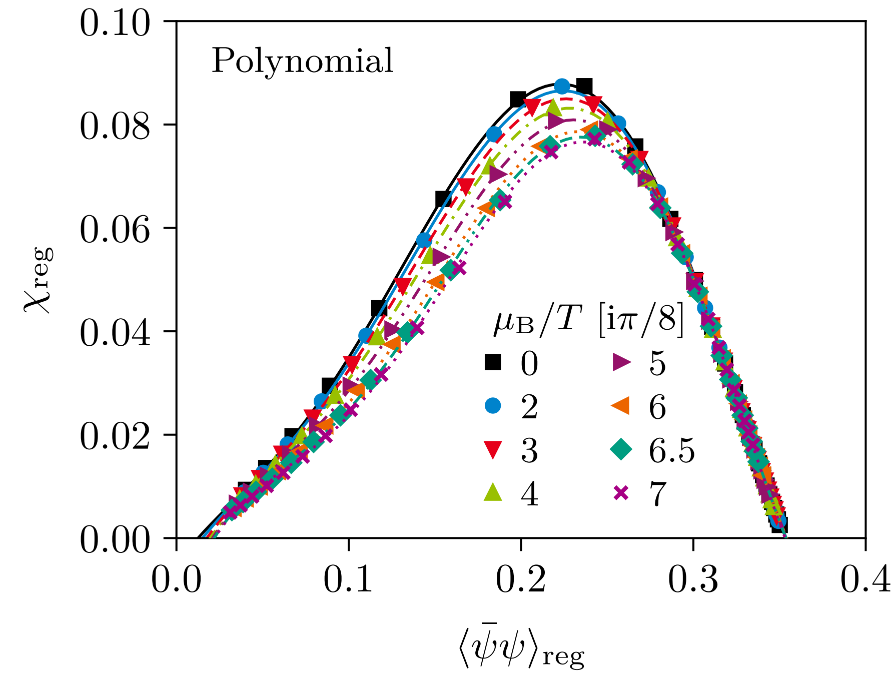


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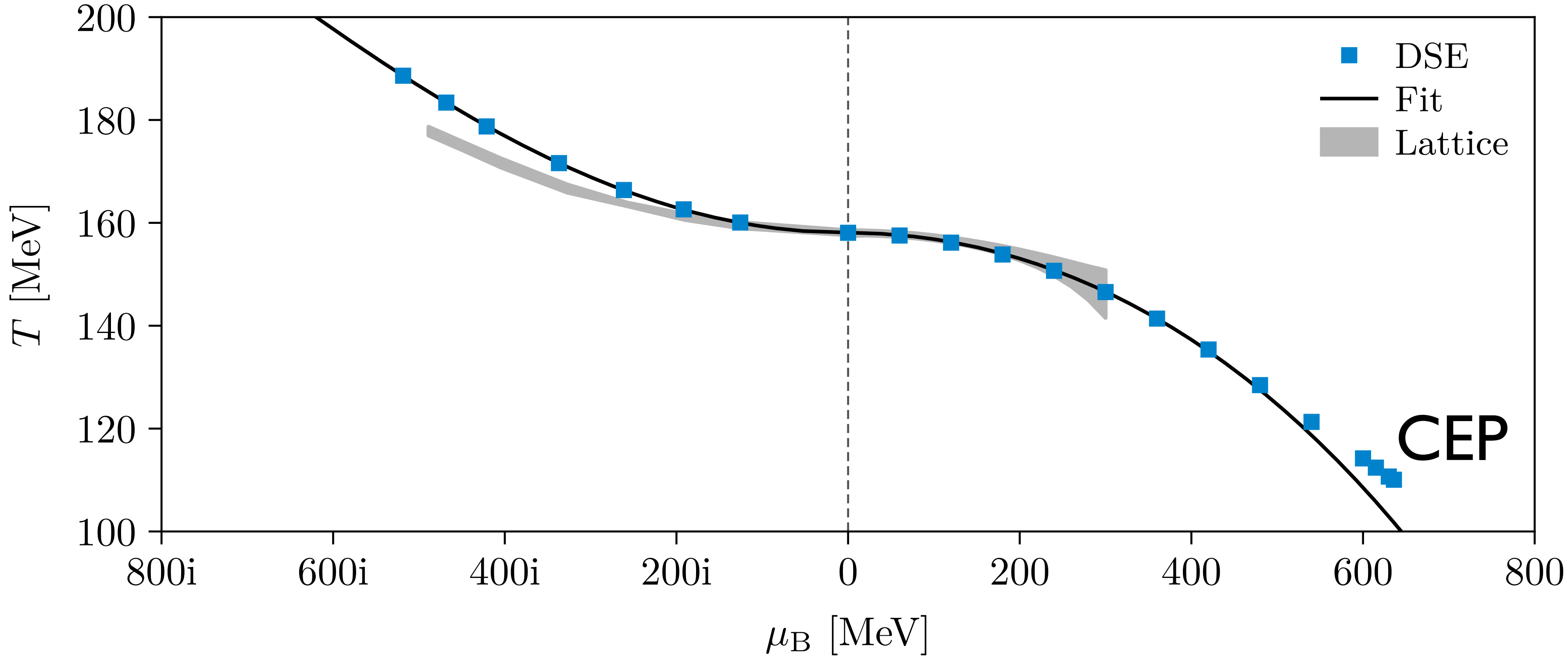


Lattice: Borsanyi et al. PRL 125 052001 (2020)
DSE: Bernhardt, CF, EPJA 59 (2023) 8, 181

- Extrapolation works very well!

$$\frac{T_c(\mu_B)}{T_c} = 1 - \kappa_2\left(\frac{\mu_B}{T_c}\right)^2 - \kappa_4\left(\frac{\mu_B}{T_c}\right)^4$$

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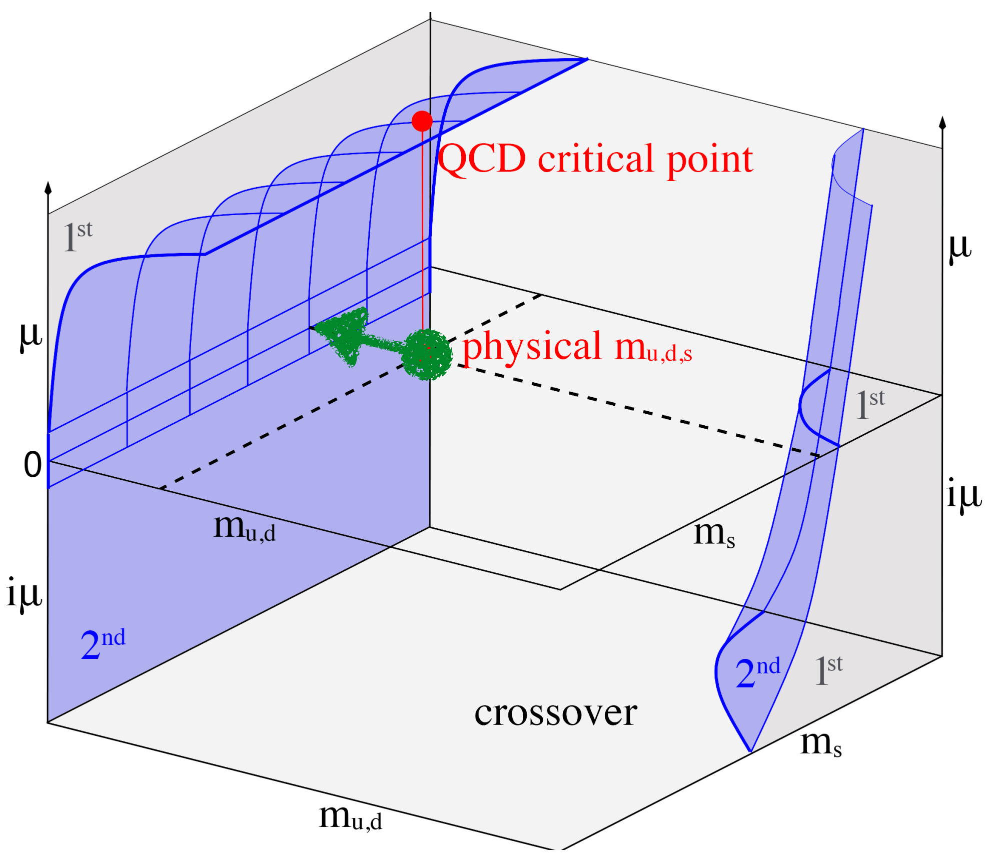
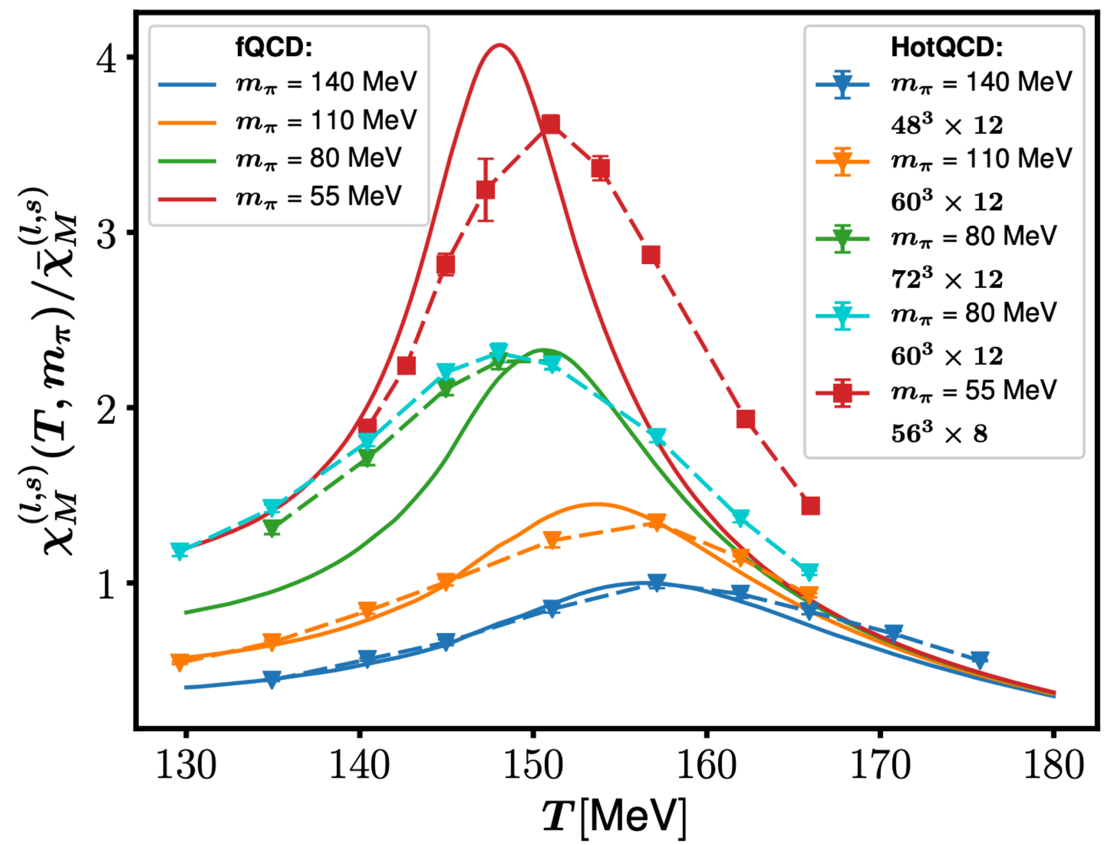
see also FRG (Nf=2): Braun et. al. PRL 106 (2011)

# Towards the chiral limit… (FRG+DSE)

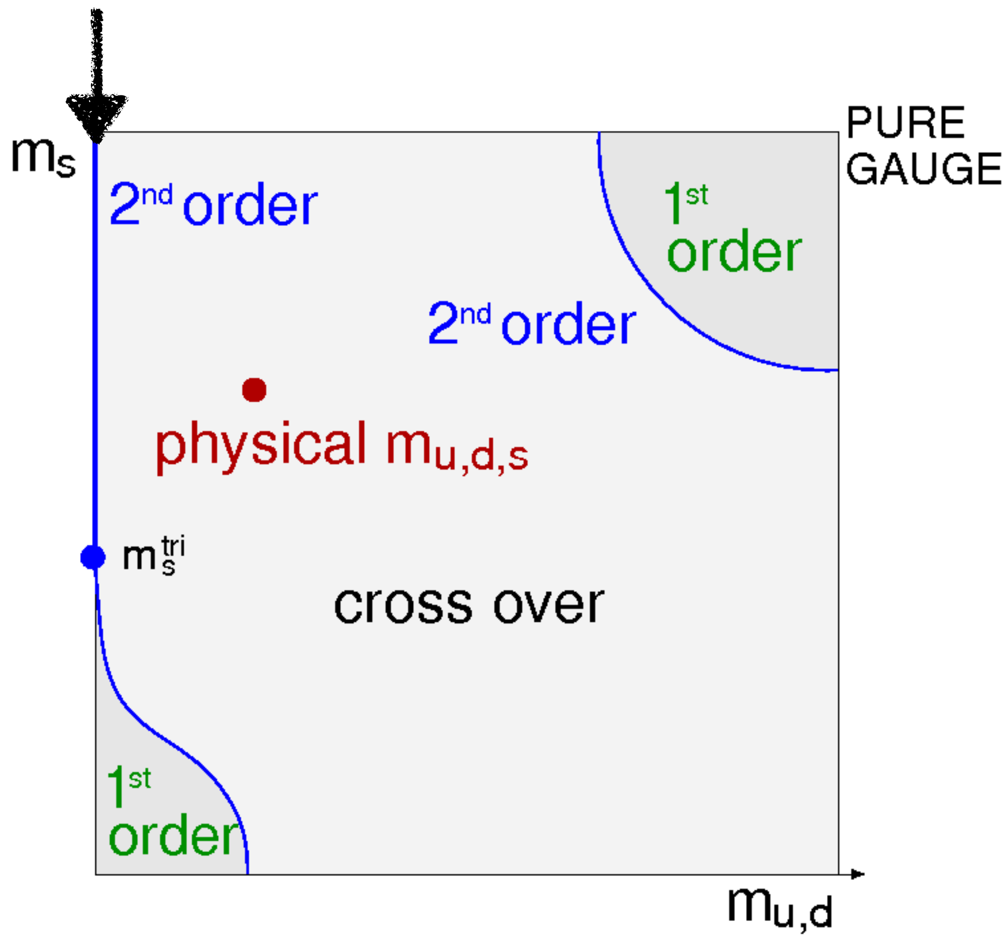


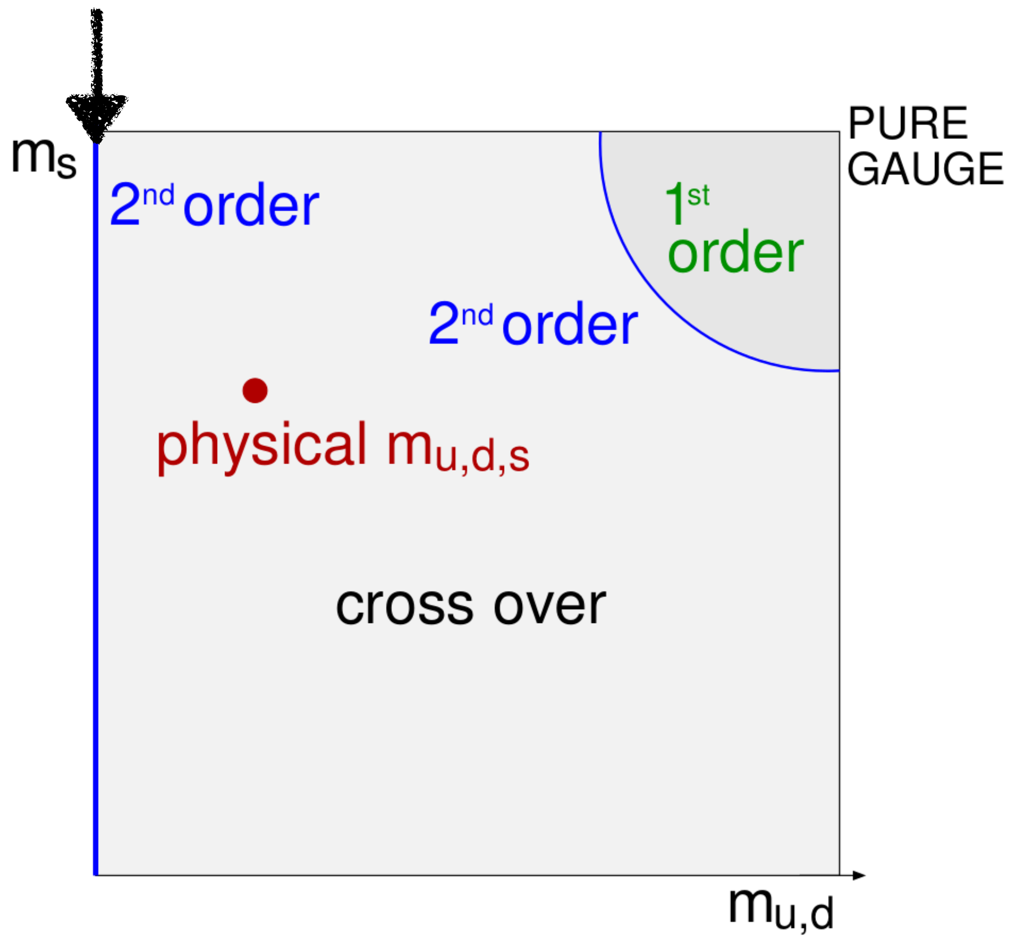

HotQCD: Ding et al. PRL 123, 062002 (2019)
FRG: Braun et al, PRD 102 (2020) 5, 056010
FRG/DSE: Gao and Pawlowski PRD 105(2022) 094020
DSE: Bernhardt and CF, PRD 108 (2023) 114018

| $m_\pi$ [MeV] | | 0 | 55 | 80 | 110 | 140 |
|---|---|---|---|---|---|---|
| $T_c$ [MeV] | DSE | 146.7 | 149.9 | 151.6 | 154.0 | 156.7 |
| | FRG [63] | 142 | 148.0 | 150.5 | 153.6 | 156.3 |
| | FRG–DSE [96] | 141.3 | 146.5 | 149.1 | 152.1 | 155.4 |
| | HotQCD ($N_\tau = 12$) [95] | - | - | $149.7^{+0.3}_{-0.3}$ | $155.6^{+0.6}_{-0.6}$ | $158.2^{+0.5}_{-0.5}$ |
| | HotQCD ($N_\tau = 8$) [95] | - | $150.9^{+0.4}_{-0.4}$ | $153.9^{+0.3}_{-0.3}$ | $157.9^{+0.3}_{-0.3}$ | $161.0^{+0.1}_{-0.1}$ |

# At the chiral limit…

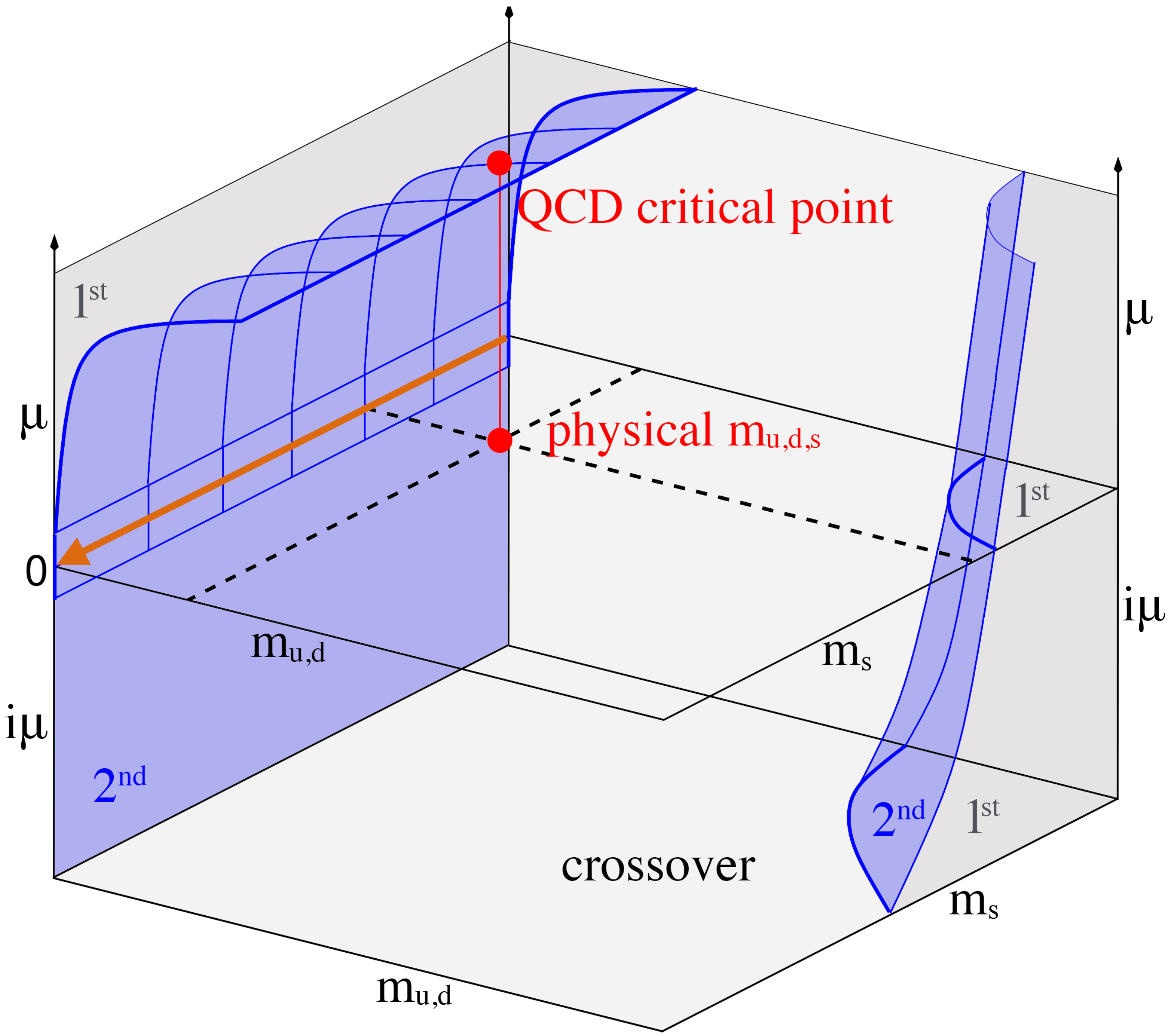



Pisarski and Wilczek, PRD 29 (1984), 338-341




Cuteri, Philipsen and Sciarra, JHEP 11 (2021), 141

# Columbia plot - chiral critical surface




DSE: Bernhardt and CF, PRD 108 (2023) 114018

Lattice: Cuteri, Philipsen and Sciarra, JHEP 11 (2021), 141
Dini, et al, PRD 105 (2022) no.3, 034510

Ding et al. PRL 123, 062002 (2019)
Bornyakov et al. PRD 82, 014504 (2010)

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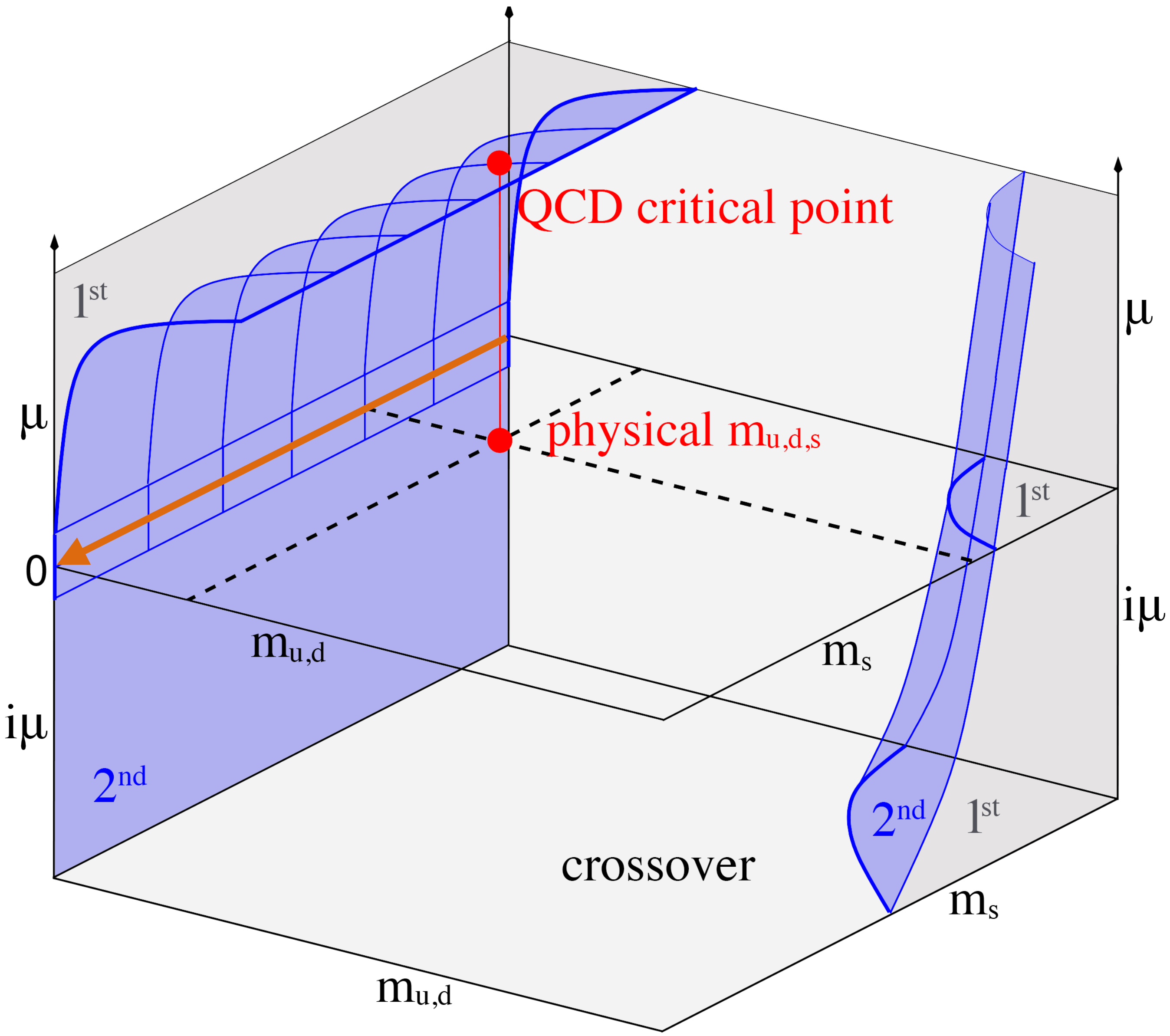


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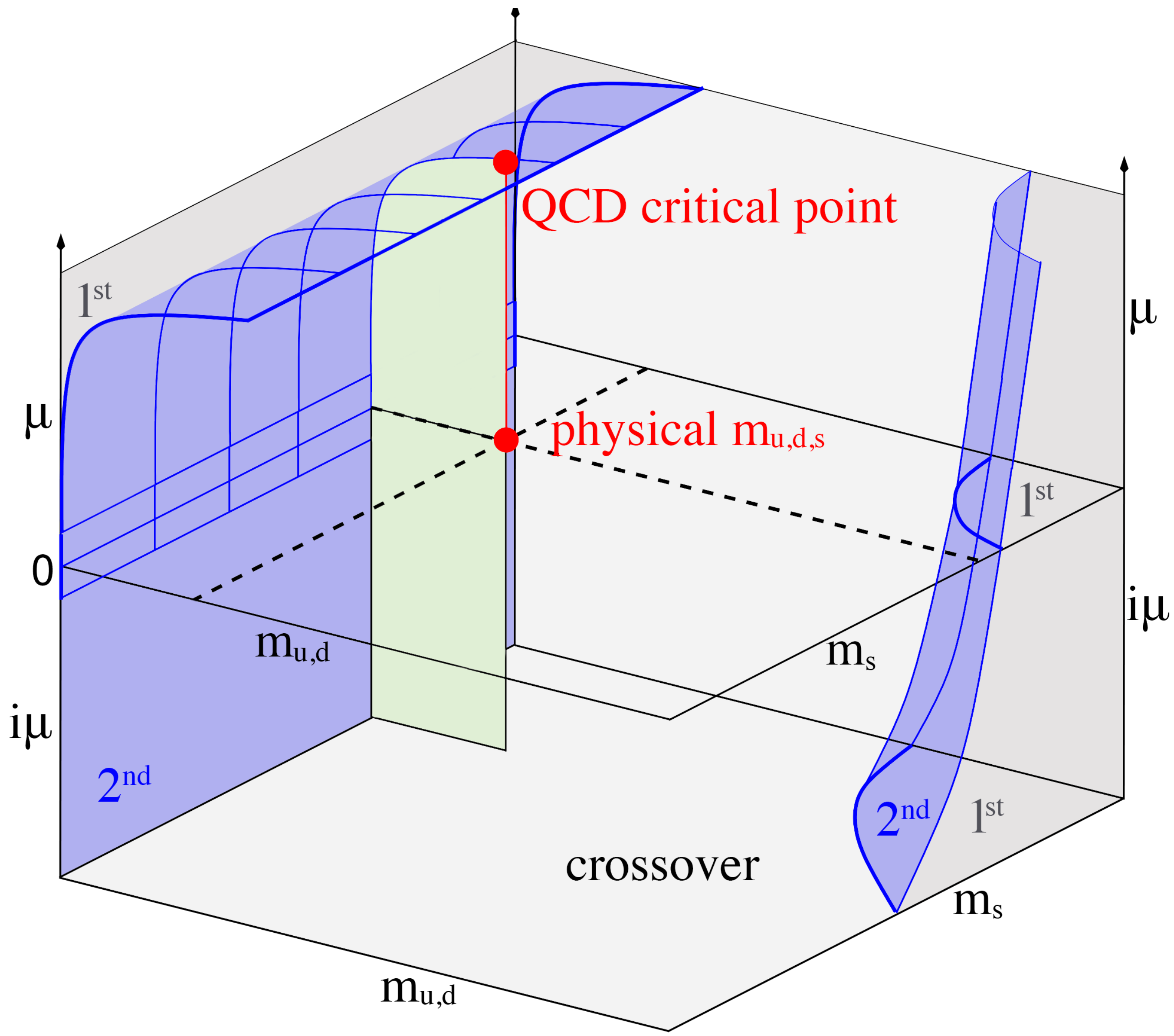


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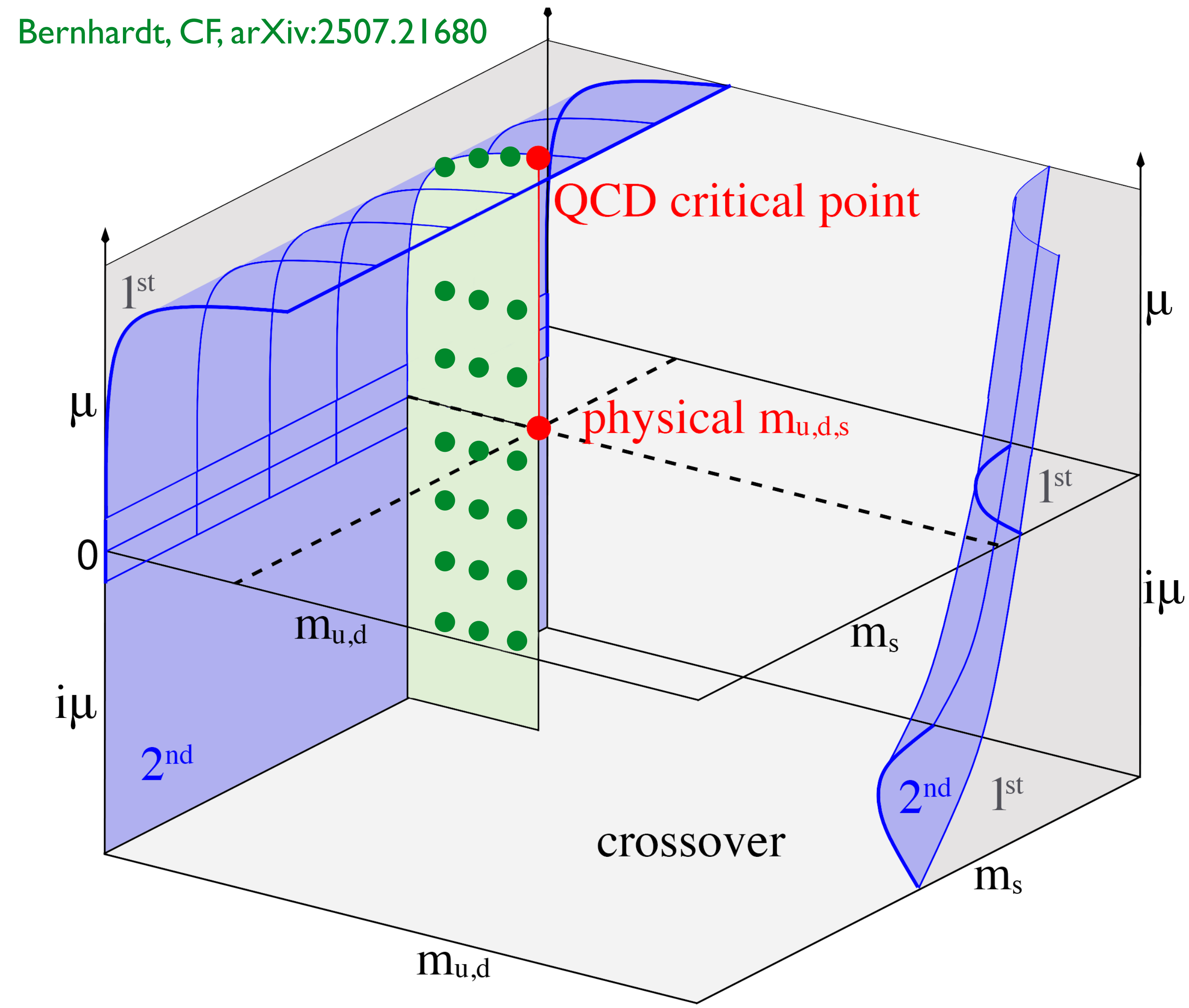


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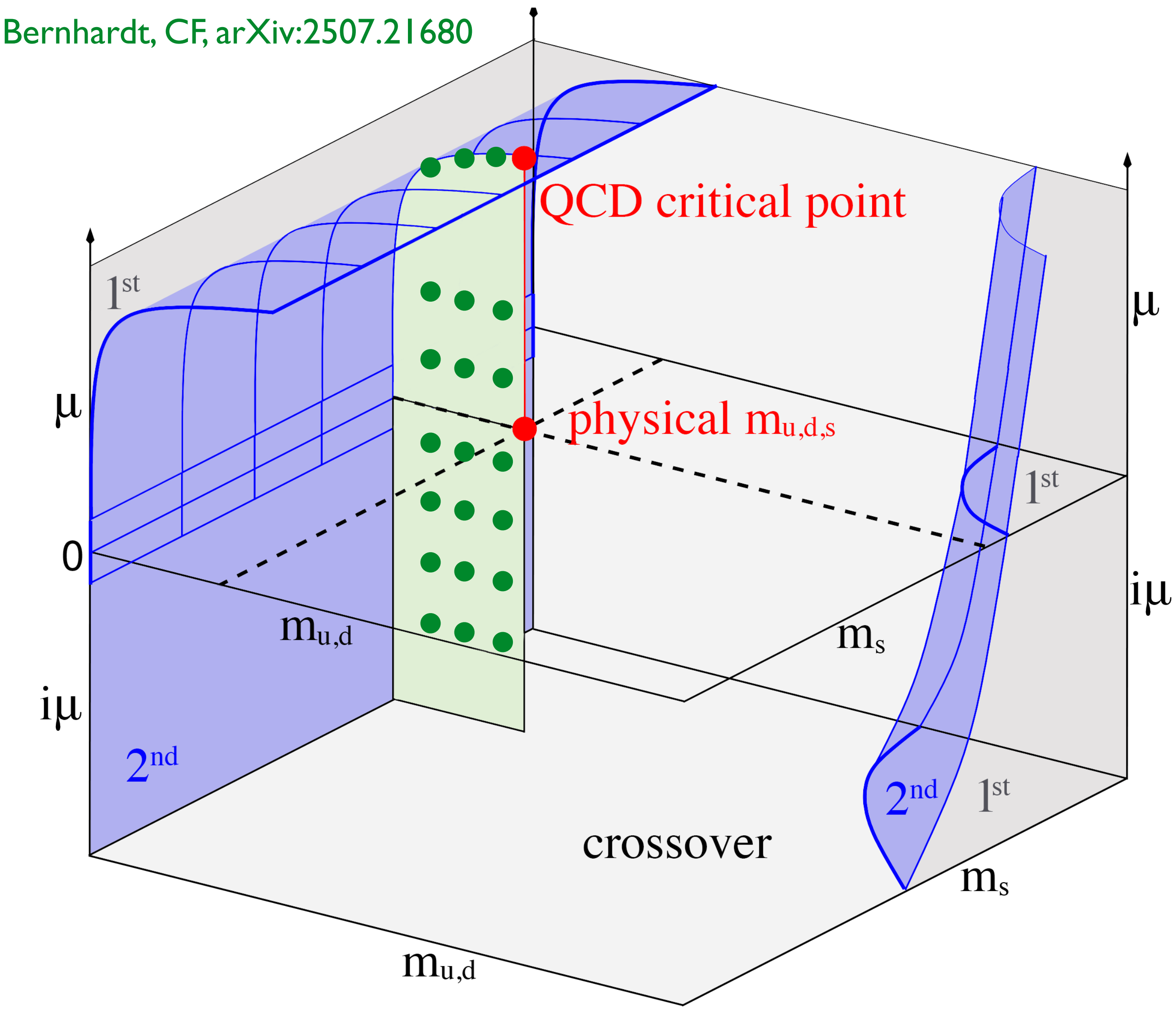


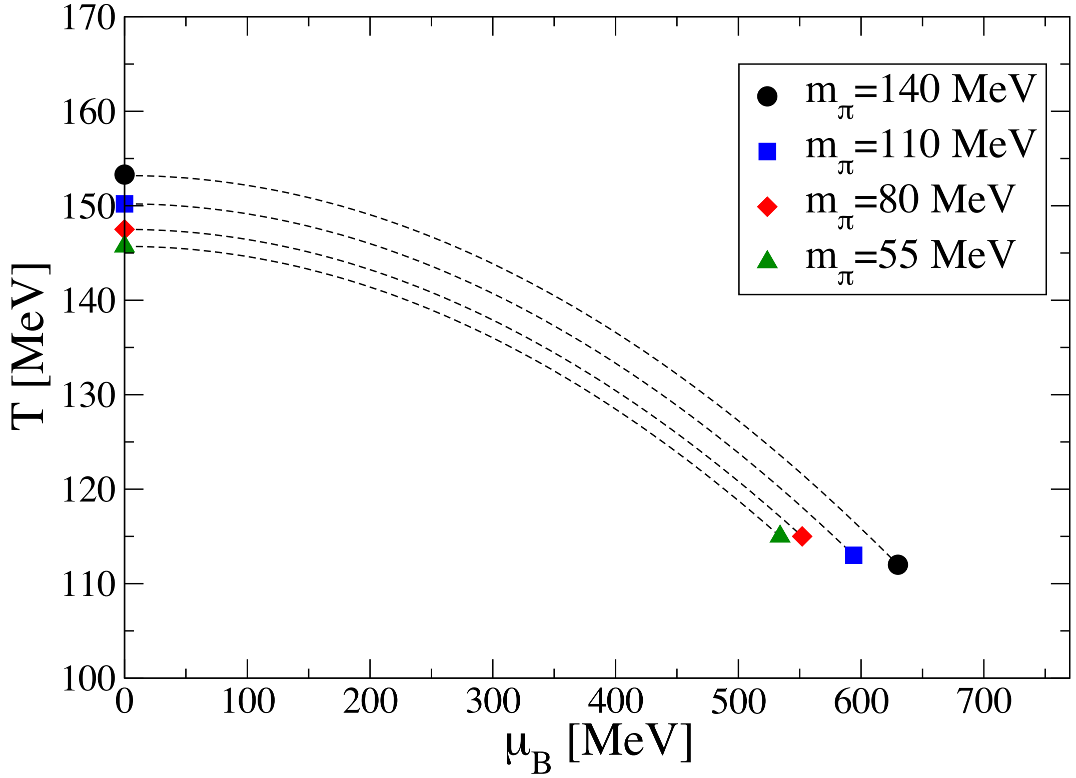


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Lattice: Cuteri, Philipsen and Sciarra, JHEP 11 (2021), 141
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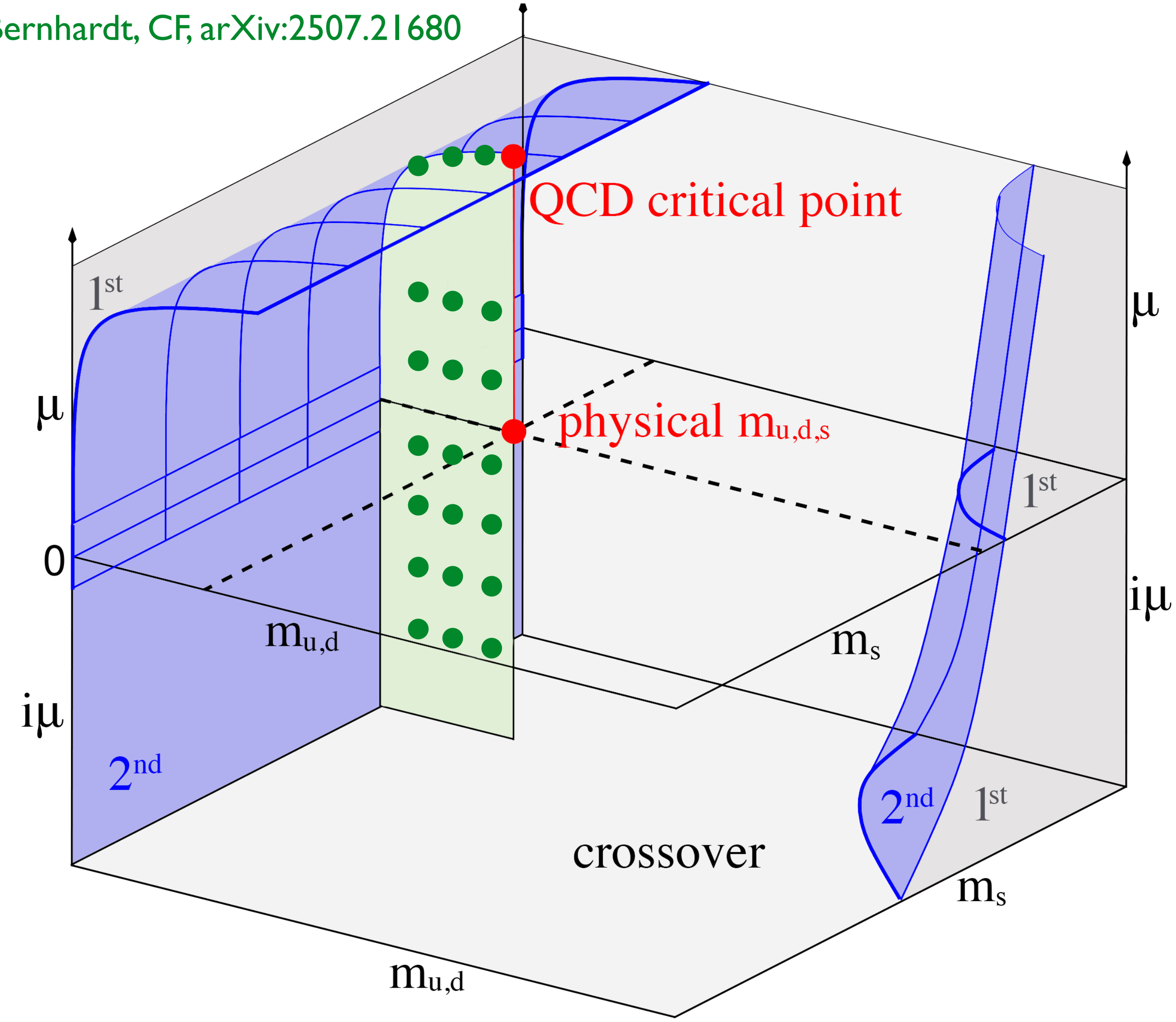
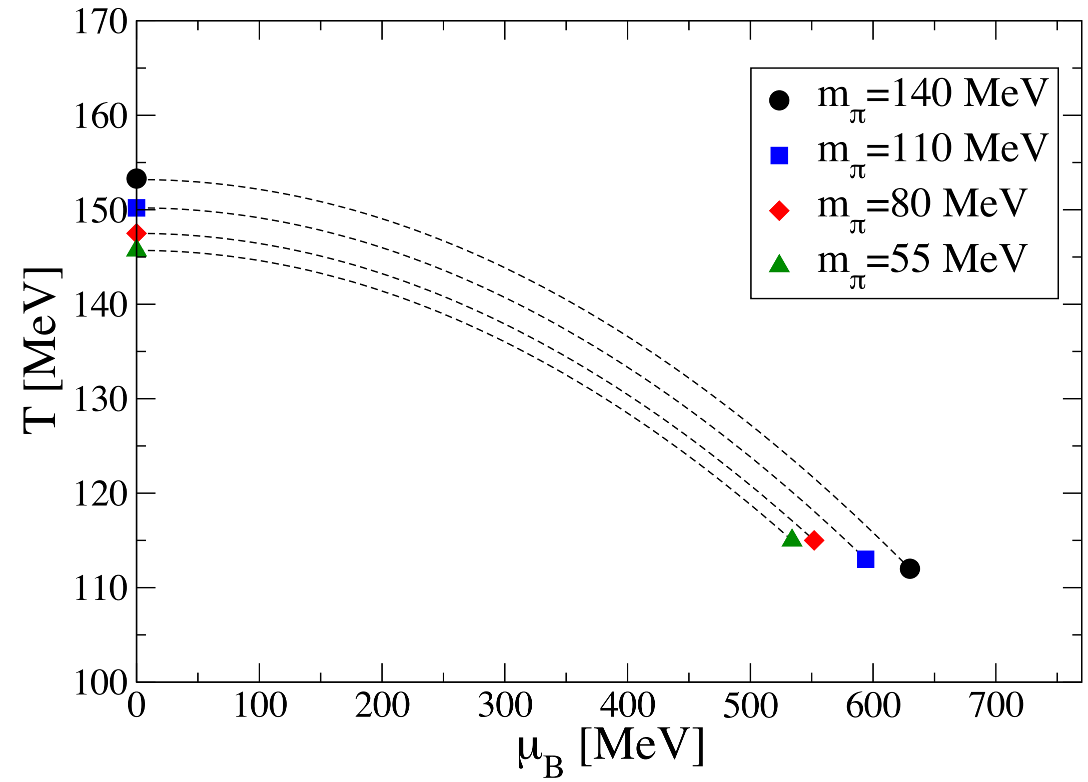
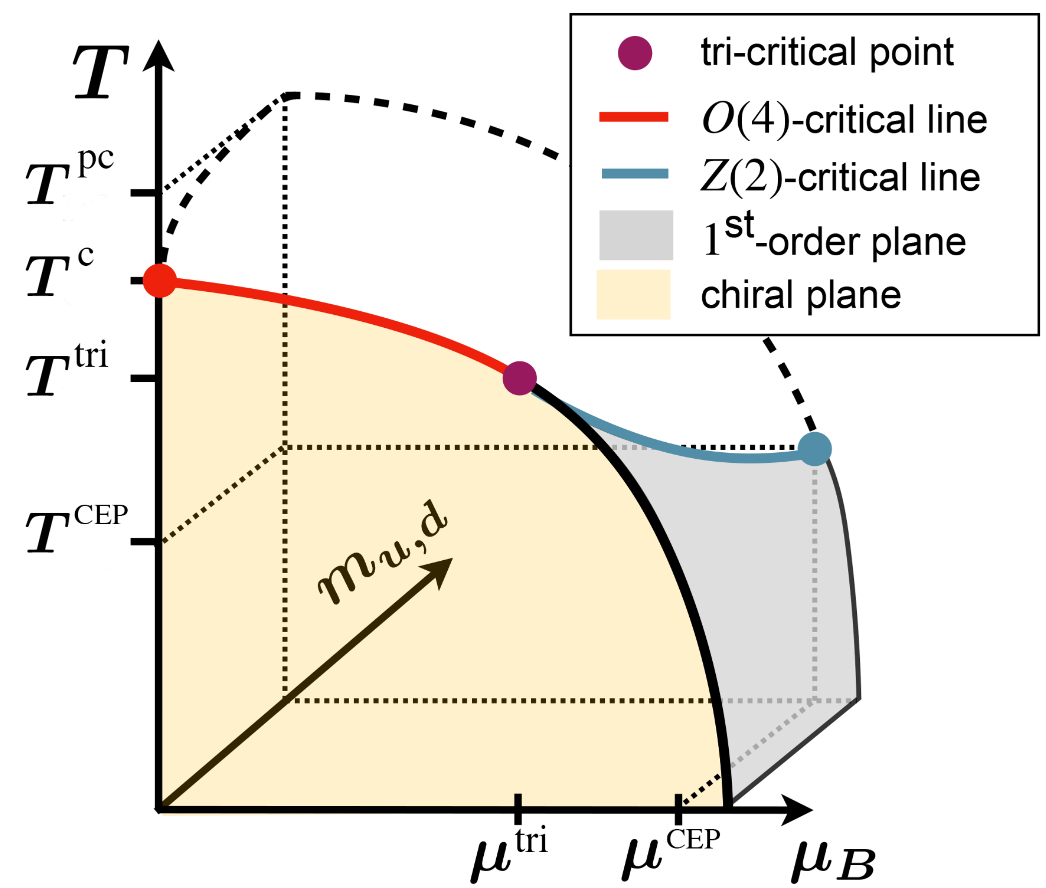
Bornyakov et al. PRD 82, 014504 (2010)

# Columbia plot - chiral critical surface

Bernhardt, CF, arXiv:2507.21680



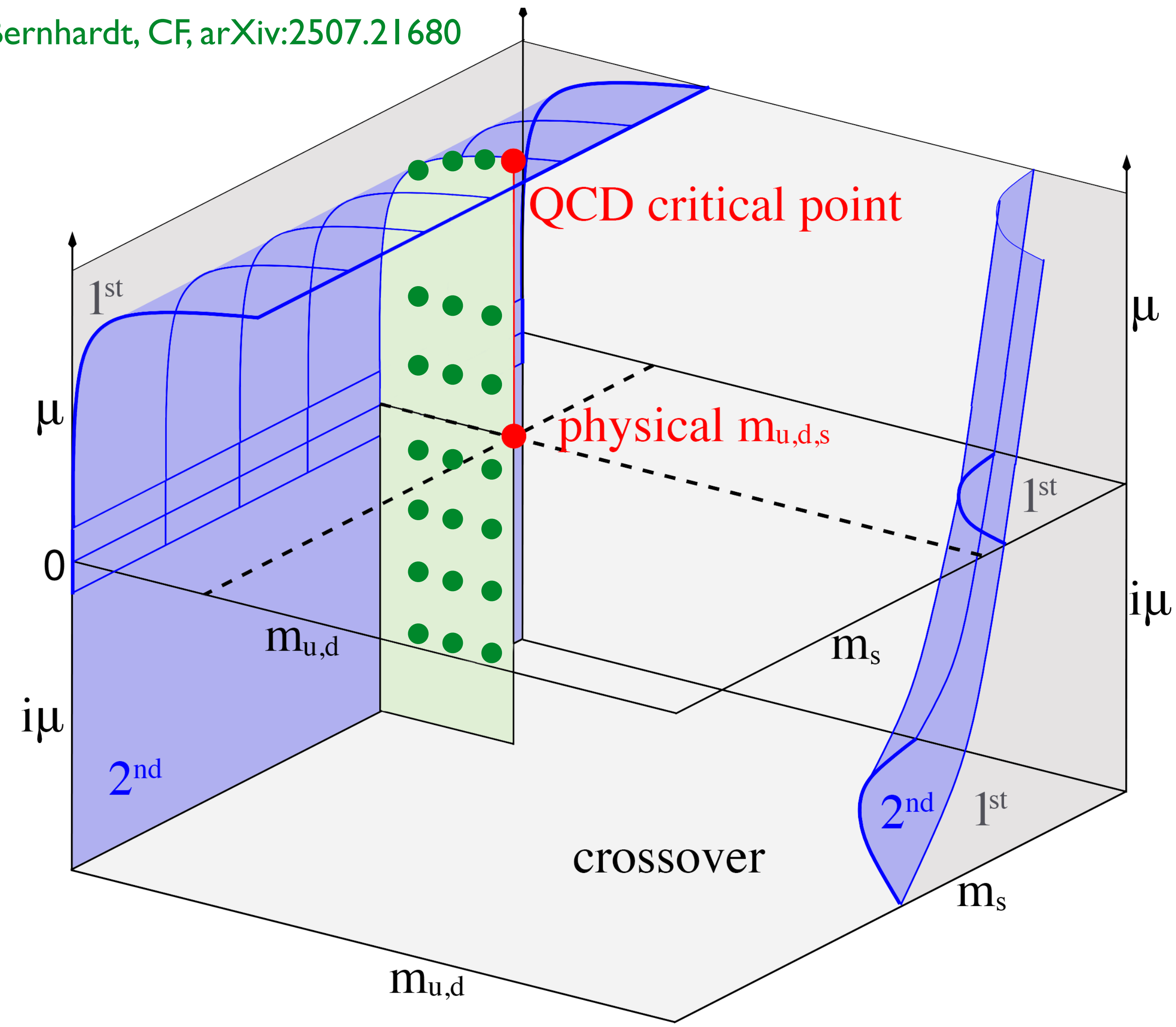



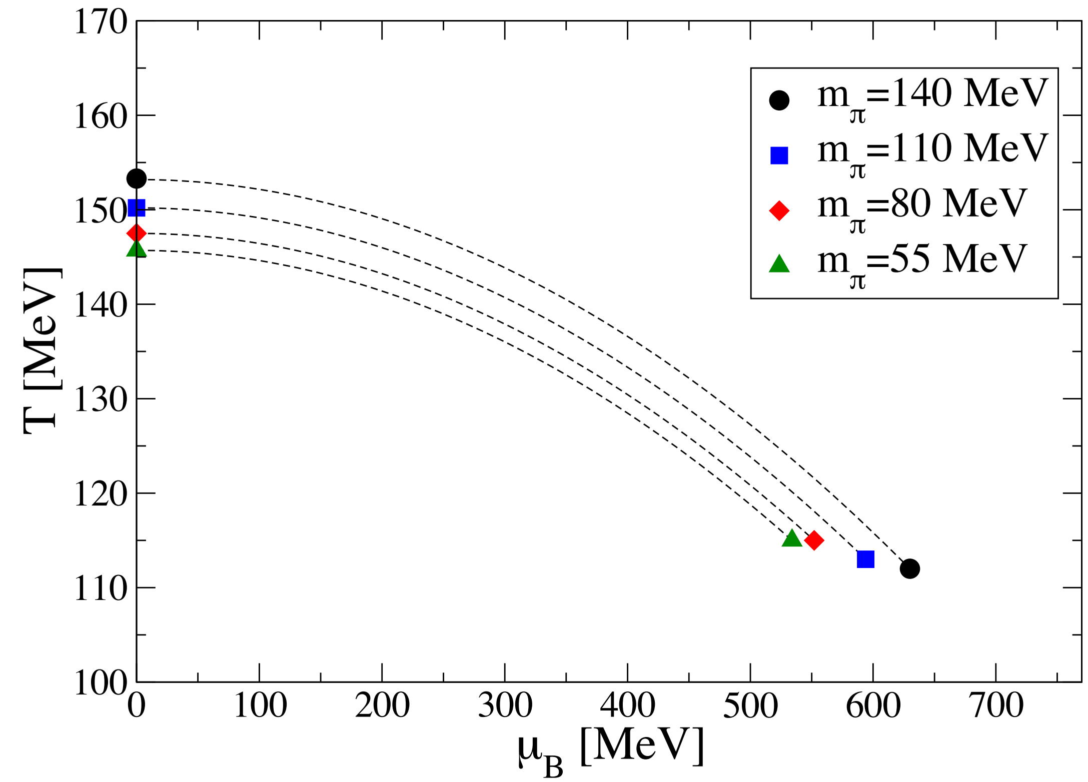

DSE: Bernhardt and CF, PRD 108 (2023) 114018

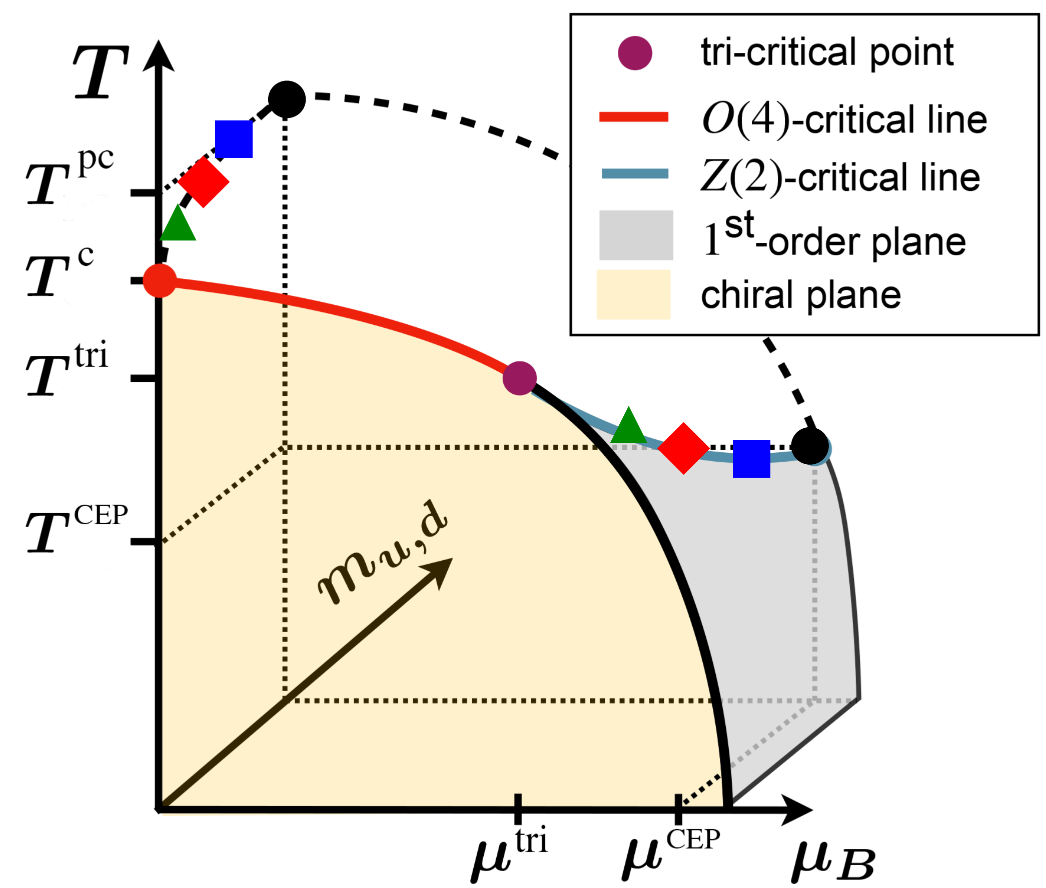
Lattice: Cuteri, Philipsen and Sciarra, JHEP 11 (2021), 141
Dini, et al, PRD 105 (2022) no.3, 034510

Ding et al. PRL 123, 062002 (2019)
Bornyakov et al. PRD 82, 014504 (2010)

Bernhardt, CF, arXiv:2507.21680

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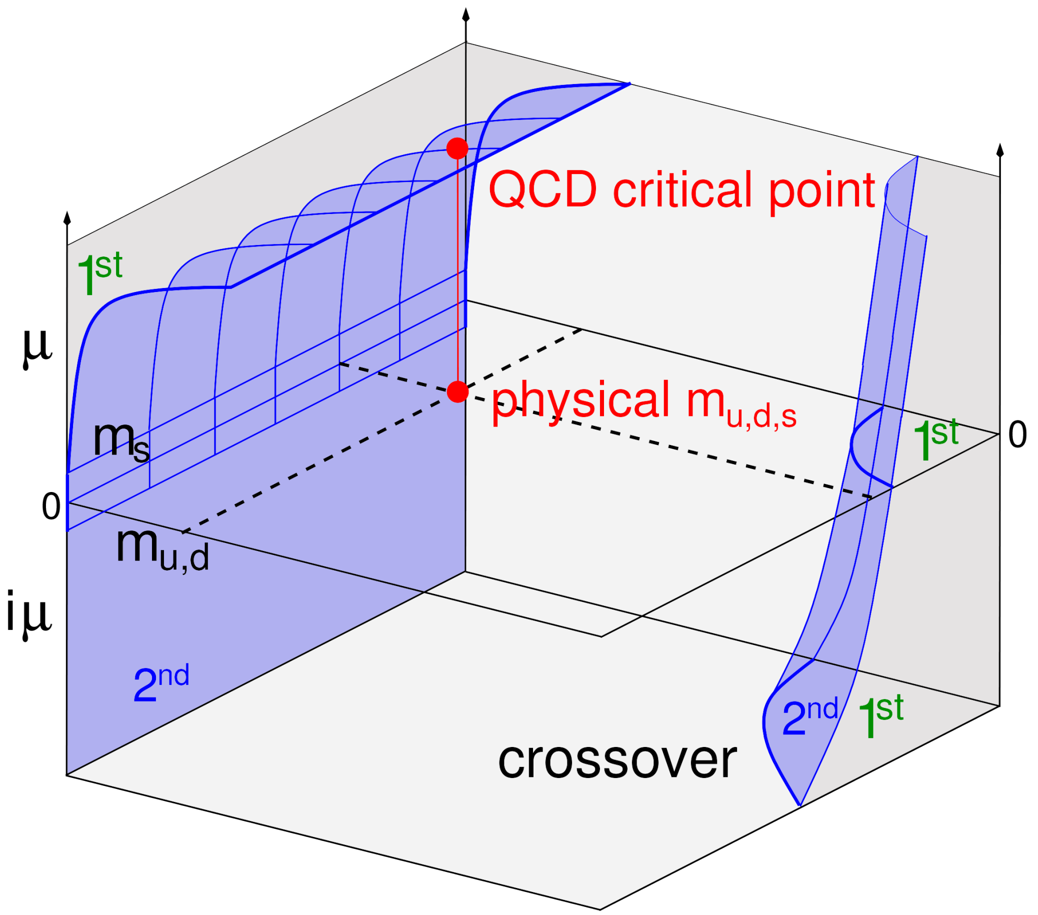
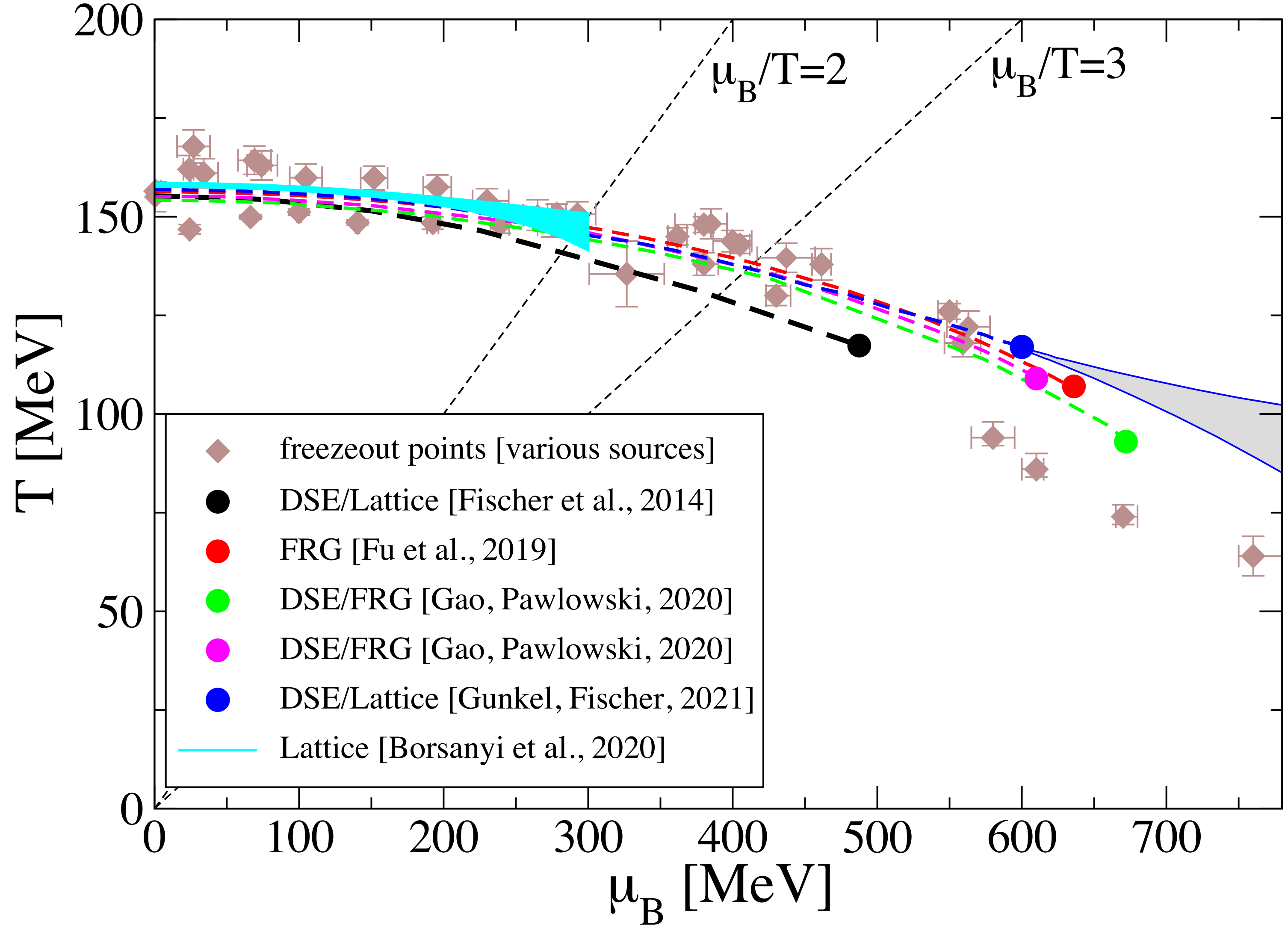
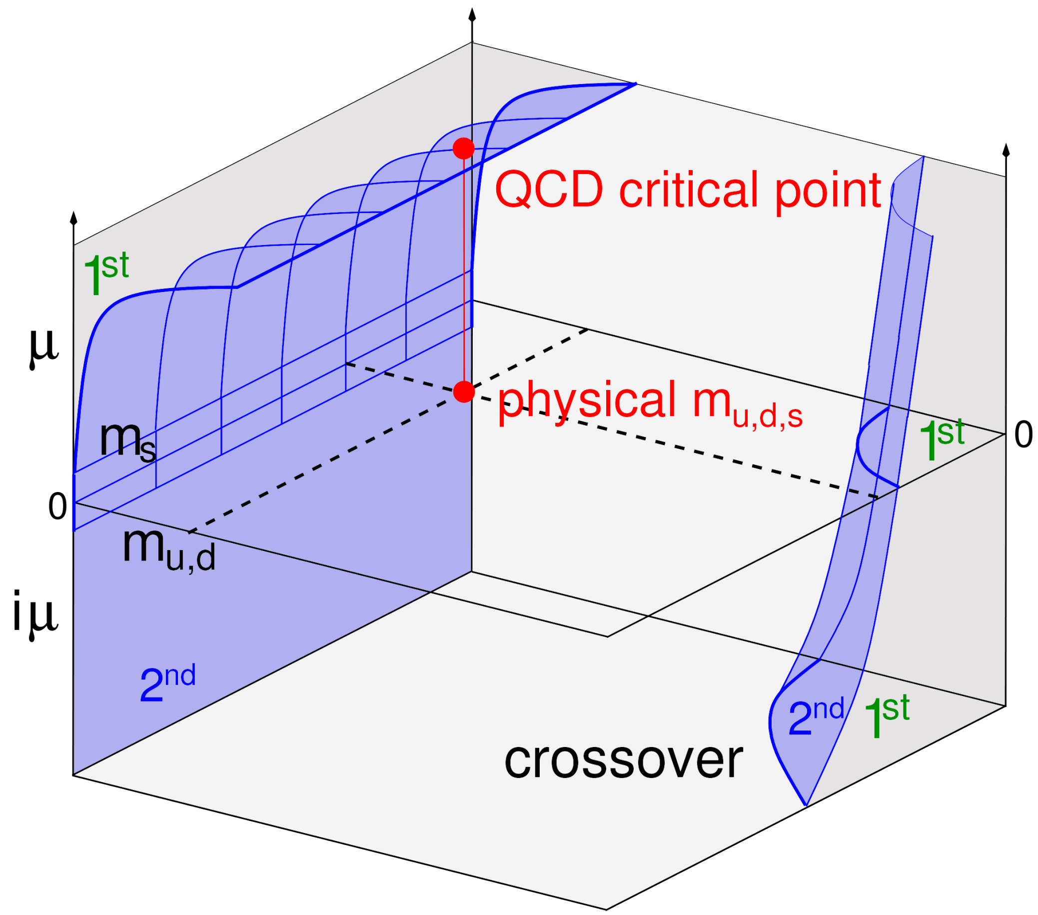
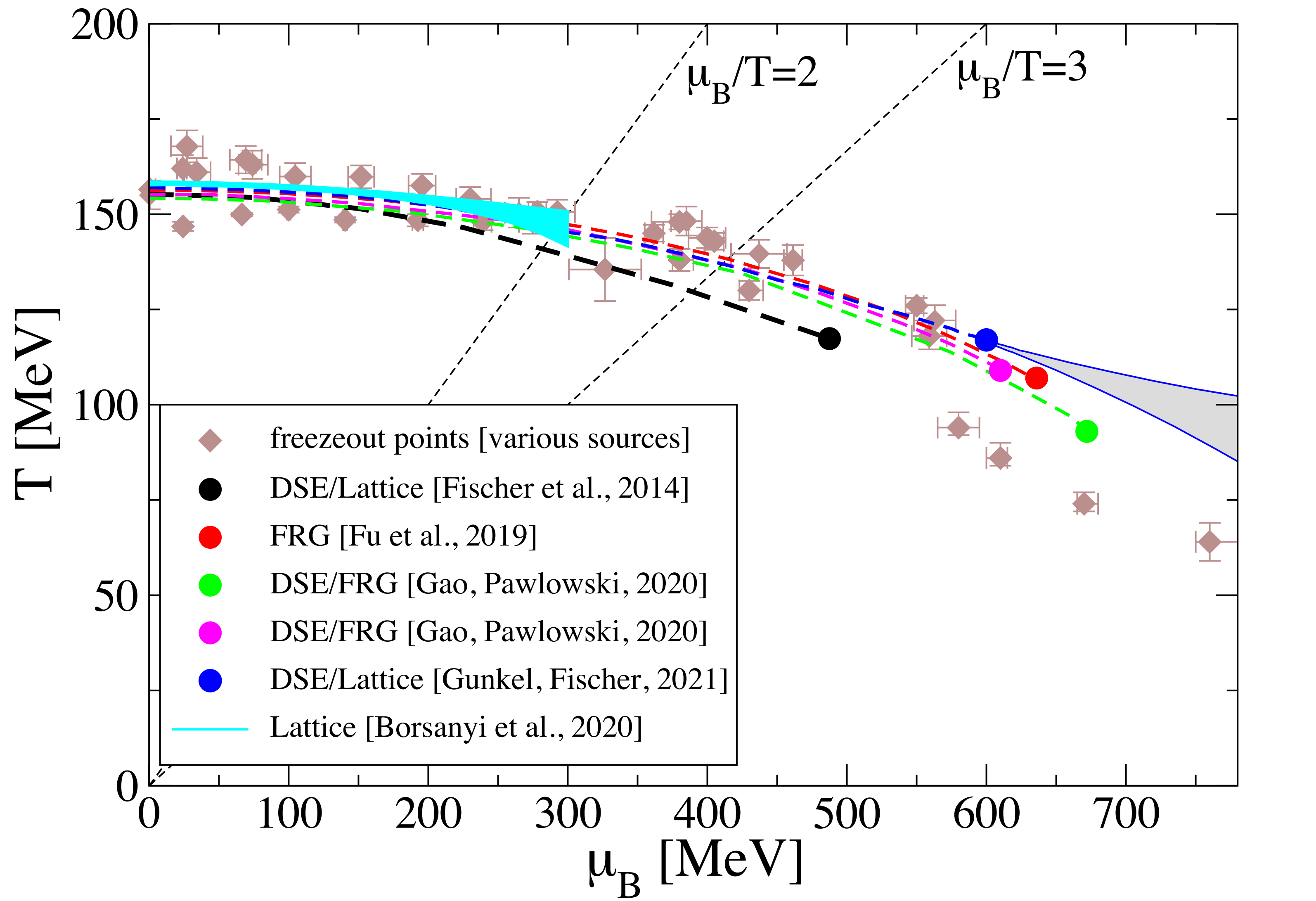
# Location of CEP




- how stable is this result ??
  - crosscheck DSE-FRG ✔

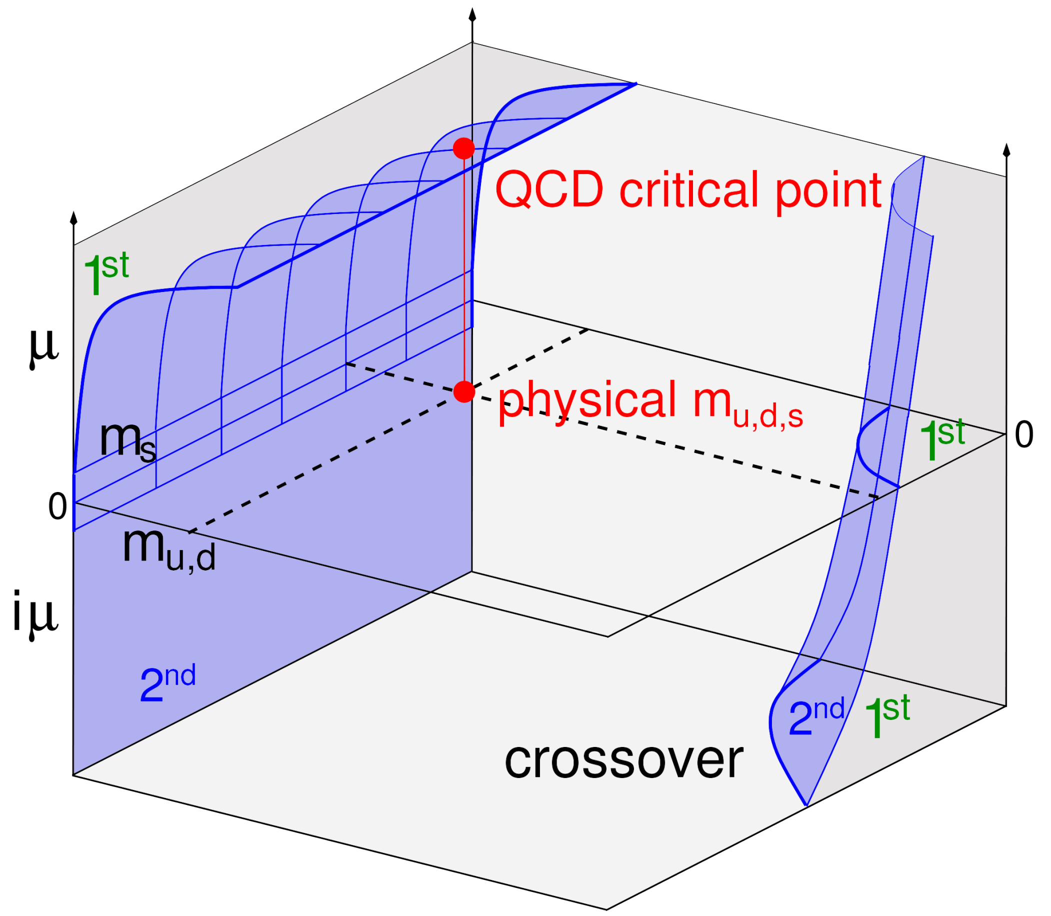


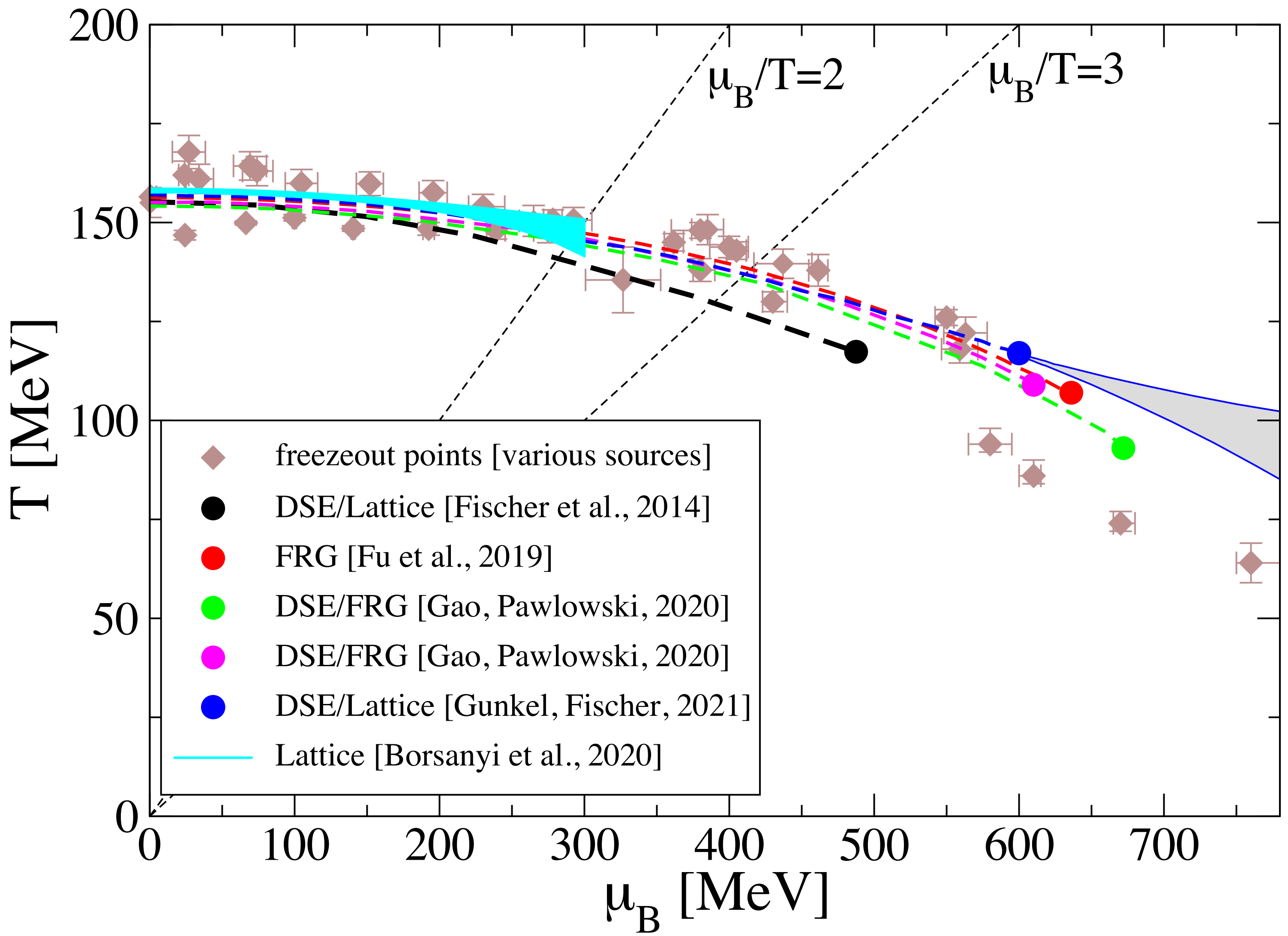




- how stable is this result ??
  - crosscheck DSE-FRG ✔
  - $N_f$=2+1+1 ✔

CF, Luecker, Welzbacher, PRD 90 (2014) 034022

# Location of CEP




- how stable is this result ??
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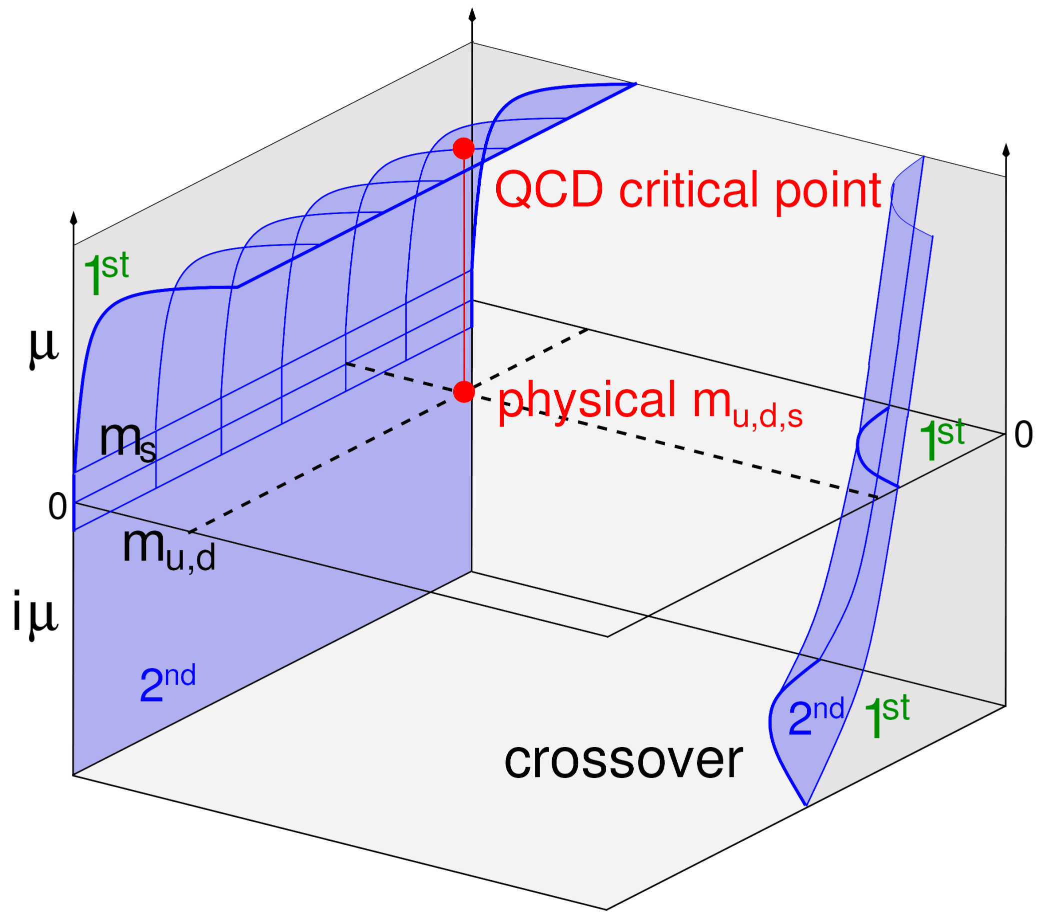
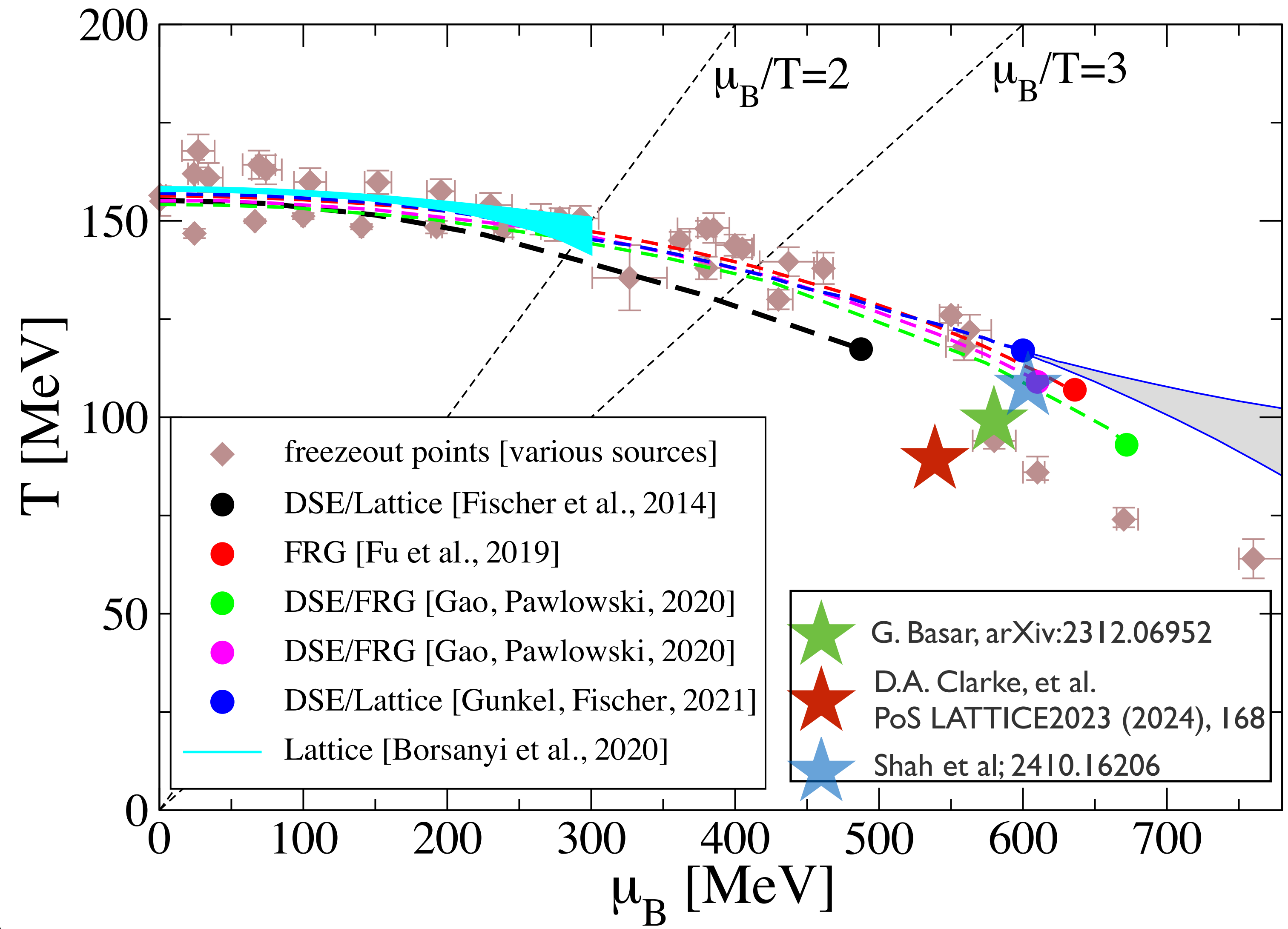
  - $N_f$=2+1+1 ✔ CF, Luecker, Welzbacher, PRD 90 (2014) 034022
  - baryon effects ✔ Eichmann, CF, Welzbacher, PRD93 (2016)
    Braun et al. PRD 101 (2020)

# Location of CEP

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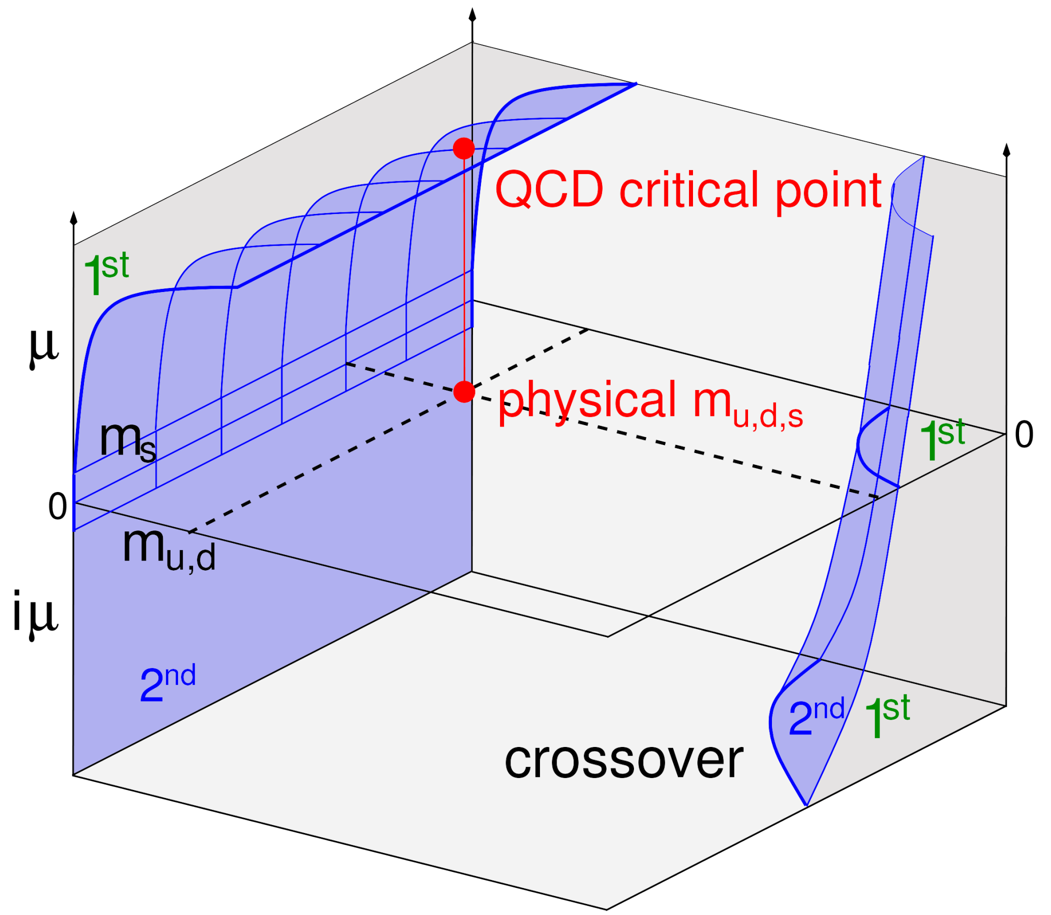
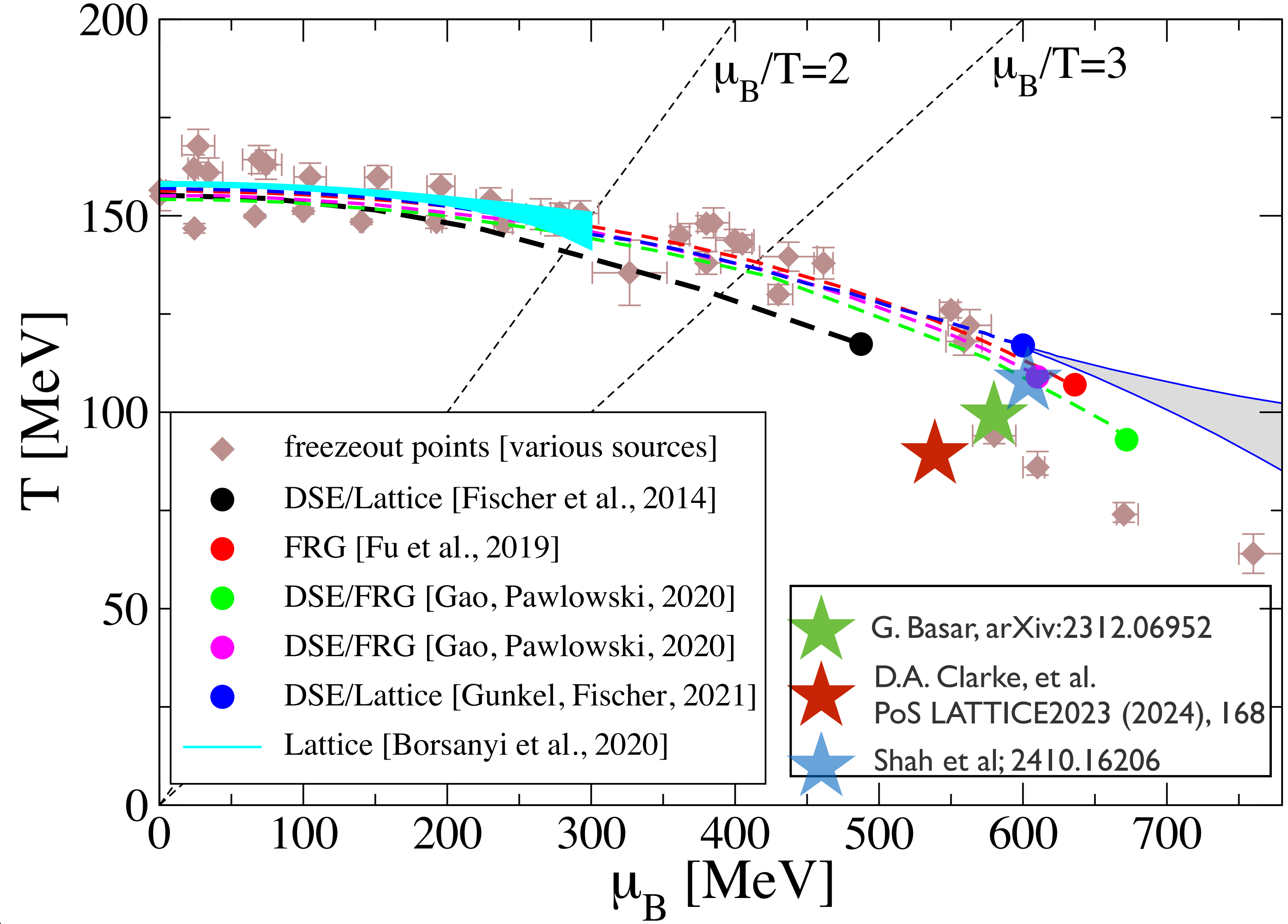
  - 'cross-check' with lattice (✔)

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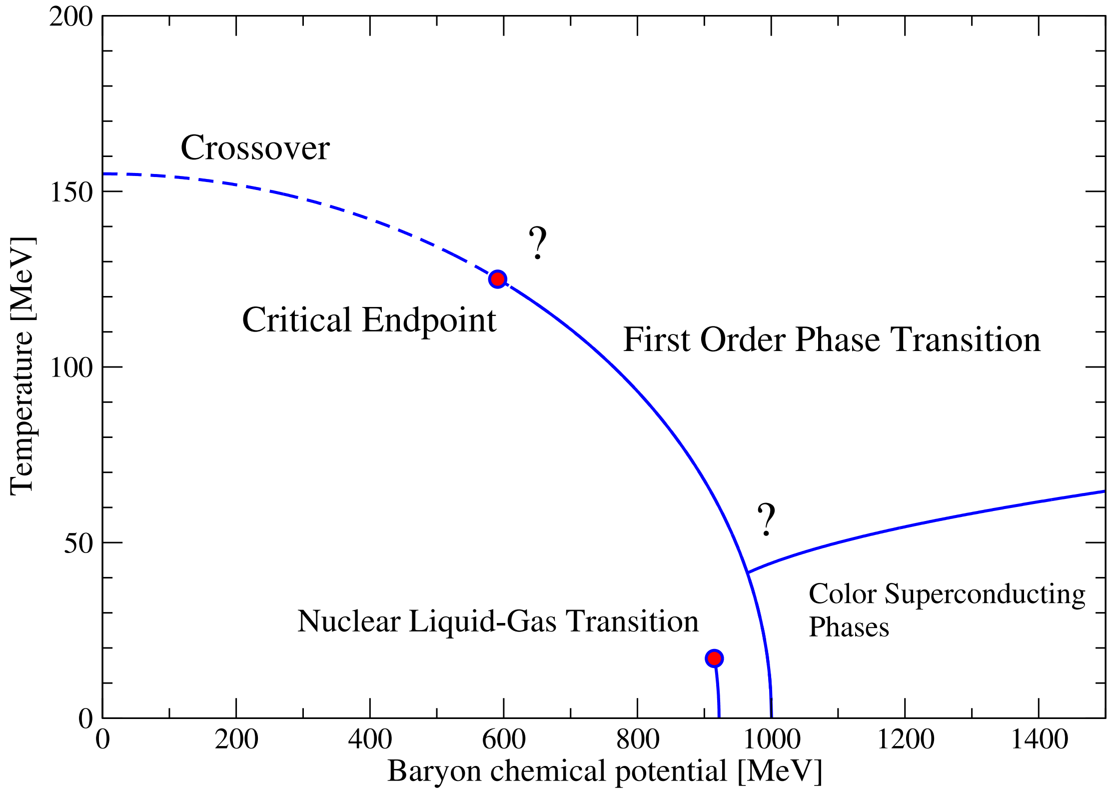
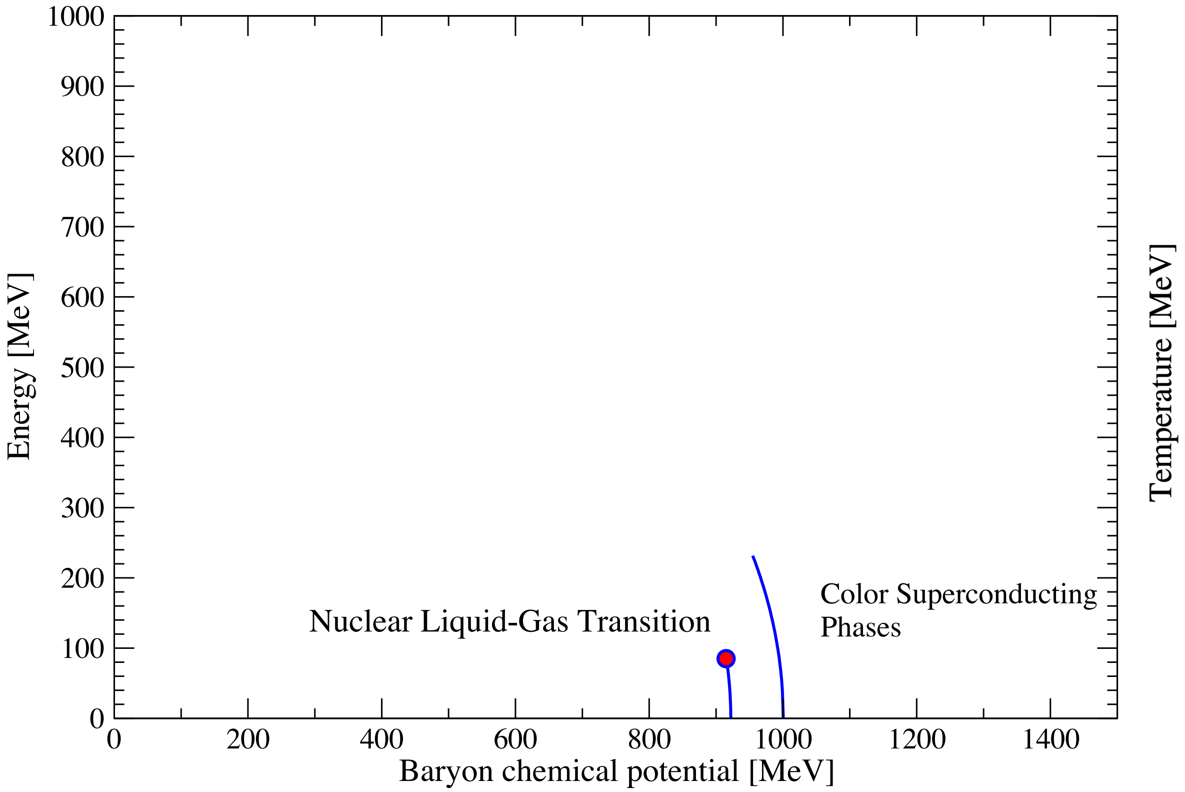
  - 'cross-check' with lattice (✔)
  - moat and inhom. phases — T. F. Motta, J. Bernhardt, M. Buballa and CF, PRD 108 (2023); T. F. Motta, J. Bernhardt, M. Buballa and CF, acc. for PRD, [2411.02285]

# Backup

# Phase transitions at zero T: what to expect…



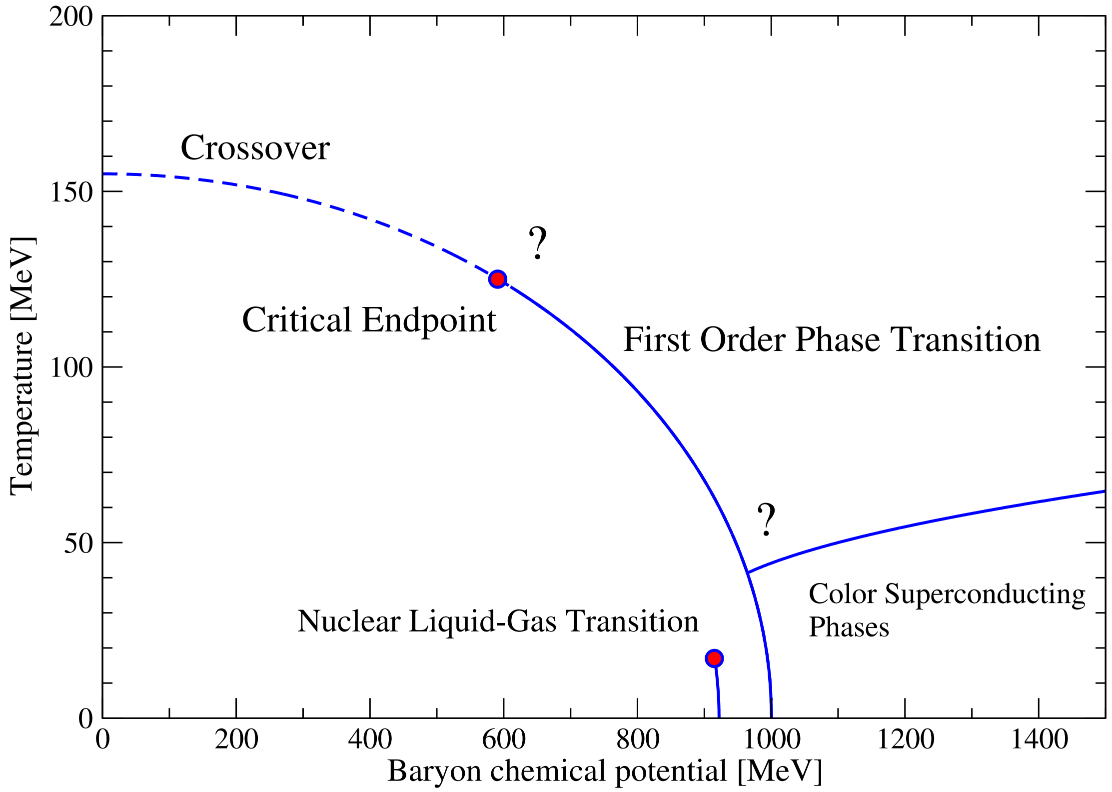




"silver blaze": no excitations until

T. D. Cohen, PRL 91 , 222001 (2003)

$$\mu_B \leq m_N^*$$

# Phase transitions at zero T: what to expect…

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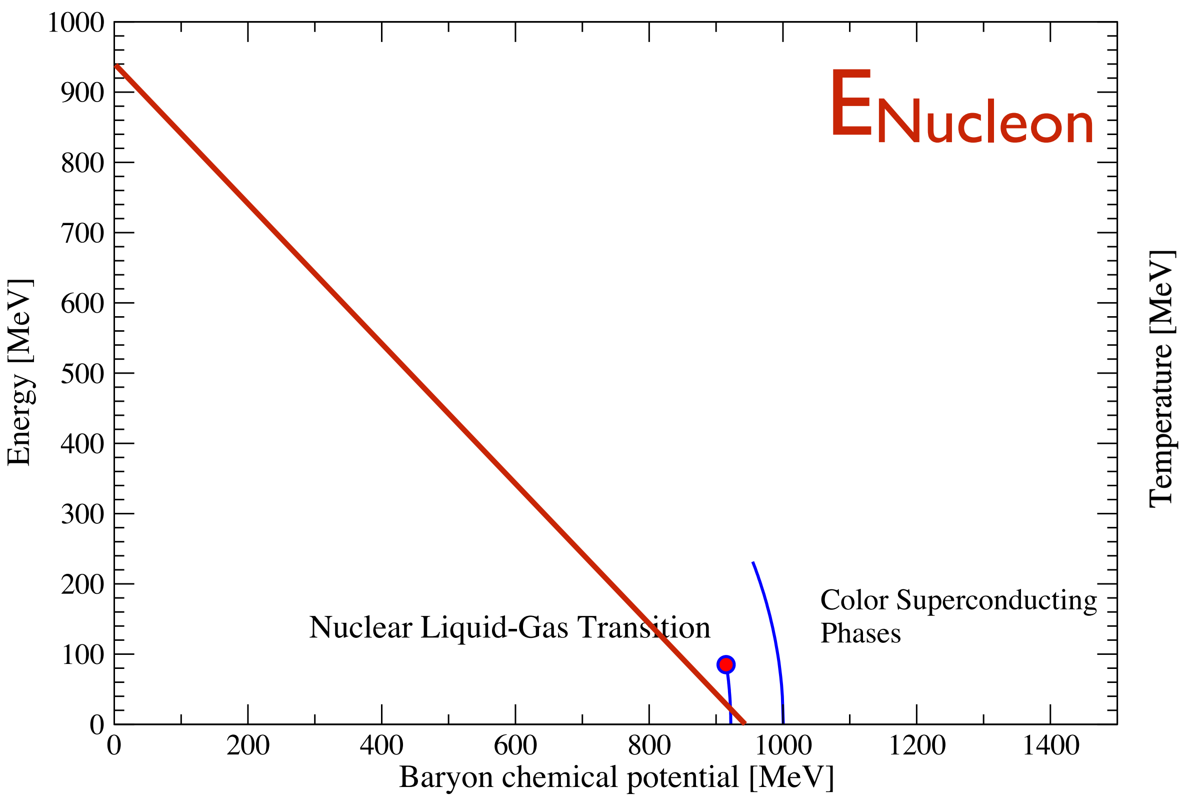

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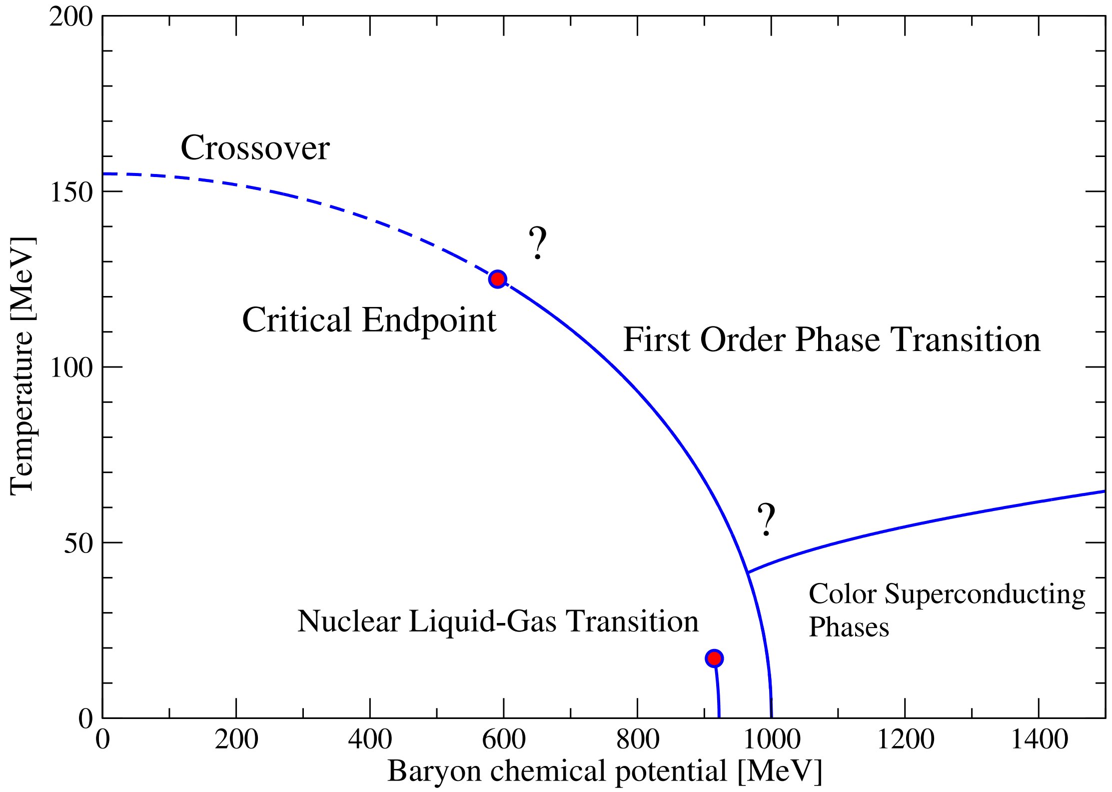


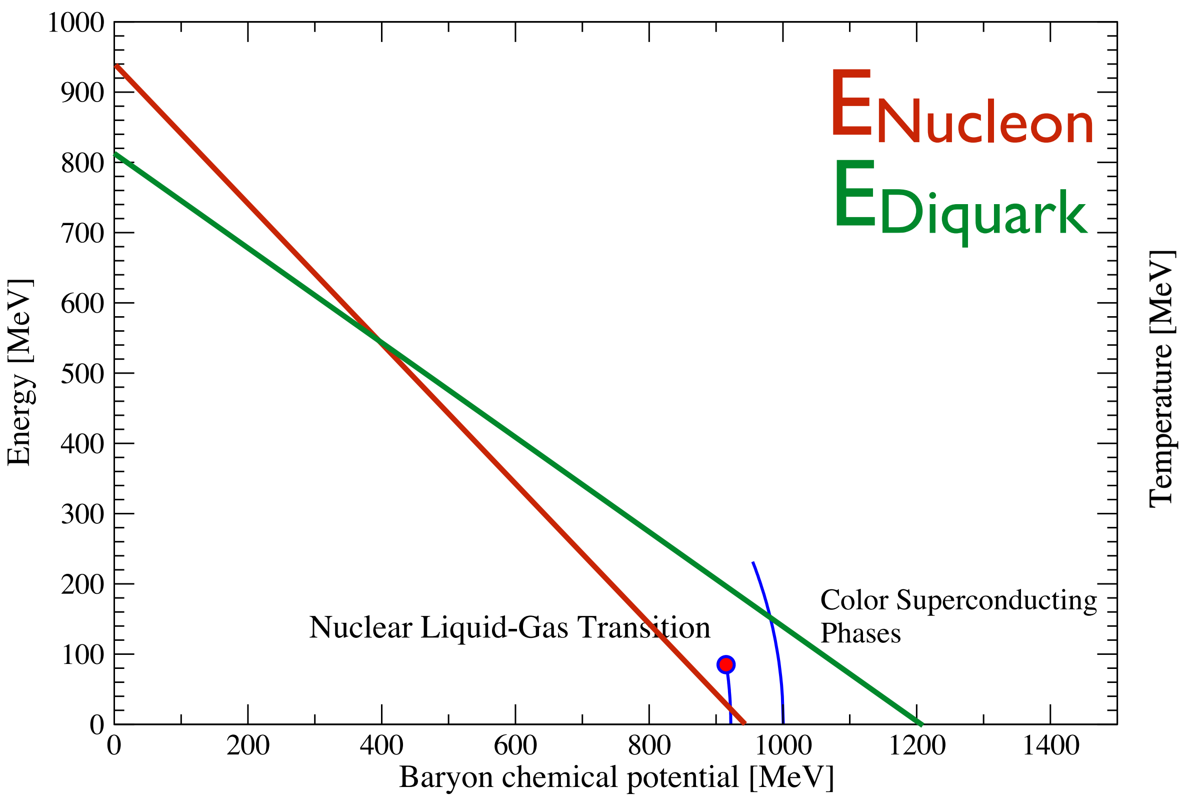


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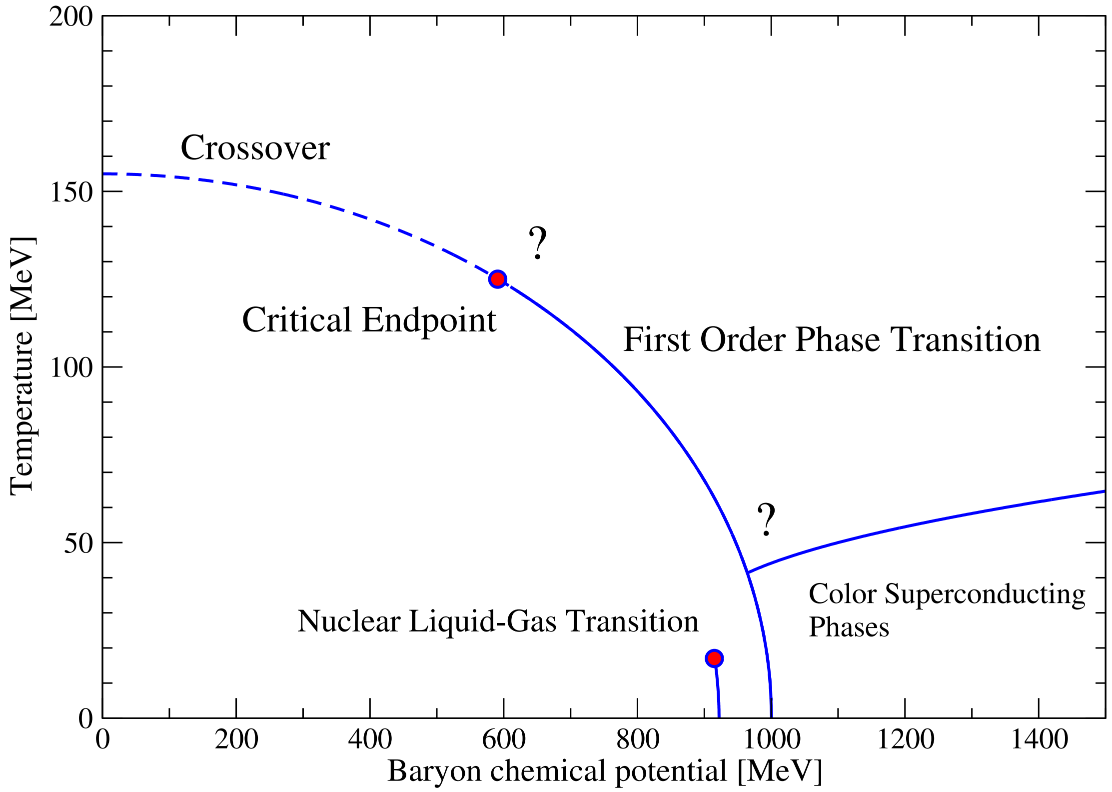


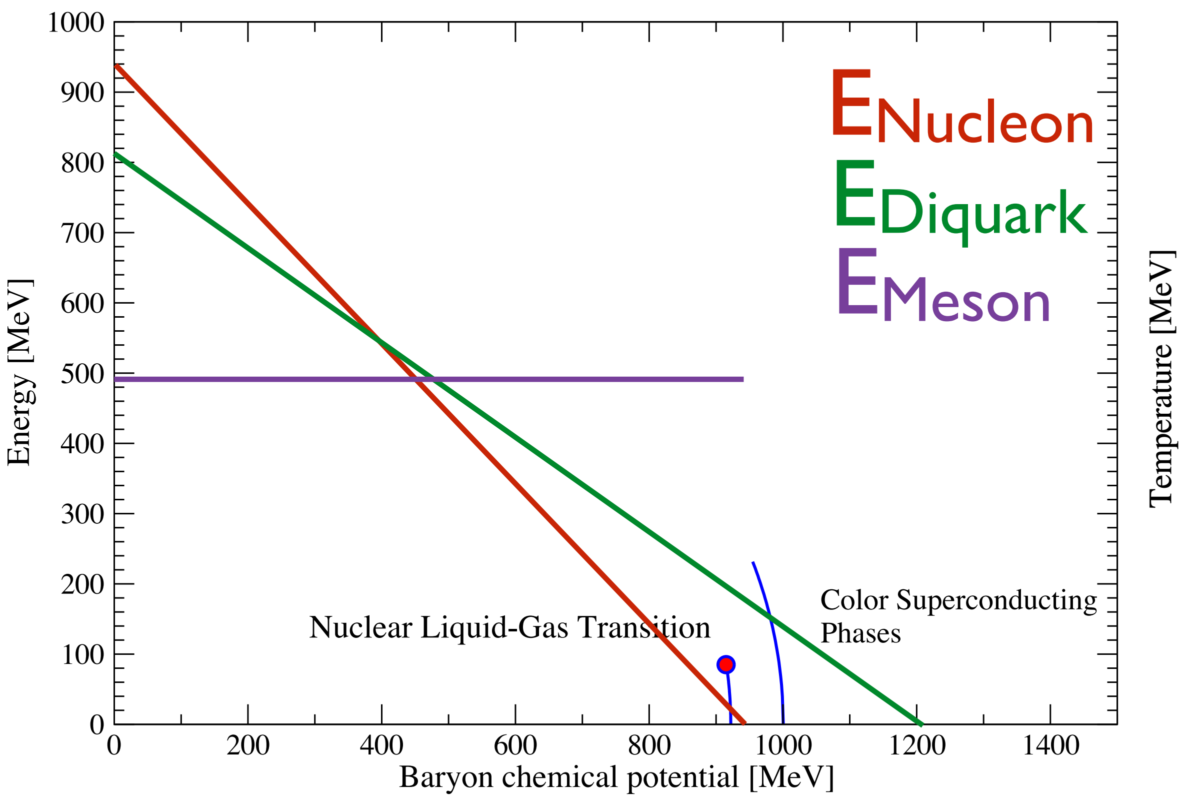


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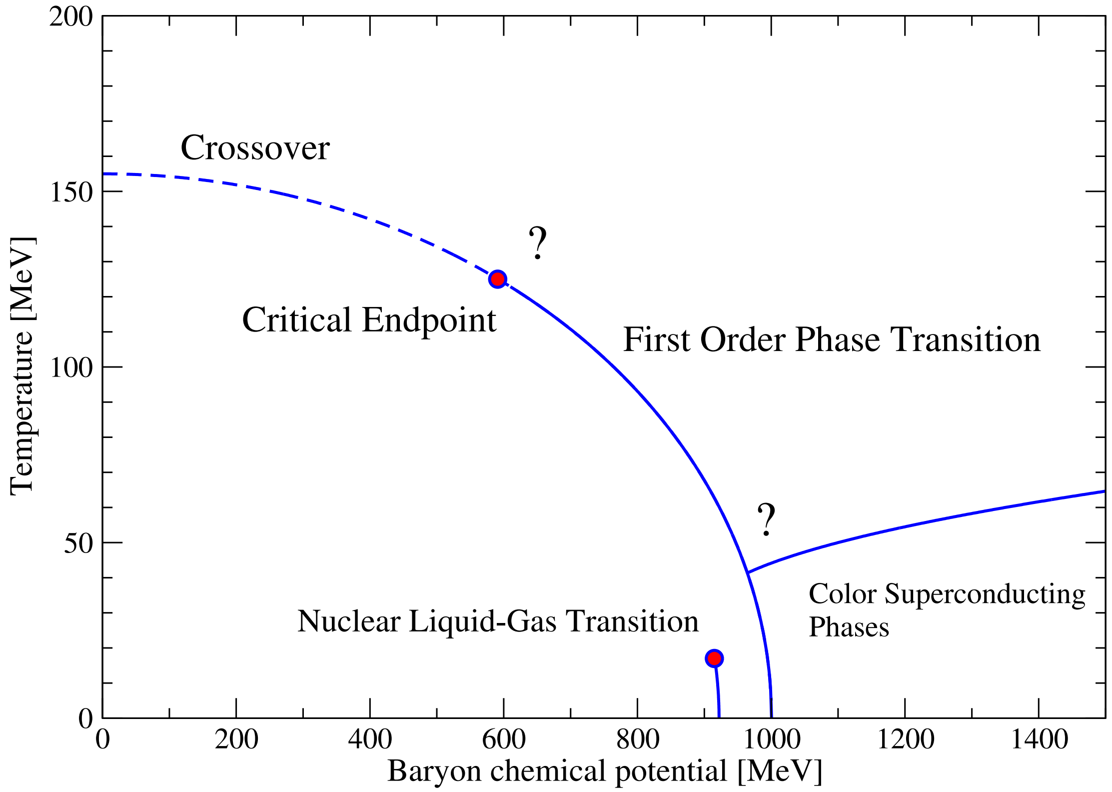


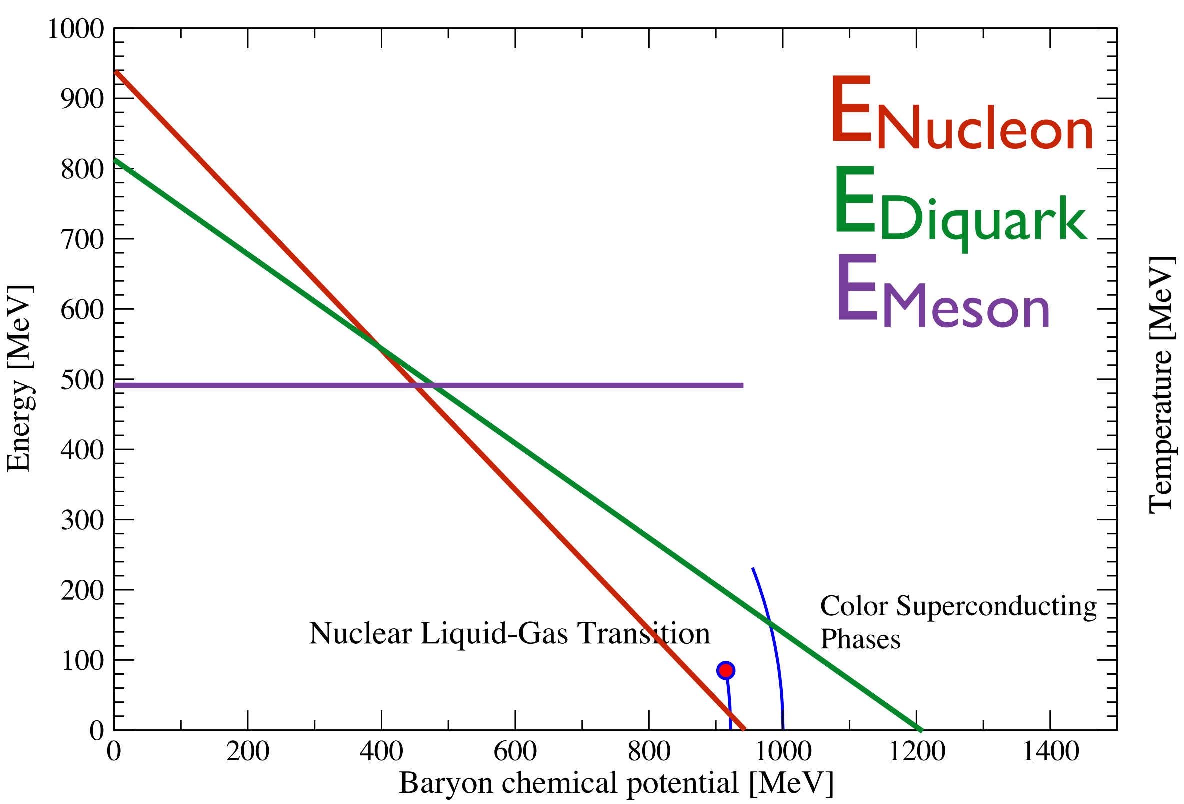


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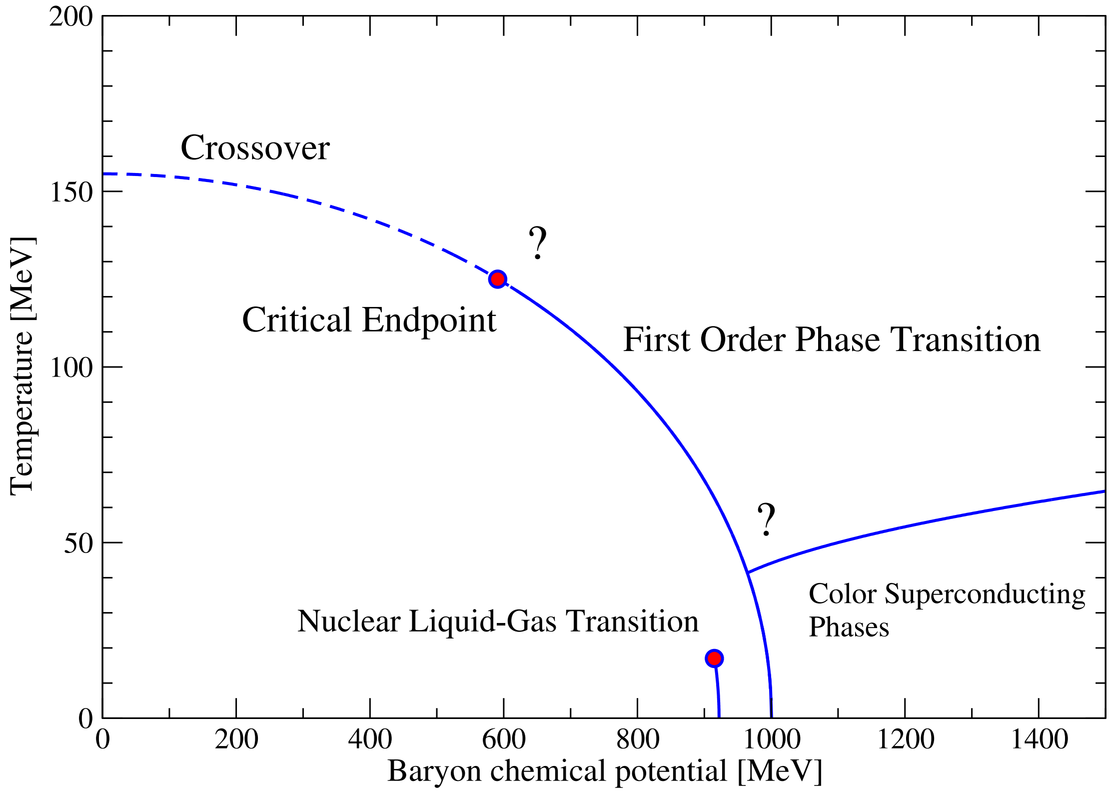
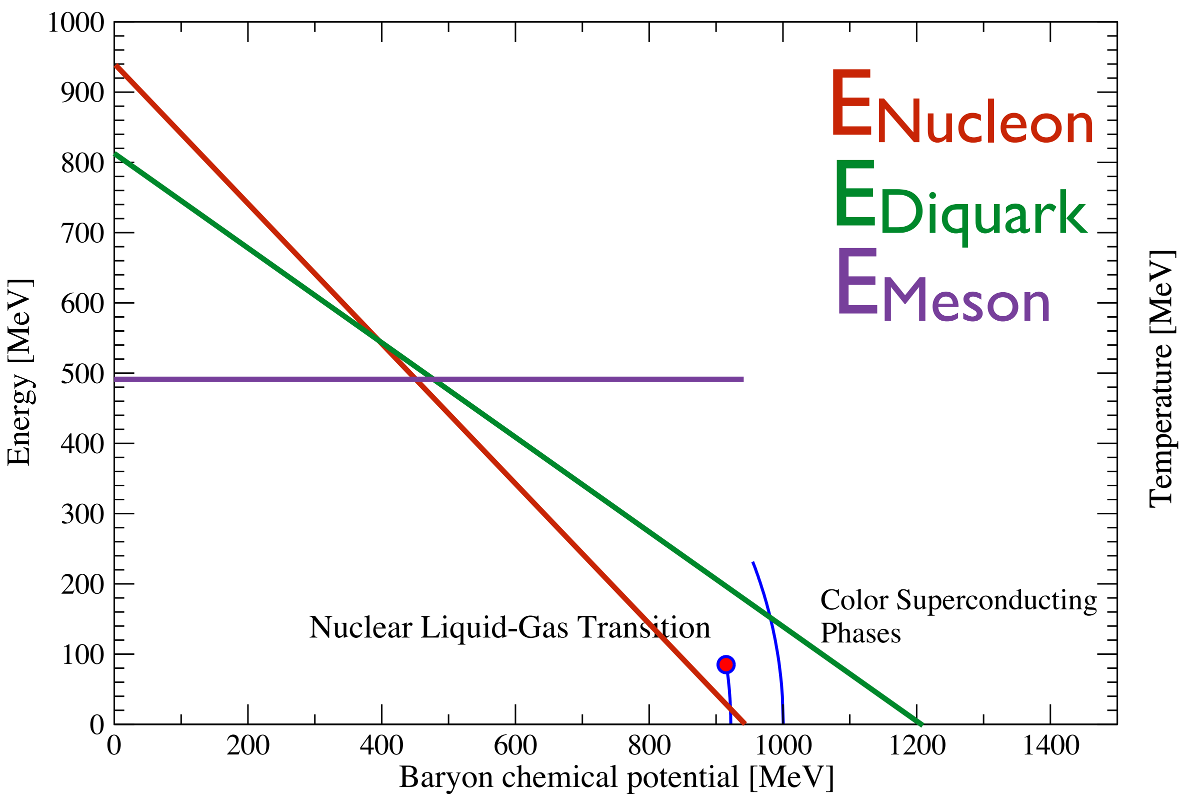
- Meson ✔

Gunkel, CF, Isserstedt, EPJ A 55 (2019) no.9, 169
Gunkel, CF, EPJ A 57 (2021) no. 4, 147

# Phase transitions at zero T: what to expect…

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- **Meson** ✔ Gunkel, CF, Isserstedt, EPJ A 55 (2019) no.9, 169; Gunkel, CF, EPJ A 57 (2021) no. 4, 147
- **Diquarks** Müller, Buballa, Wambach, arXiv:1603.02865; Graetz, CF work in progress

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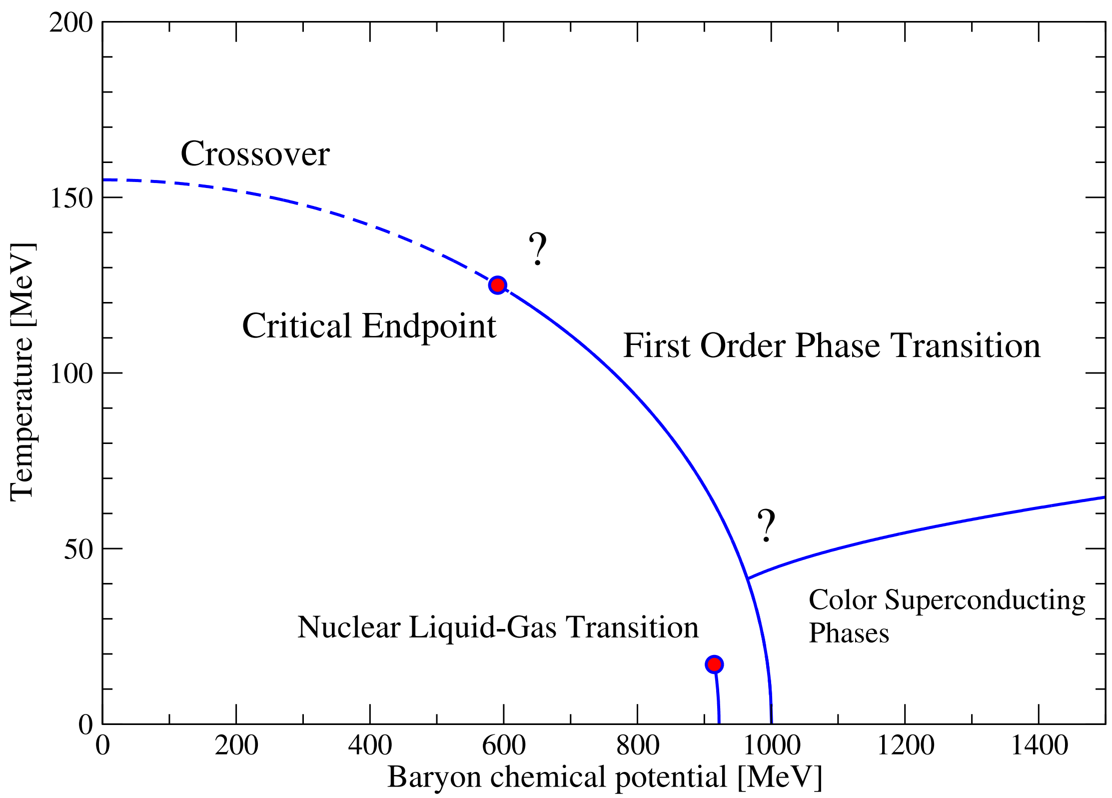


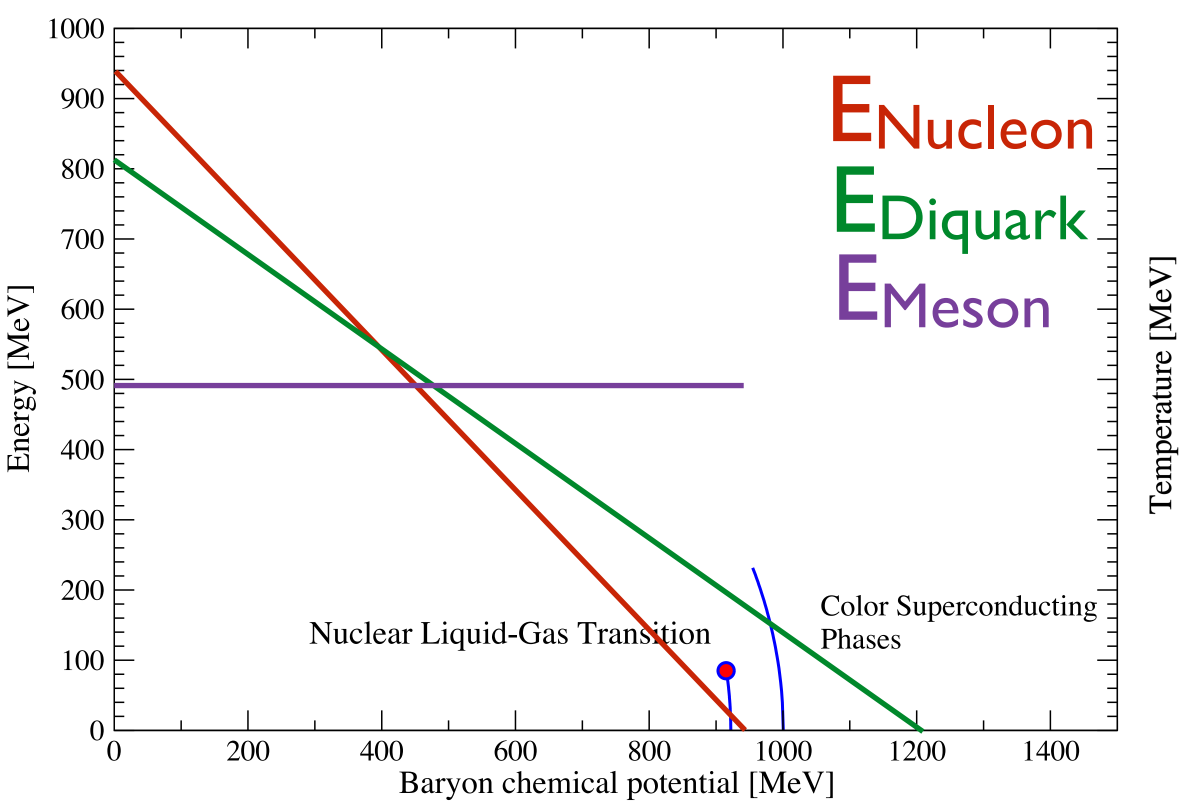


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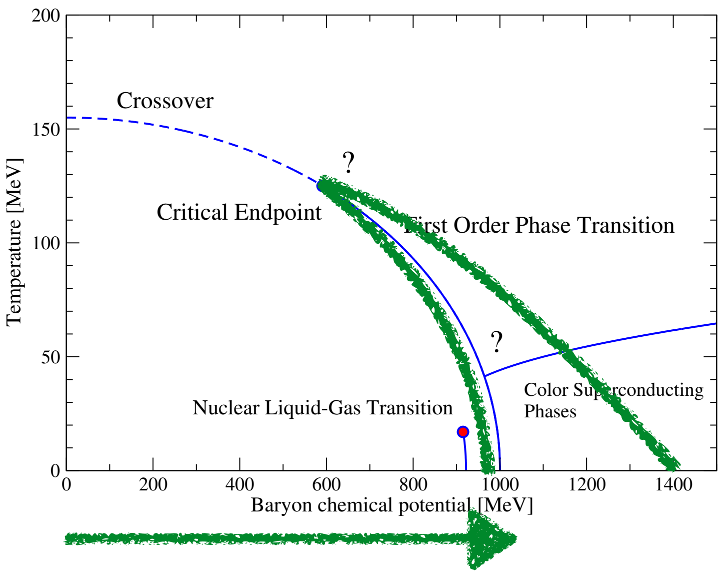
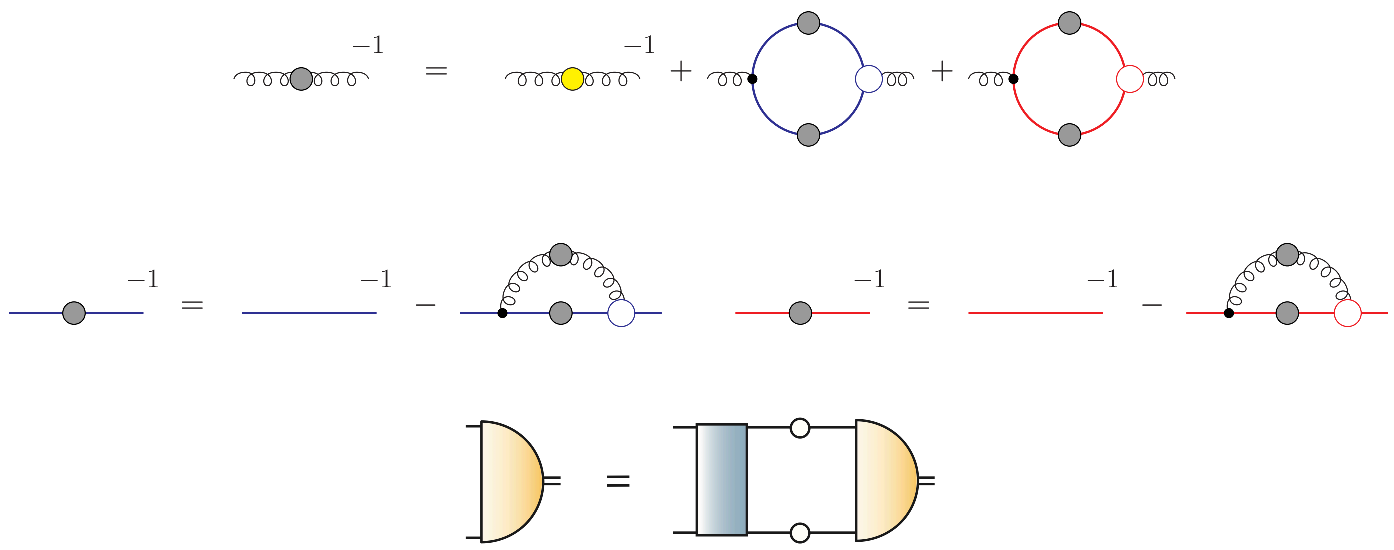
  Gunkel, CF, EPJ A 57 (2021) no. 4, 147
- Diquarks
  Müller, Buballa, Wambach, arXiv:1603.02865
  Graetz, CF work in progress
- Baryons
  Eichmann, CF, Welzbacher, PRD93 (2016)
  Fukushima, et al., PRD. D 110 (2024) no.7, 076022
  Gao at al, arXiv:2504.00539
  Graetz, CF work in progress

# Meson properties at finite chemical potential

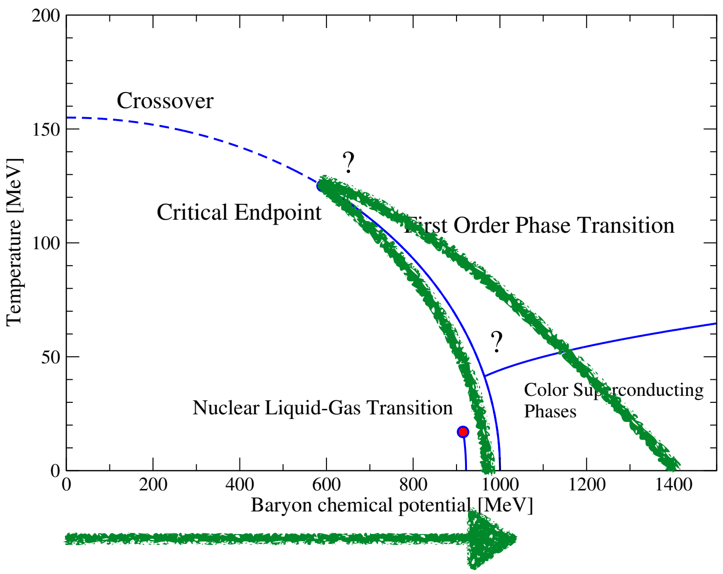


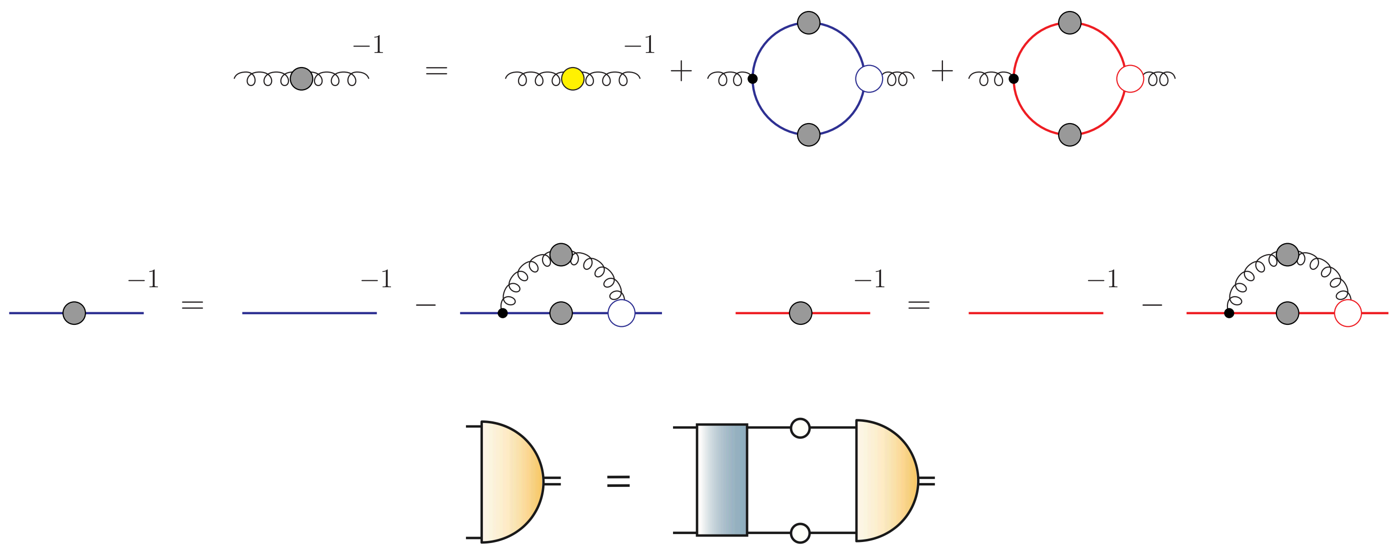



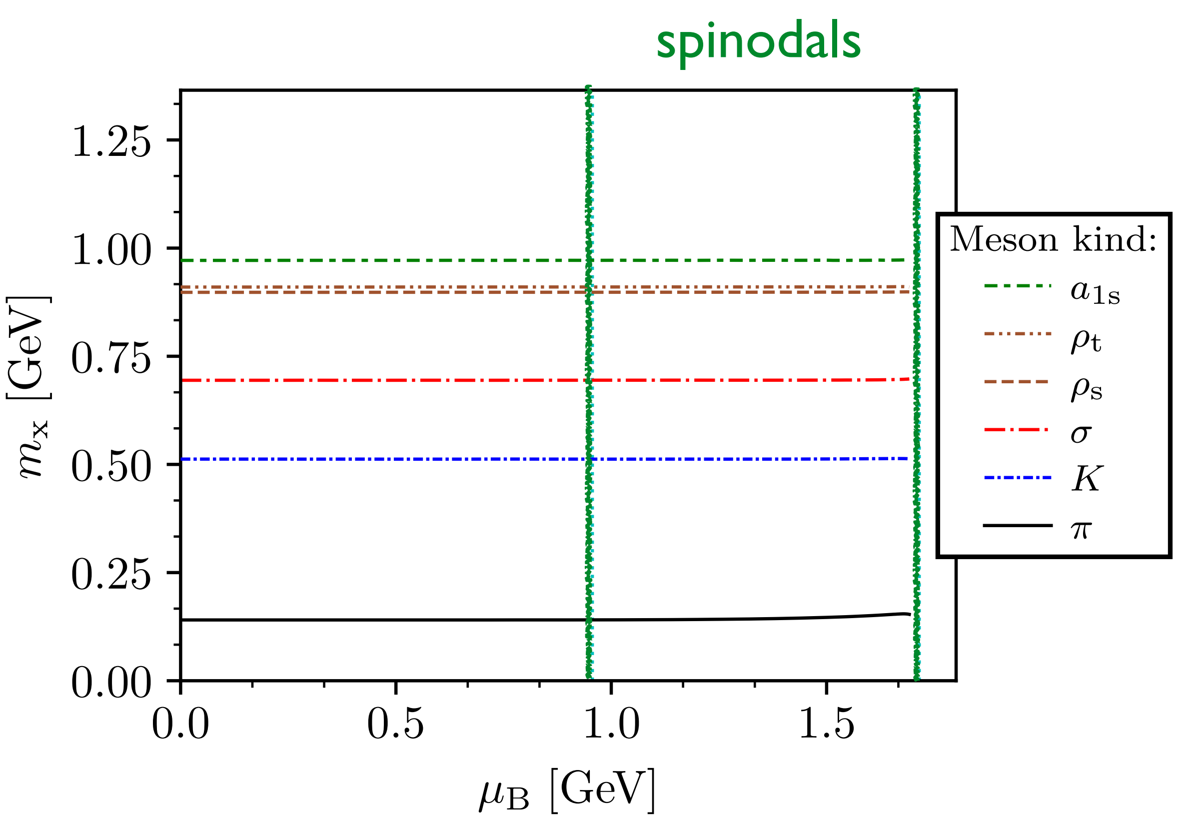
- Quarks/meson wave functions do change !

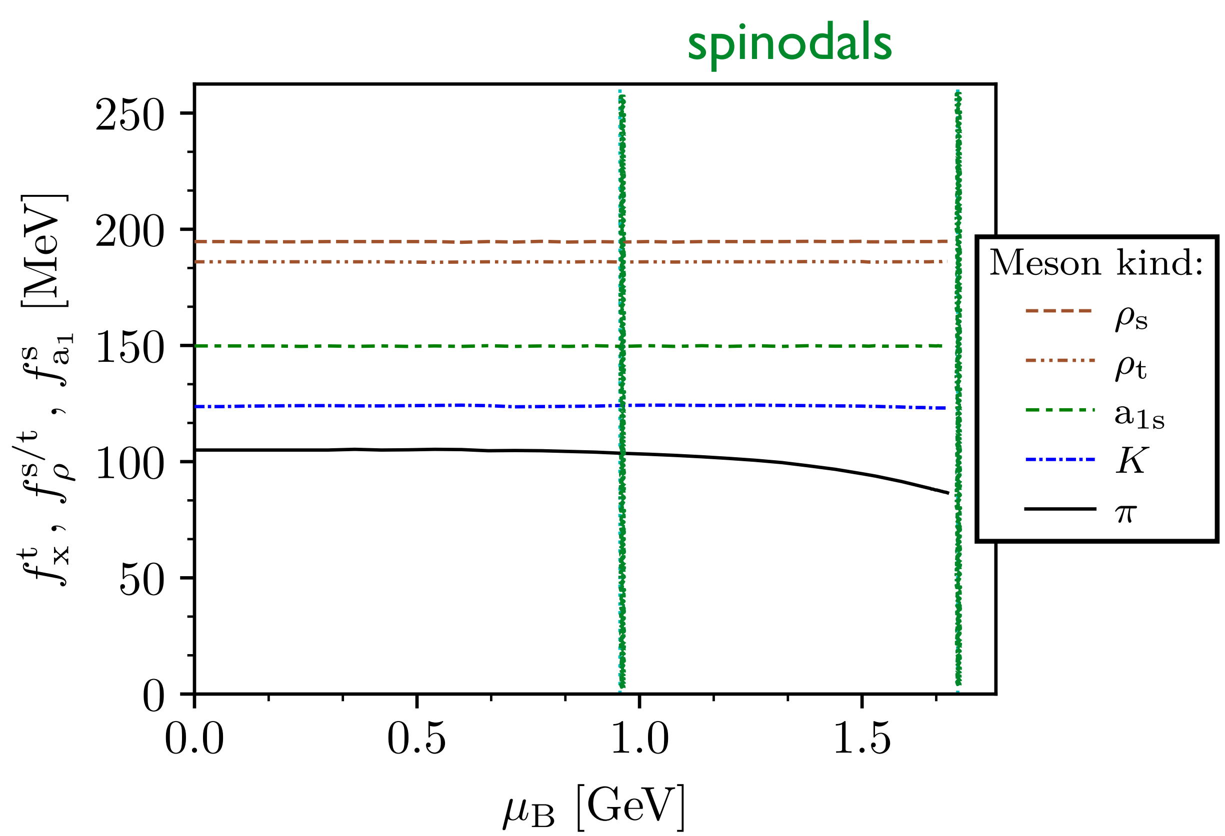
Gunkel, CF, Isserstedt, EPJ A 55 (2019) no.9, 169
Gunkel, CF, EPJ A 57 (2021) no. 4, 147

# Meson properties at finite chemical potential



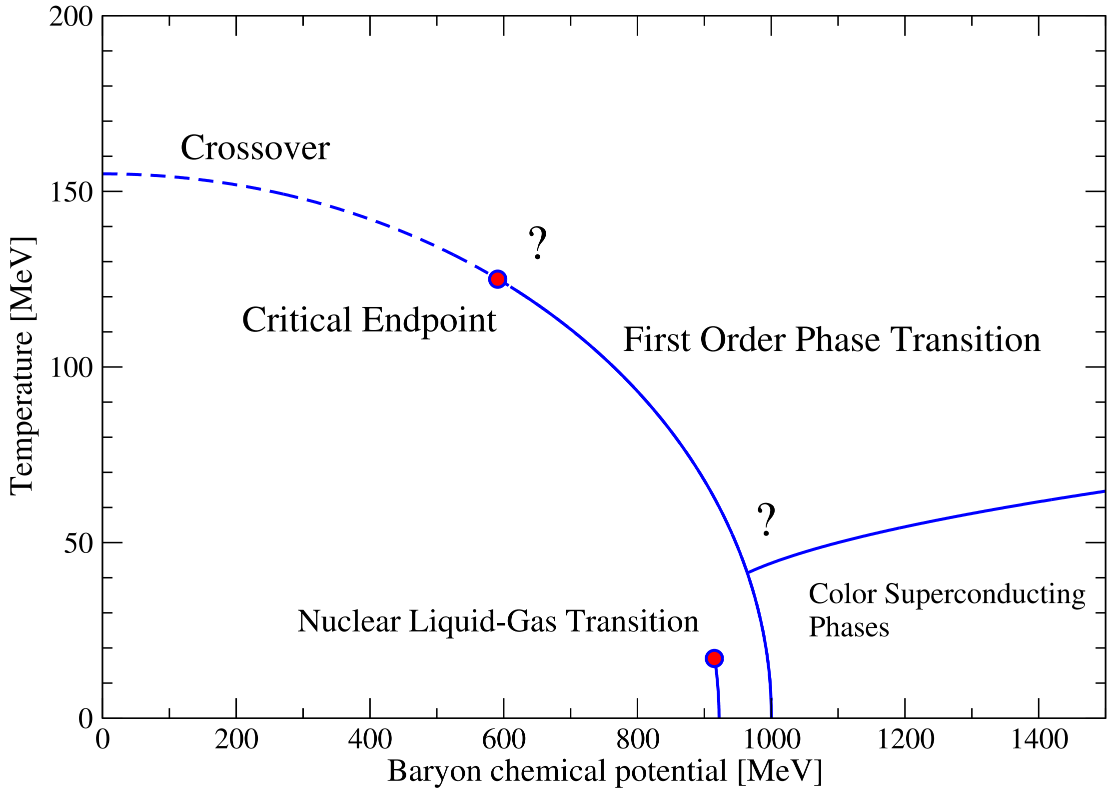



- Quarks/meson wave functions do change !
- But: Silver blaze satisfied

Gunkel, CF, Isserstedt, EPJ A 55 (2019) no.9, 169
Gunkel, CF, EPJ A 57 (2021) no. 4, 147

# Phase transitions at non-zero T: what to expect…



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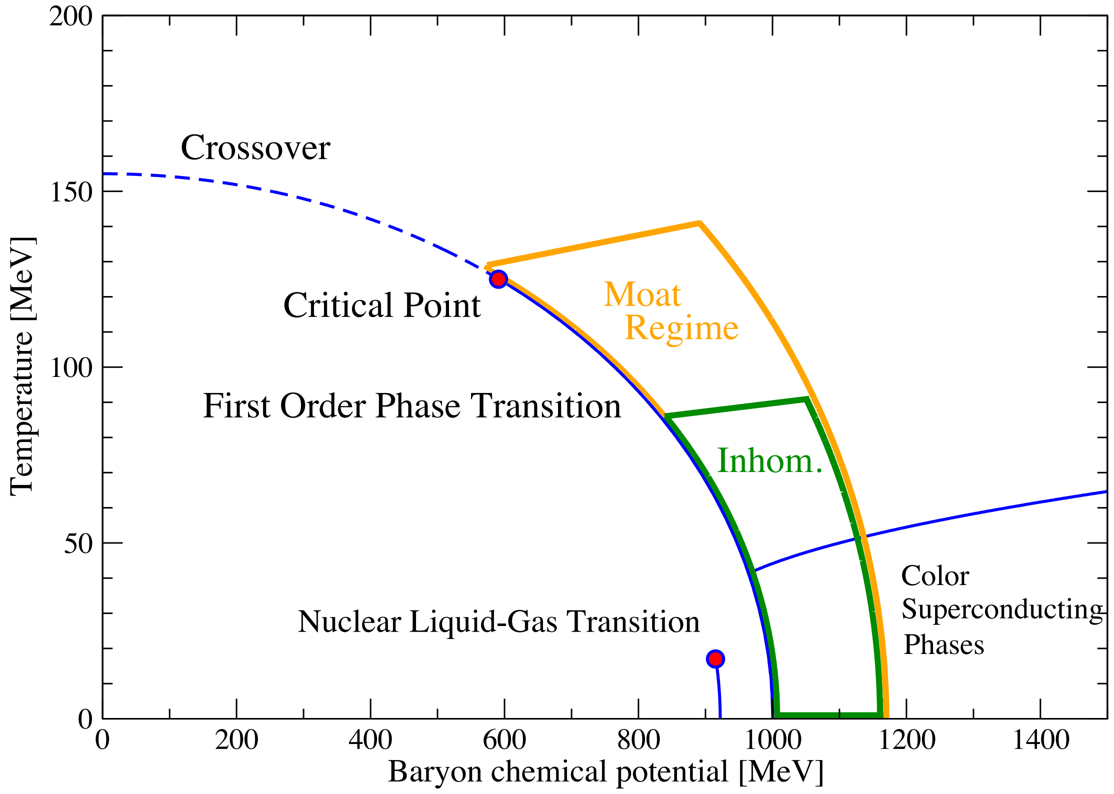

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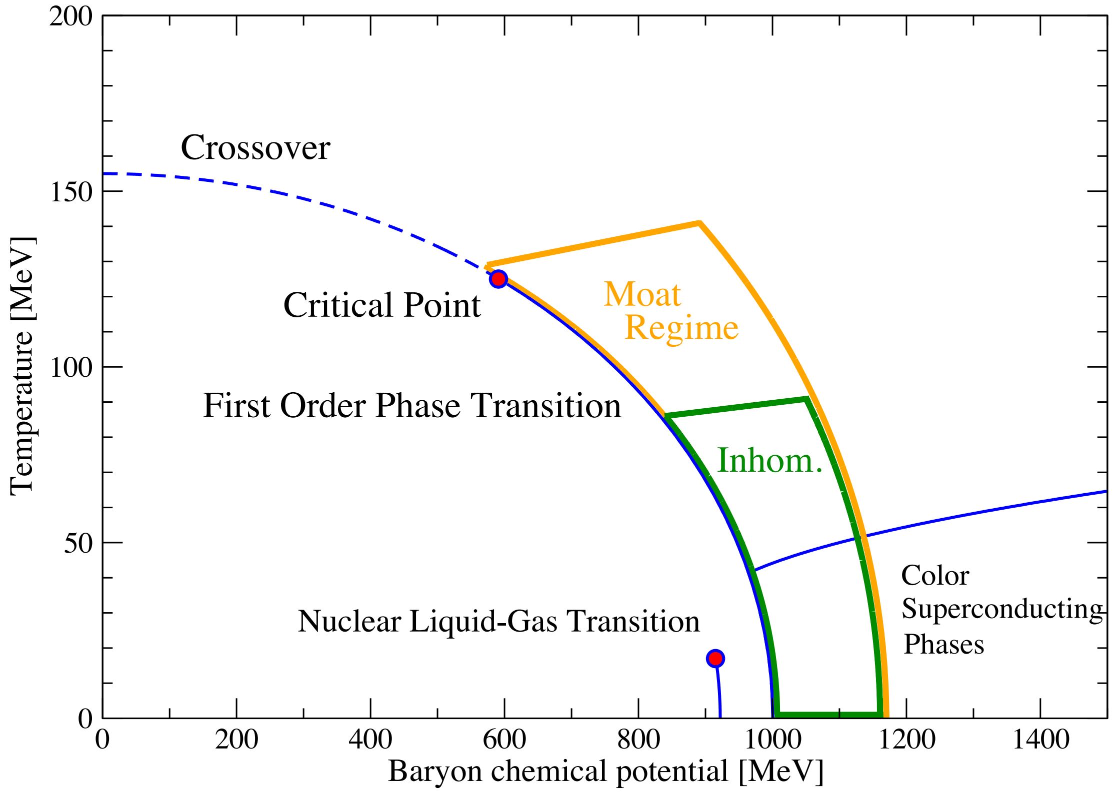
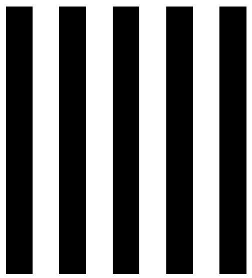
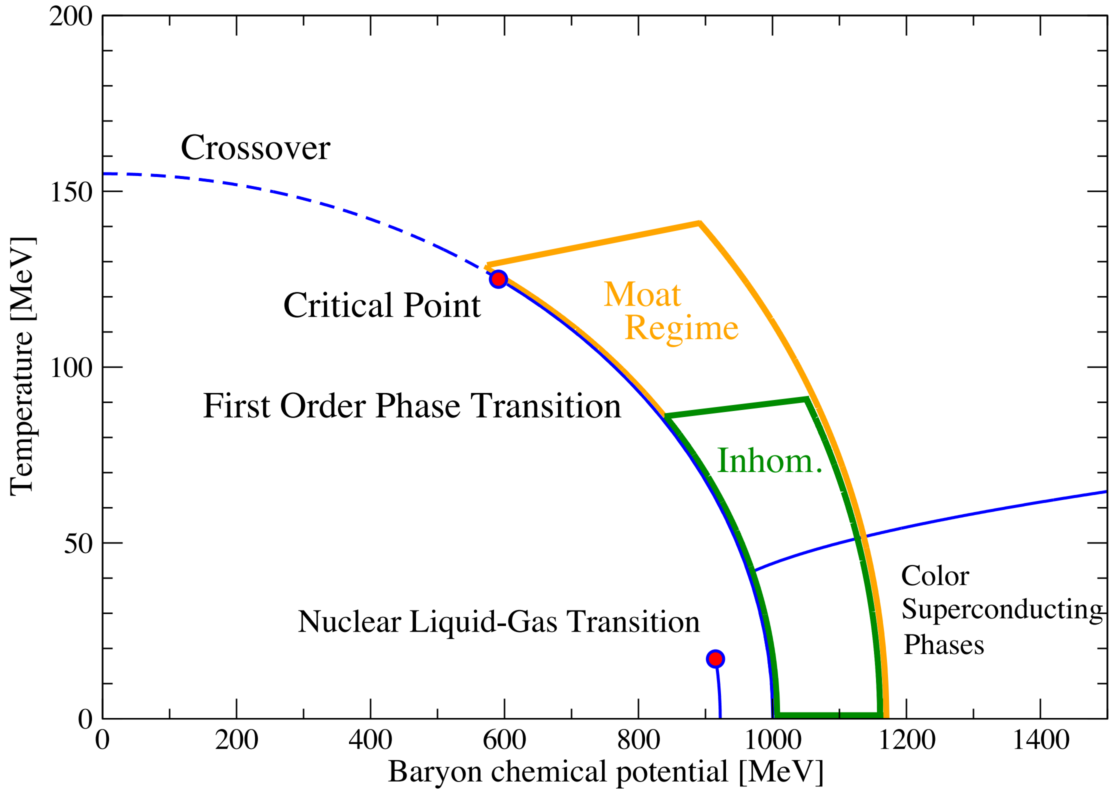



$\langle \bar{\Psi} \Psi \rangle$



$\langle \bar{\Psi} \Psi \rangle (x)$

# Phase transitions at non-zero T: what to expect…

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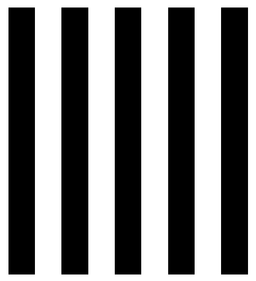

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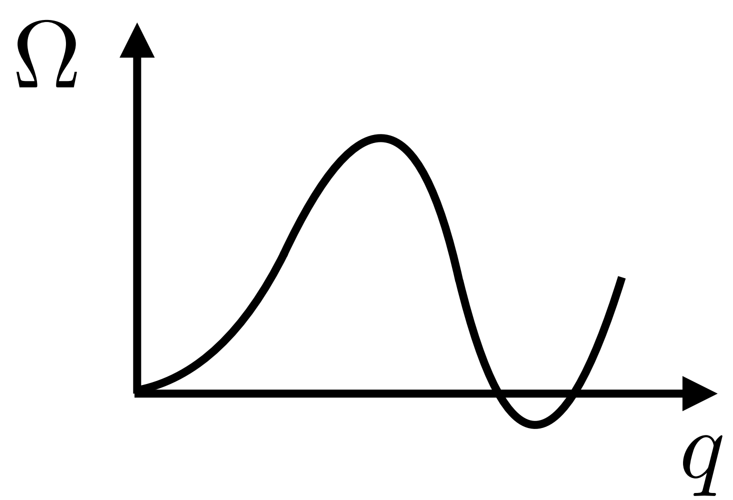
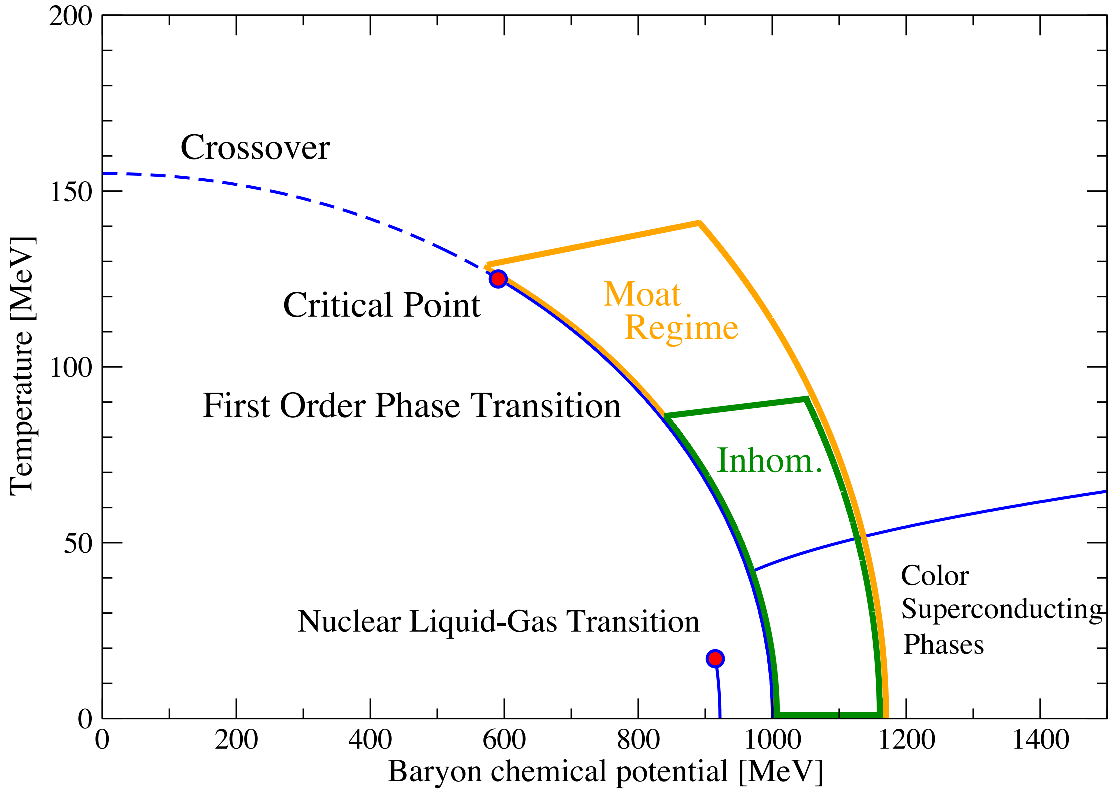



inhom. phase

Buballa and Carignano, PPNP 81 (2015), 39-96

# Phase transitions at non-zero T: what to expect…

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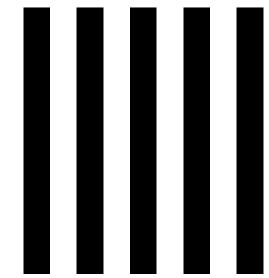

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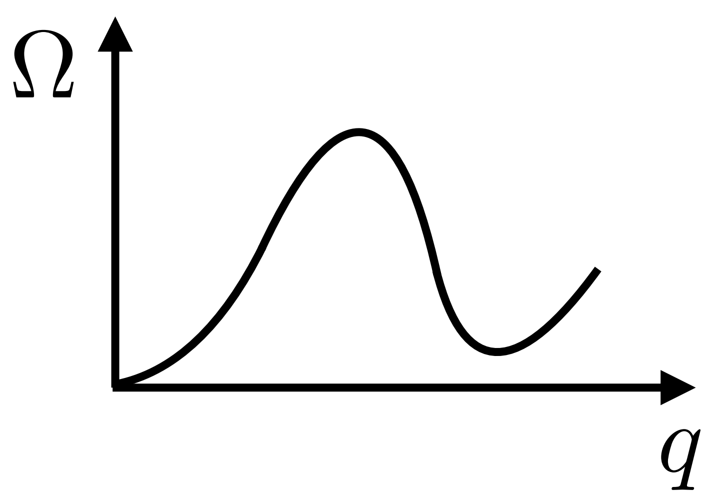
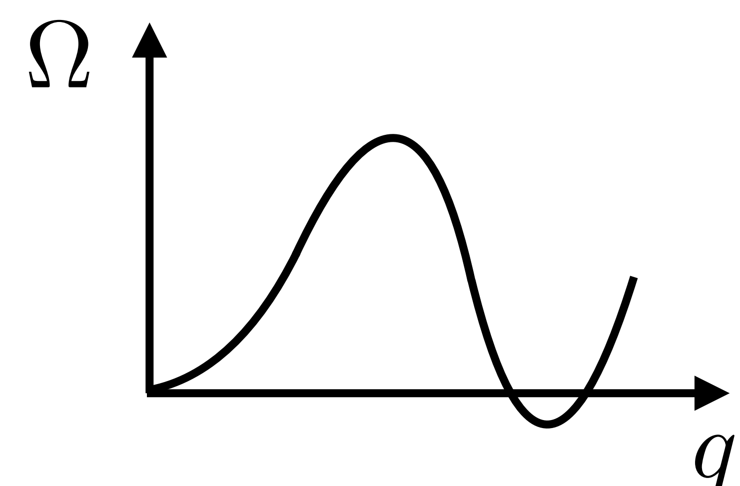



moat regime

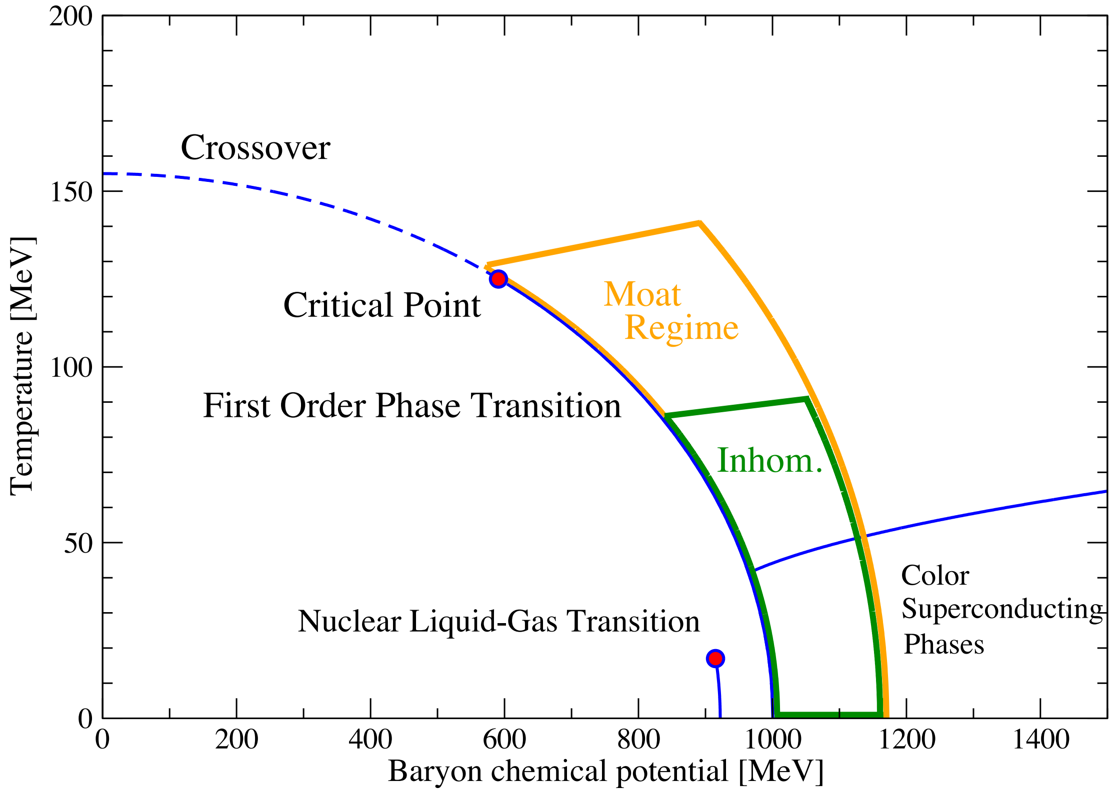



inhom. phase

Fu, Pawlowski, Rennecke, PRD 101, (2020) 5 054032
Fu, Pawlowski, Pisarski, Rennecke, Wen, Yin, arXiv:2412.15949

Buballa and Carignano, PPNP 81 (2015), 39-96

# Phase transitions at non-zero T: what to expect…

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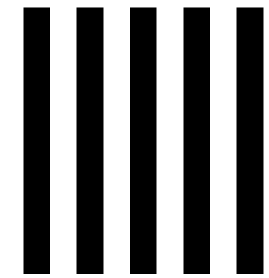
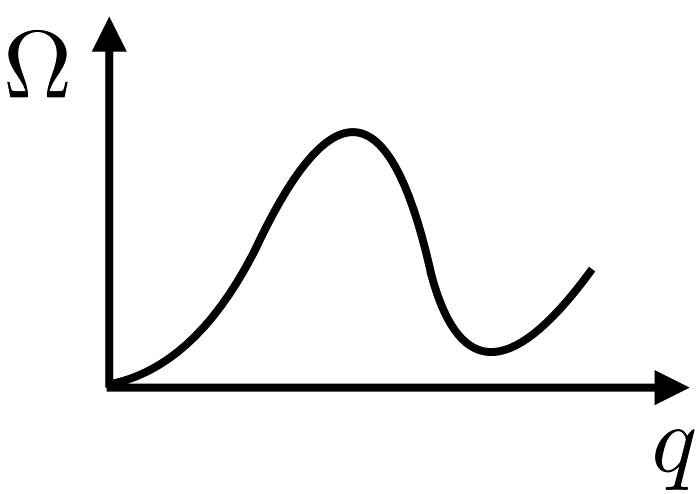
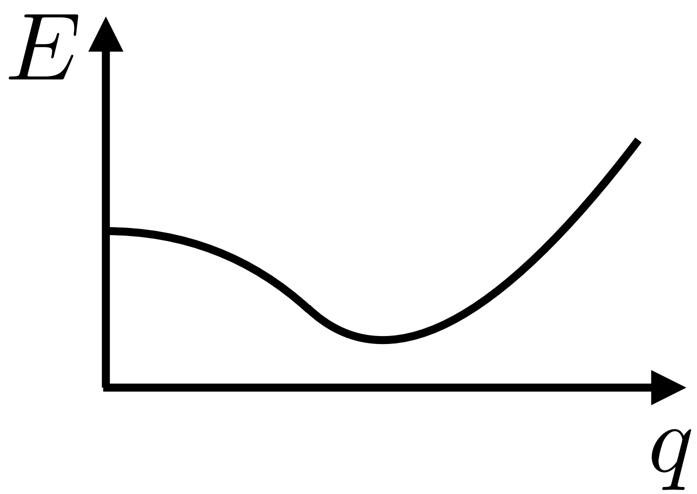


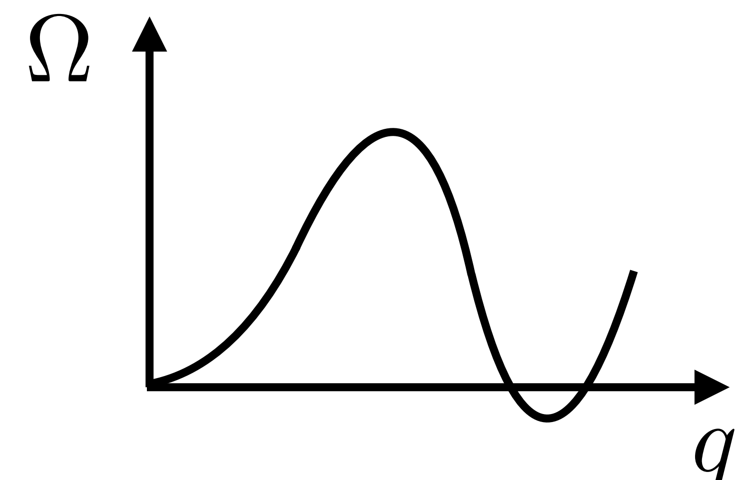
$\langle \bar{\Psi}\Psi \rangle(x)$




moat regime




nontrivial
dispersion relation

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Fu, Pawlowski, Rennecke, PRD 101, (2020) 5 054032
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Pisarski, Rennecke, PRL 127 (2021), 15 152302

Buballa and Carignano, PPNP 81 (2015), 39-96

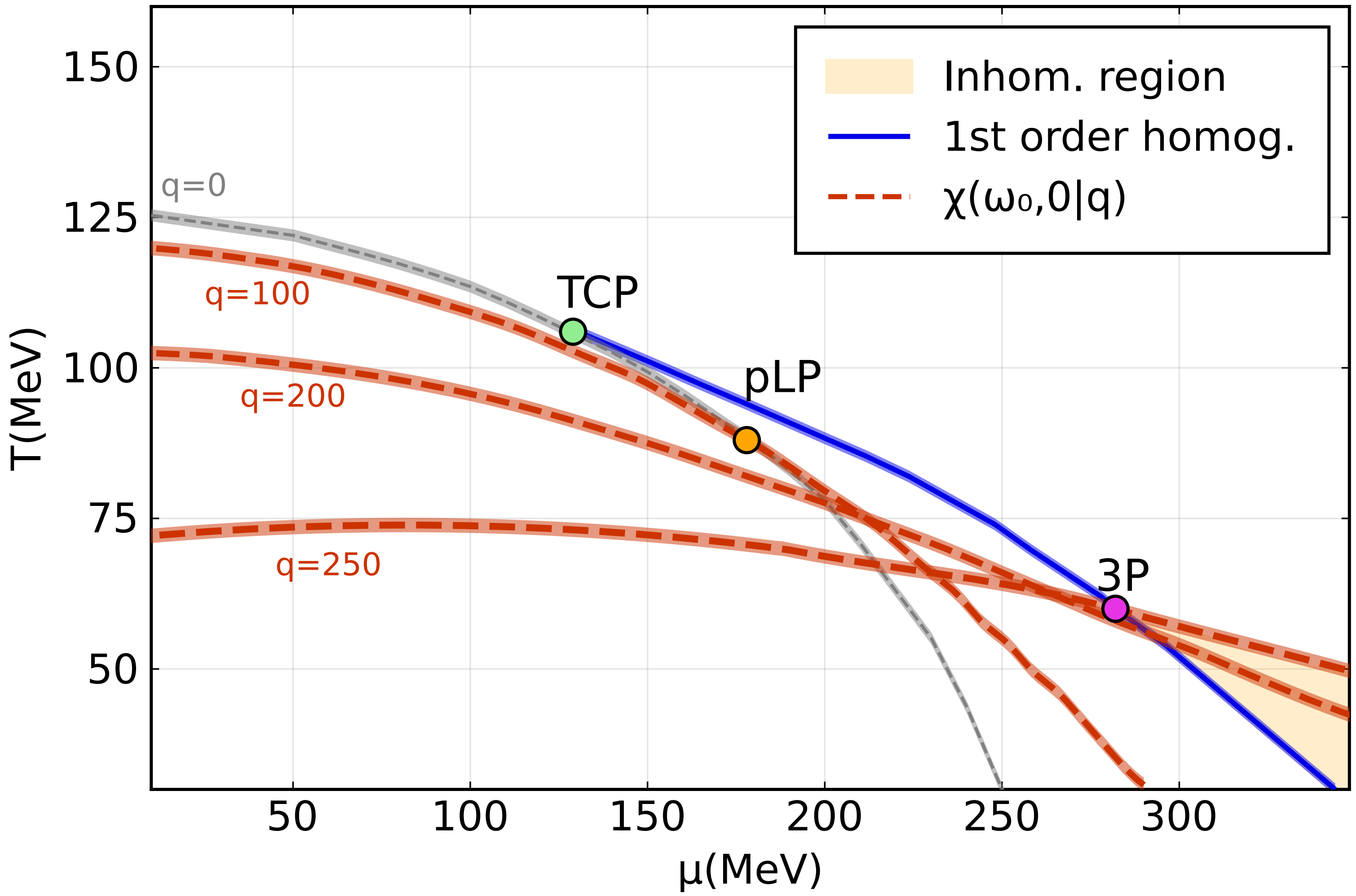
# Inhomogeneous phases (simple truncation)

$$\xi = e^{i\gamma_5\tau_3 q\cdot x}\psi$$

$$\langle\bar{\psi}(x)\psi(x)\rangle = \cos(2q\cdot x)\,\langle\bar{\xi}\xi\rangle$$
$$\langle\bar{\psi}(x)i\gamma_5\tau_3\psi(x)\rangle = \sin(2q\cdot x)\,\langle\bar{\xi}\xi\rangle$$

stability analysis

$$\chi(q) = \frac{\partial M}{\partial m} < 0$$




- unambiguous identification of inhomogeneous region
- locations of CEP and 3P are far apart!

T. F. Motta, J. Bernhardt, M. Buballa and CF, PRD 108 (2023), PRD 110 (2024), PRD 111 (2025)