Turbulence

State of the theory and challenges for experiments.

Personal view of a physicist.

Gregory Falkovich

State of the theory and challenges for experiments. Personal view of a physicist.

Gregory Falkovich June 2025

General motifs.

- Are there universality classes of developed turbulence?

  Tolstoy: **All happy families are alike, each unhappy family is miserable in its own way.** Is that really true?
- **Cascade**. Underappreciated and unexplored differences between direct and inverse cascades. Information perspective.
- Diminishing power of the flux laws.
- Beyond flux laws and spectra.
- Can physicists count to three?

State of the theory and challenges for experiments.
Personal view of a physicist.

Part I

- Turbulence onset and turbulence-flow interaction.
- Develop turbulence and **cascade**. Underappreciated and unexplored differences between direct and inverse cascades..
- Diminishing power of the flux laws.
- Beyond flux laws and spectra.

Part II

- Turbulence of particles and quasiparticles.
- Information perspective.
- Renormalizing weak turbulence into strong.

Theoretician dream picture –
homogeneous isotropic incompressible turbulence.

Real turbulence had boundaries and contain different classes of interacting excitations.

Let us start from boundaries.

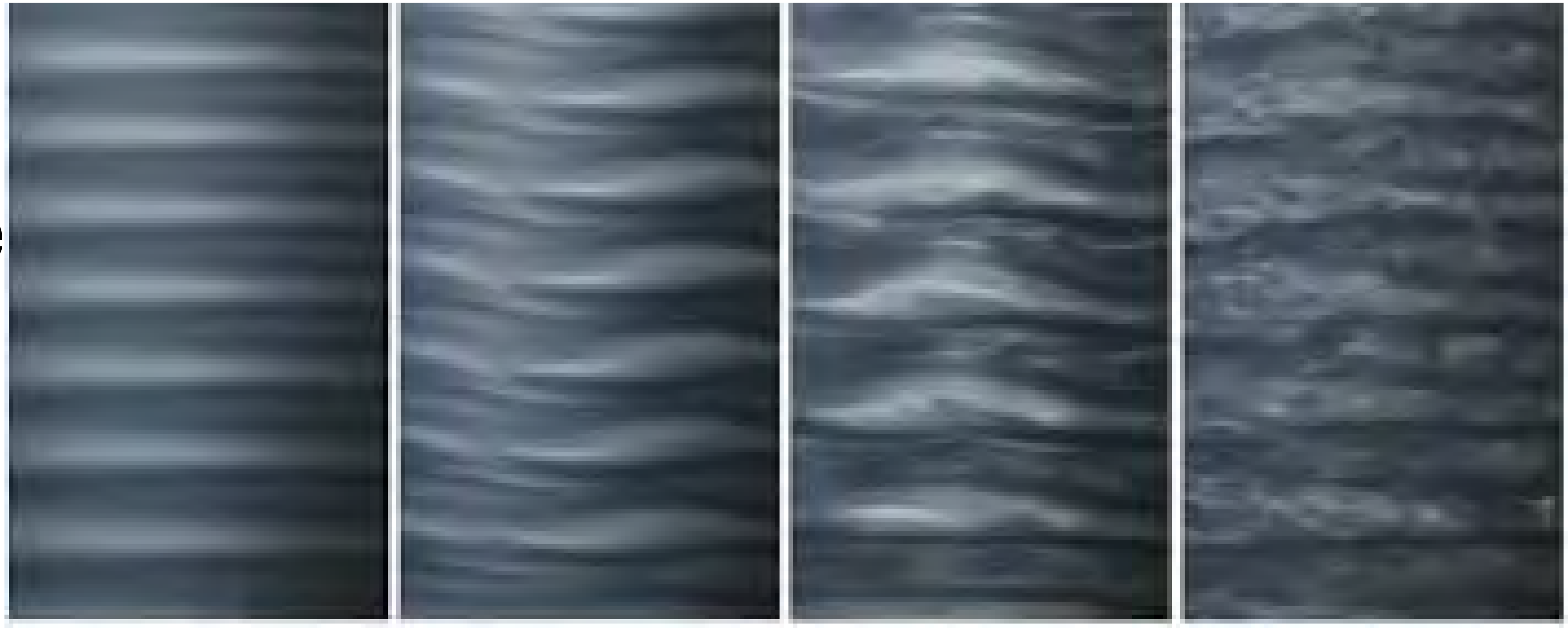
Laminar flow in the whole domain is determined by **boundaries.** Is the small-scale structure of turbulence far from the boundaries independent of them? Turbulence onset crucially depends on boundaries. Most basically, there are two ways to transit to turbulence, depending on whether

A: the laminar flow loses stability at a finite Re or

B: stays stable all the way to $Re \to \infty$.

A: random in time

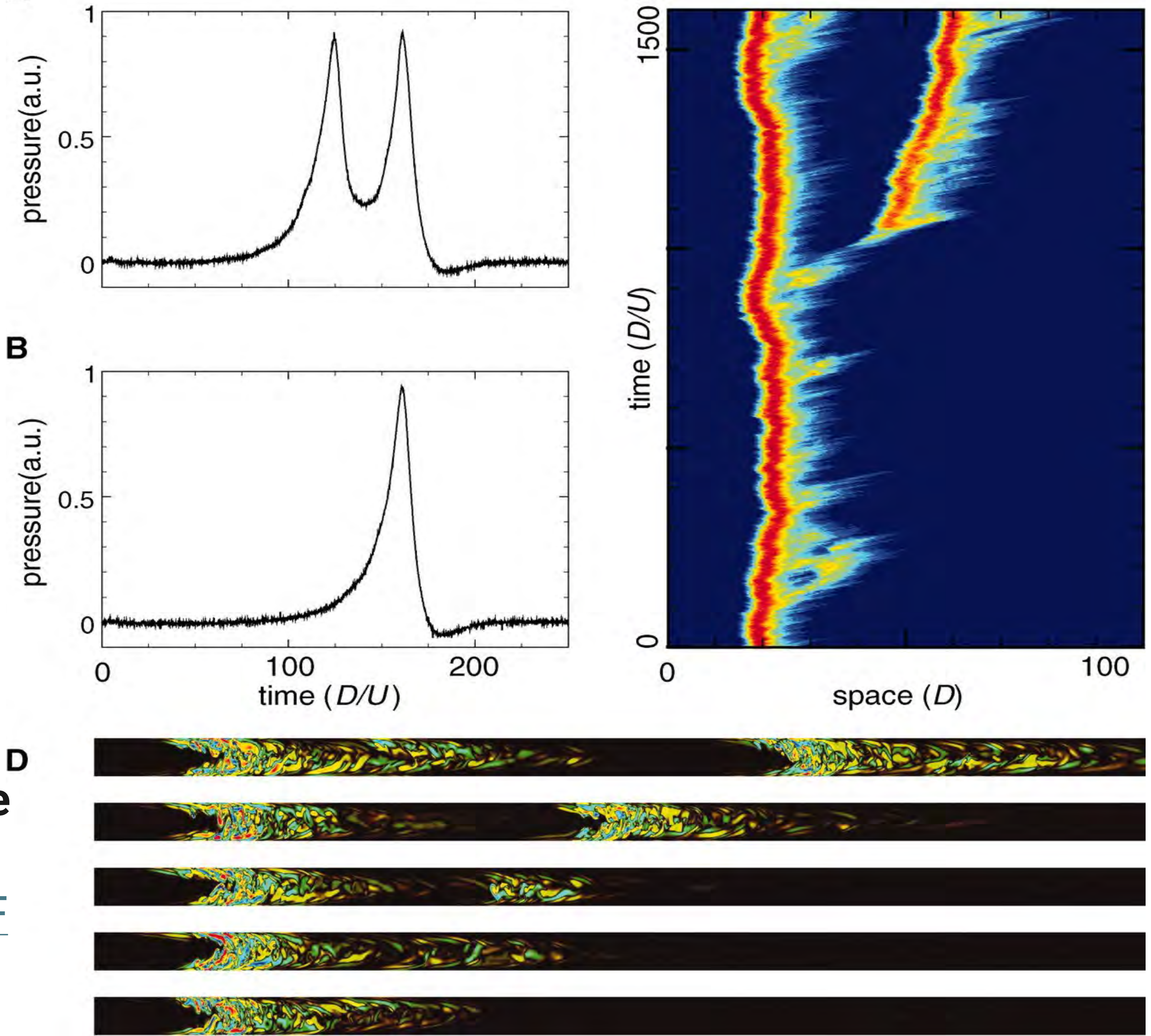
B: random in space



B: flow in a circular pipe stays stable at $Re \to \infty$.
Puffs of turbulence fill the pipe as *Re* increases

Reynolds 1895

The Onset of Turbulence in Pipe Flow AVILA, MOXEY, DE LOZAR, BARKLEY, BJÖRN HOF ***SCIENCE*** 2011



Is the small-scale structure of turbulence far from the boundaries independent of the domain shape?

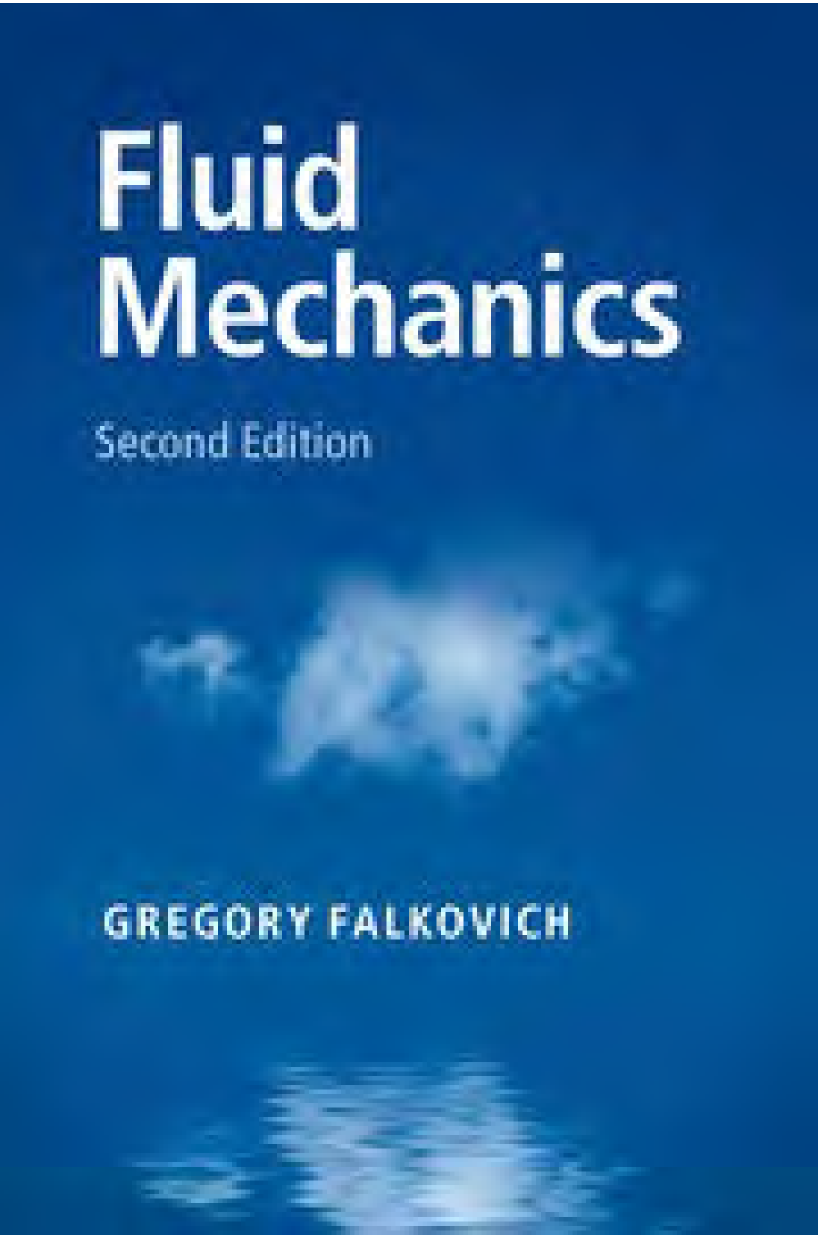
Do all roads lead to the same developed turbulence at least for incompressible fluid? My personal answer: **No.** Turbulence is generally nonlocal in k-space.

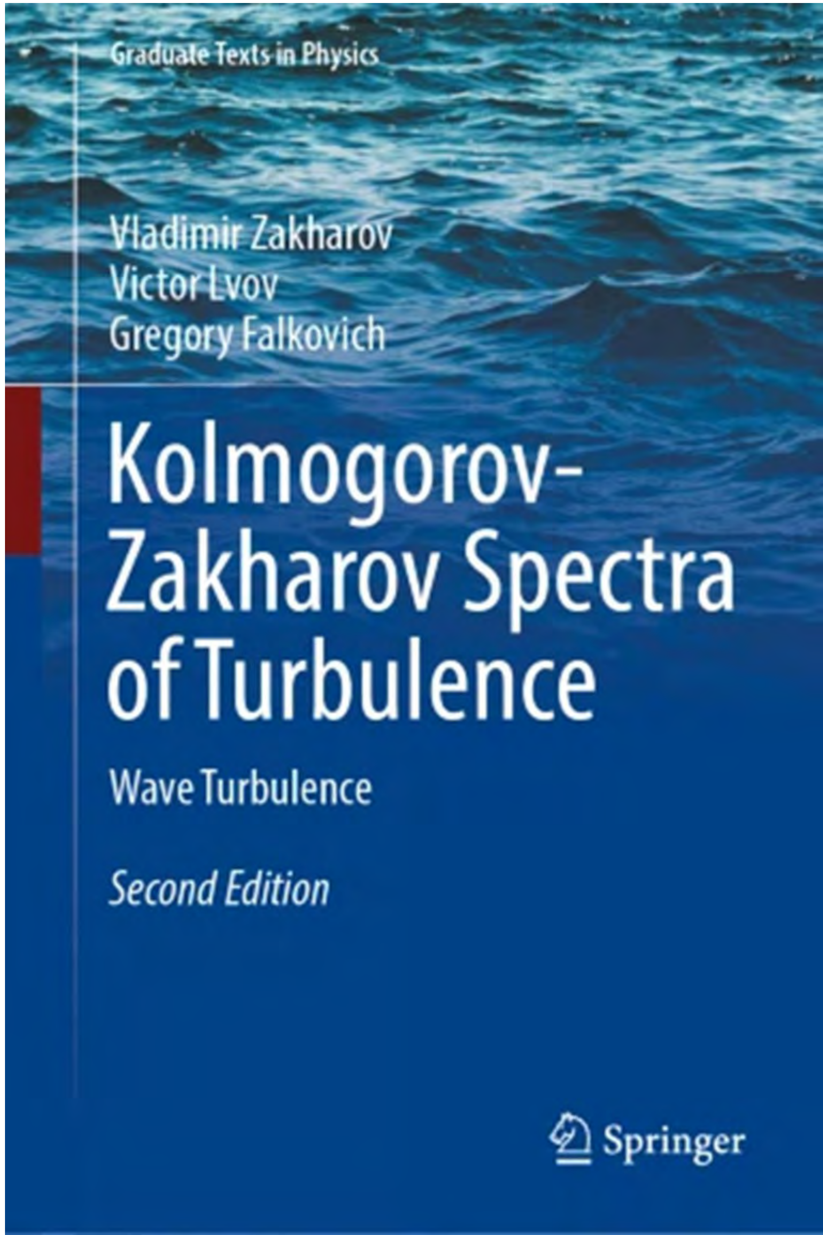
How turbulence and flow interact?
How can we suppress turbulence by modifying large-scale arrangements.
Geophysics, astrophysics and thermonuclear plasma.

Compressible developed turbulence

Acoustic waves, vortices and their interaction

Compressible turbulence
Theory review





Euler equations

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \boldsymbol{v} = 0 \, ,$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\boldsymbol{\nabla})\boldsymbol{v} = -\frac{\boldsymbol{\nabla} p}{\rho}$$

$$\mathcal{H} = \int \left[\rho v^2/2 + \varepsilon(\rho)\right] d\boldsymbol{r}$$

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$$d\mu/dt = [\partial/\partial t + (\boldsymbol{v}\boldsymbol{\nabla})]\mu = 0$$

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Another conservation law is that of velocity circulation around a “fluid” contour. This means that there is some scalar function $\mu(r, t)$ transferred together with the fluid

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$$d\mu/dt = [\partial/\partial t + (\boldsymbol{v}\nabla)]\mu = 0$$

$$\mathcal{L}(t) = \int \left[\rho \frac{v^2}{2} - \varepsilon(\rho) + \Phi\left(\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \boldsymbol{v}\right) - \lambda\left(\frac{\partial \mu}{\partial t} + (\boldsymbol{v}\nabla)\mu\right)\right] d\boldsymbol{r}$$

$$\partial \rho/\partial t = \delta \mathcal{H}/\delta \Phi \,,$$

$$\partial \Phi/\partial t = -\delta \mathcal{H}/\delta \rho$$

$$\partial \lambda/\partial t = \delta \mathcal{H}/\delta \mu \,,$$

$$\partial \mu/\partial t = -\delta \mathcal{H}/\delta \lambda$$

$$\boldsymbol{v} = \lambda \frac{\nabla \mu}{\rho} + \nabla \Phi$$

$$v = \lambda \frac{\nabla \mu}{\rho} + \nabla \Phi$$

In general, all four variables are hopelessly intertwined.
And so are vortices and waves. They are decoupled for M<<1.

Consider a fluid of uniform density $\rho = 1$. For the velocity, we now have $\boldsymbol{v} = \lambda \nabla \mu + \nabla \Phi$. From the condition $\operatorname{div} \boldsymbol{v} = 0$, one can find $\Phi = -\Delta^{-1} \operatorname{div} \lambda \nabla \mu$. The formula for the velocity may be rewritten as

$$\boldsymbol{v} = -\Delta^{-1} \operatorname{rot}[\nabla \lambda, \nabla \mu]$$

$$\partial \lambda / \partial t = \delta \mathcal{H} / \delta \mu \ ,$$

$$\partial \mu / \partial t = - \delta \mathcal{H} / \delta \lambda$$

Alternatively, for pure acoustic modes, the canonical variables: density and velocity potential

$$\Phi(\boldsymbol{k}) = -(i/k)(\omega_k / 2\rho_0)^{1/2}[b(\boldsymbol{k}) - b^*(-\boldsymbol{k})]$$

$$\delta\rho(\boldsymbol{k}) = k(\rho_0 / 2\omega_k)^{1/2}[b(\boldsymbol{k}) + b^*(-\boldsymbol{k})] \ ,$$

$$\omega(\boldsymbol{k}) = k c_s, \quad c_s^2 = (\partial p / \partial \rho) \ .$$

$$\partial \rho / \partial t = \delta \mathcal{H} / \delta \Phi \ ,$$

$$\partial \Phi / \partial t = - \delta \mathcal{H} / \delta \rho$$

Cascade idea and the flux laws in two limits

Consider in parallel the incompressible NS equation

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{\nabla P}{\rho} + \nu \Delta \mathbf{v} + \mathrm{f}$$

and 1D Burgers equation describing acoustic waves

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + w w_x = \nu w_{xx} + f(x,t)\,.$$

$$\frac{\partial}{\partial t}\langle w_1 w_2\rangle = \langle w_1\big(f_2 + \nu w_{2xx} - w_2 w_{2x}\big) + w_2\big(f_1 + \nu w_{1xx} - w_1 w_{1x}\big)\rangle$$

For $|x_1 - x_2| \ll L$ we can put $\langle f_1 w_2\rangle = \langle f_2 w_1\rangle \approx \langle f w\rangle = \epsilon$. Since the derivatives are large on the scale η, they are uncorrelated at the distances $|x_1 - x_2| \gg \eta$, so we can neglect $\langle w_1 w_{2xx}\rangle = \partial_2\langle w_1 w_{2x}\rangle = -\partial_1\langle w_1 w_{2x}\rangle = -\langle w_{1x} w_{2x}\rangle$. We consider the force statistics to be uniform in space, then any moment like $\langle w_1 w_2^2\rangle$ is a function of $x_1 - x_2$ so that $\partial_2\langle w_1 w_2^2\rangle = -\partial_1\langle w_1 w_2^2\rangle$. Adding (zero) term $\partial_1\langle w_1^3 - w_2^3\rangle$ we derive $2\epsilon = -\partial_1\langle (w_1 - w_2)^3\rangle/6$ which gives

$$\mathbf{3\langle w_2 w_1(w_2 - w_1)\rangle = \langle (w_1 - w_2)^3\rangle = -12\epsilon\,(x_1 - x_2)}$$

4/5-law in 3D

$$\mathbf{\langle (v_1 - v_2)^3\rangle = -\,{}^{12\epsilon\,(x_1 - x_2)}\!\big/_{d(d+2)}}$$

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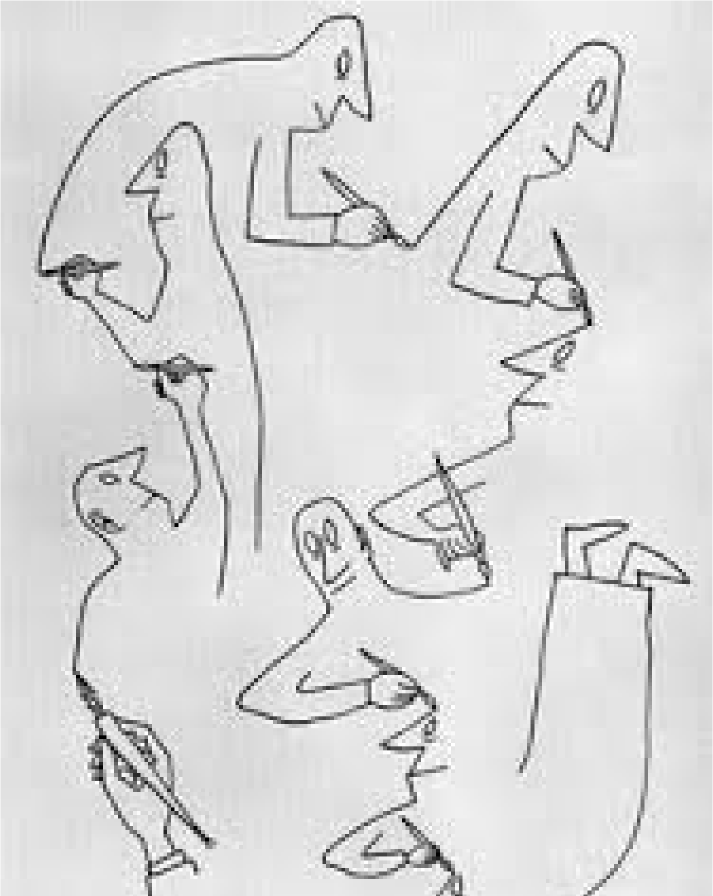

$$\frac{\text{kinetic energy } (\delta v_r)^2}{\text{time } r/\delta v_r} = \text{energy flux } \epsilon$$

Lagrangian form
for both cases

$$\frac{d|\delta v|^2}{dt} = -4\epsilon$$

$$\langle (w_1 - w_2)^3 \rangle = -12\epsilon\,(x_1 - x_2)$$

$$\langle (v_1 - v_2)^3 \rangle = -\,{12\epsilon\,(x_1 - x_2)}\Big/{d(d+2)}$$

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Lagrangian form for both cases $\quad \dfrac{d|\delta \boldsymbol{v}|^2}{dt} = -4\epsilon$

$$\langle (w_1 - w_2)^3 \rangle = -12\epsilon\,(x_1 - x_2)$$

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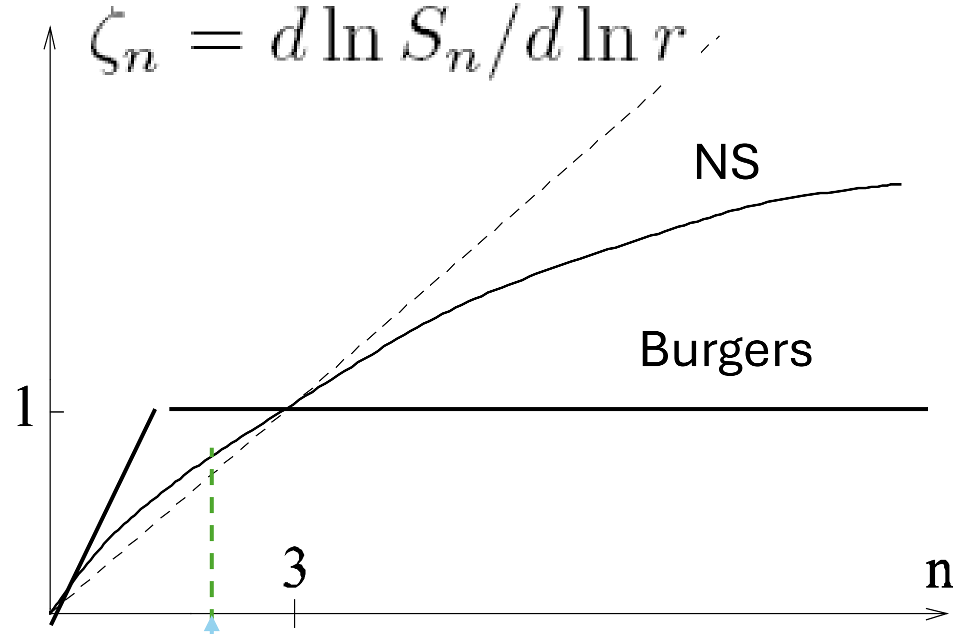
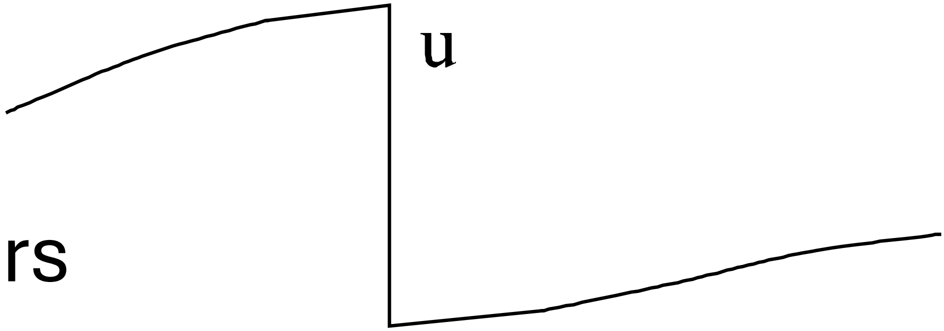
Despite having the same flux laws, incompressible and acoustic turbulence are completely different, which is reflected in the rest of statistics, particularly other moments.



Burgers



$$u_k \propto 1/k \text{ at } k \ll v/\nu$$

$$S_2(x) \propto |x|$$

$$E(k) \propto k^{-2}$$

NS

KO41 $\quad S_2(r) \sim (\epsilon r)^{2/3} \quad E_k \sim \epsilon^{2/3} k^{-5/3}$

$$S_2 \sim (\epsilon r)^{2/3} \, (r/L)^{\Delta_2} \qquad \zeta_2 = \frac{2}{3} + \Delta_2 \qquad ?$$

21st century challenges for experimental measurements needed to understand the physics

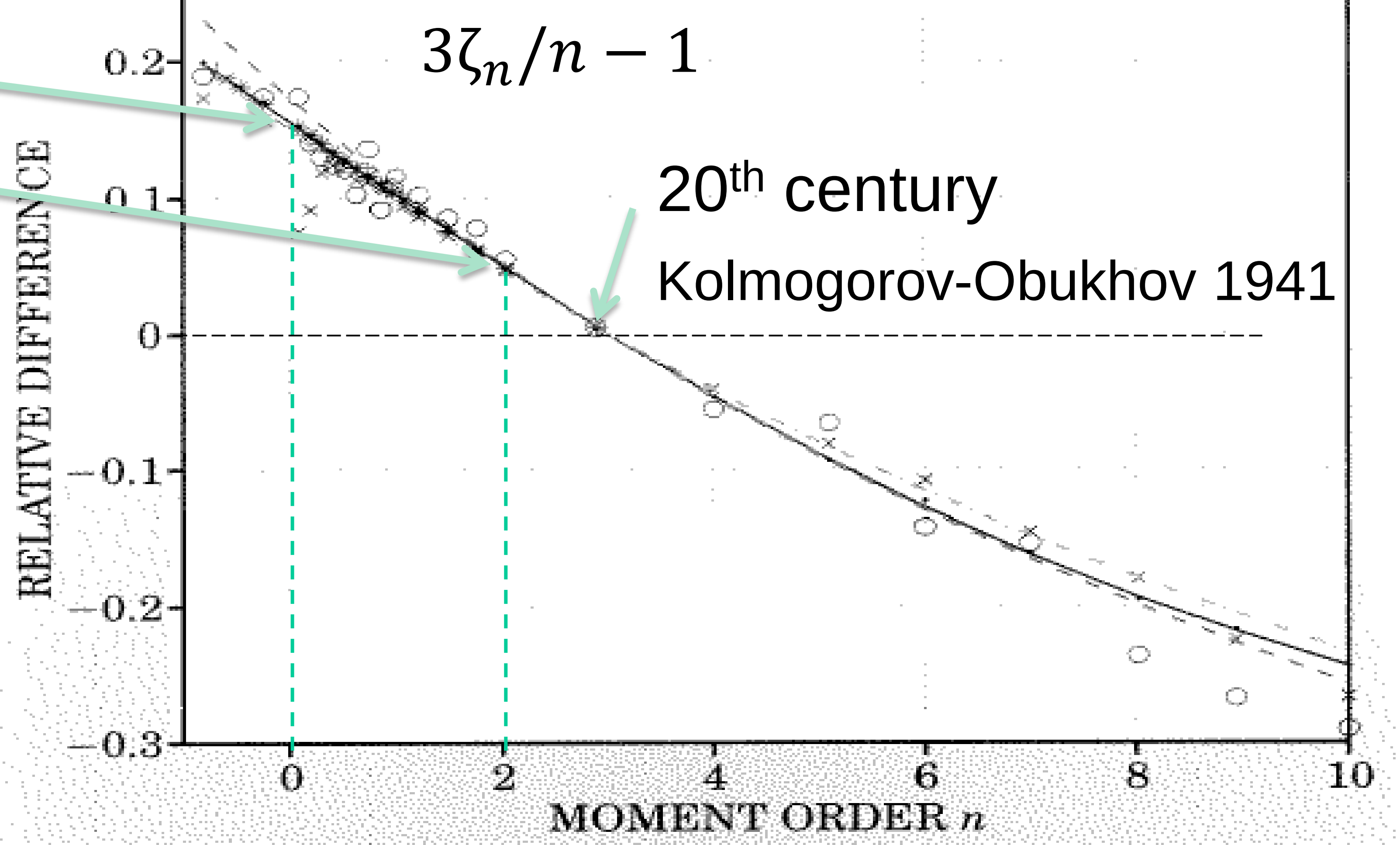


Figure 3. Scaling exponents ζ_n differ from the scale-invariant, linear Kolmogorov value of $n/3$. In this plot, the relative difference is defined as $(\zeta_n - n/3)/(n/3)$, open circles show experimental results, crosses and stars show results of simulations, and the curves give various theoretical results. Note that the relative difference passes through $n = 0$ without showing any special feature. Nor does there appear to be any special behavior as n approaches -1, notwithstanding that structure functions of order -1 and lower are undefined. (Adapted

Statistical integrals of motion (martingales) and the anomalous scaling.

Another way to measure ζ_2 - following two fluid trajectories.

Can we find a function of $R_{12}(t), {u_{12}}^2(t)$, time-independent on average?

$$I(t) =< f(R_{12}(t)/r){u_{12}}^2(t) >$$

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$$R_{12}(0) = r$$

$$I(0) =< {u_{12}}^2(0) >\propto r^{\zeta_2}$$

Consider $\lim\limits_{t\to\infty} I(t) =< f(R_{12}(t)/r){u_{12}}^2(t) >$.

We expect in most realizations $R_{12}(t) \gg r$ and fluctuations of $R_{12}(t)$ and $u_{12}(t)$ to be r-independent:

$$\lim_{x\to\infty} f(\mathrm{x}) \propto x^{-\zeta_2} \rightarrow \quad \lim_{t\to\infty} I(t) = r^{\zeta_2} < R_{12}^{-\zeta_2} u_{12}^2 >$$

Comparing incompressible and acoustic turbulence from a Lagrangian perspective

The Kolmogorov scaling $\delta v^3 \propto r$ means that the velocity field is spatially non-smooth. It is then non-Lipshits and allows multiple trajectories for the same initial conditions. Indeed, the equation $dx(t)/dt = x^{1-a}$, with the initial condition $x(0) = 0$, admits two solutions

$$x(t) = 0 \text{ and } x(t) = (at)^{1/a}.$$

However, non-uniqueness is qualitatively different in the two cases.

The Richardson law of incompressible turbulence for the distance between two particles, $R^{2/3}(t) - R^{2/3}(0) \sim \epsilon t$, means particle **splitting**.

In compressible turbulence, shocks mean particle **sticking**, not splitting.

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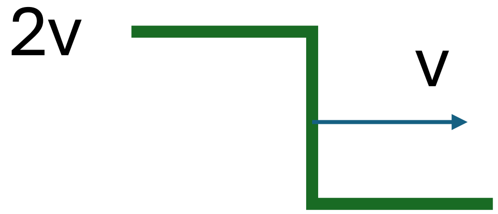

In both cases, we have dissipative anomaly –
finite dissipation in an inviscid limit. It is trivial in shocks:
flying particle hits a standing one, stick to it, and continue flying together with half velocity, preserving momentum and dissipating energy.
But we know little about the flow configurations providing the main dissipation at large Re in incompressible and compressible turbulence.

Anomalous scaling and intermittency in turbulence

$$< (\delta v)^n > = (\epsilon r)^{n/3}\ (L/r)^{\Delta_n}$$

Δ_n changes sign at n=3

$\Delta_2 < 0$ means that the level of small-scale turbulence needed to carry a fixed flux goes to zero as the cascade length (L/r) increases.
In other words, cascade and dissipation are realized by a smaller and smaller numbers of stronger and stronger events – extreme intermittency.

Cascades and the flux laws at finite Mach

The very existence of a cascade needs a demonstration due to spectral non-locality of interactions. Aluie (2011, 2013) has proven the existence of an inertial range for highly compressible turbulence produced by any type of driving, solenoidal or compressive.

A fundamental idea for the scaling of supersonic turbulence was proposed by Lighthill (1955) and later refined by Henriksen (1991), Fleck (1996) and Kritsuk et al. (2007). Based on the dimensional analysis by K41 and the assumption of a constant flux of the kinetic energy density, $e_{\mathrm{kin}} = (1/2)\rho v^2$ in the inertial range, we can write

$$\frac{\mathrm{d}e_{\mathrm{kin}}}{\mathrm{d}t} \propto \frac{\rho v^2}{t} \propto \frac{\rho v^3}{\ell} \overset{!}{=} \text{constant.}$$

$$P(\rho^{1/3} v) \propto \mathrm{d}(\rho^{1/3} v)^2/\mathrm{d}k \propto k^{-5/3}$$

Pure hand-waving.

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Surprisingly, two exact relations were derived at about the same time, likely independently:

1. G. Falkovich, I. Fouxon, and Y. Oz, *J. Fluid Mech*. 644, 465 (**2010**).
2. S. Galtier and S. Banerjee, *Phys. Rev. Lett.* 107, 134501 (**2011**).

Cascades and the flux laws at finite Mach

1. GF, Fouxon, Oz, *J Fluid Mech* 2010.
The Kolmogorov flux relation is a particular case of the general relation on the current-density correlation function.
Using that, one can derive new exact relations for compressible turbulence, which are of higher orders:

$$\partial_t q^a + \nabla \cdot \boldsymbol{j}^a = f^a$$

$$j_i^a = F_i^a(\{\rho\}) + \sum_{jb} G_{i,jb}^a(\{\rho\})\nabla_j \rho^b + \ldots$$

Wagner, GF, Kritsuk, Norman, *J Fluid Mech* 2012.

$$\nabla_i \langle q^a(0,t) F_i^a(\boldsymbol{r},t)\rangle = \langle q^a(0,t) f^a(\boldsymbol{r},t)\rangle$$

$$\langle q^a(0) F_i^a(\boldsymbol{r})\rangle = \frac{\epsilon_a r_i}{d}$$

$$\langle [\rho(0)\boldsymbol{u}(0)\cdot\boldsymbol{u}(\boldsymbol{r})\rho(\boldsymbol{r})]u_\parallel(\boldsymbol{r})\rangle + \langle \rho(0)u_\parallel(0)\rho(\boldsymbol{r})\rangle = \frac{\bar{\epsilon} r}{d},$$

Cascade and the flux laws at finite Mach

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$$\langle [\rho(0)\boldsymbol{u}(0)\cdot\boldsymbol{u}(\boldsymbol{r})\rho(\boldsymbol{r})]u_\parallel(\boldsymbol{r})\rangle + \langle \rho(0)u_\parallel(0)\rho(\boldsymbol{r})\rangle = \frac{\bar{\epsilon}r}{d},$$

2. Galtier, Banerjee, *Phys Rev Lett* 2011, *Phys Rev E* 2013.

$$-\frac{2}{3}\varepsilon_{\text{eff}}\, r = \mathcal{F}(r) \;\propto\; \rho v^3$$

$$\varepsilon_{\text{eff}}(r) = \varepsilon + \frac{3}{8}r\frac{\partial}{\partial r}\mathcal{S}|_{r\to 0} \qquad \mathcal{S}(r) = \langle (\nabla\cdot\boldsymbol{v})'(R-E)\rangle_x + \langle (\nabla\cdot\boldsymbol{v})(R'-E')\rangle_x$$

$$\text{with} \quad R = \rho(\boldsymbol{v}\cdot\boldsymbol{v}'/2 + e') \quad \text{and} \quad E = \rho(\boldsymbol{v}\cdot\boldsymbol{v}/2 + e)$$

If $\varepsilon_{\text{eff}} =$ constant, then the spectrum $P(\rho^{1/3}v) \propto k^{-5/3}$

for turbulence with a strong $\nabla\cdot\boldsymbol{v}$ component $\quad P(\rho^{1/3}v) \propto k^{-19/9}$

C Federrath, **On the universality of supersonic turbulence**, MNRAS 436, 1245–1257 (2013)

Inverse cascade as self-organization of far-from-equilibrium state

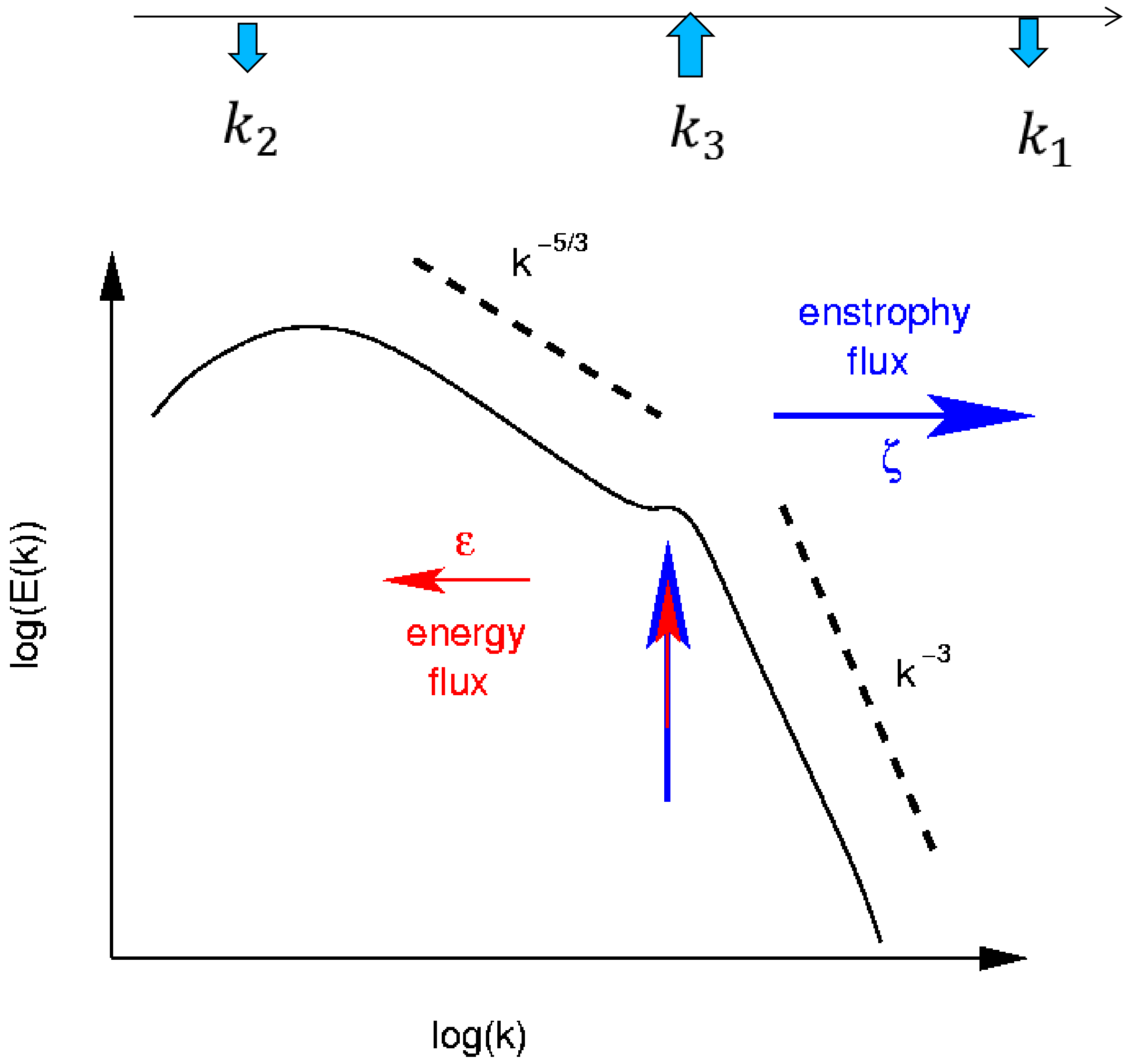
Double cascade in 2D incompressible fluid:
balance of energy v_k^2 and enstrophy $\omega_k^2 = k^2 v^2$

$$v_1^2 = v_2^2 + v_3^2$$

$$k_1^2 v_1^2 = k_2^2 v_2^2 + k_3^2 v_3^2$$

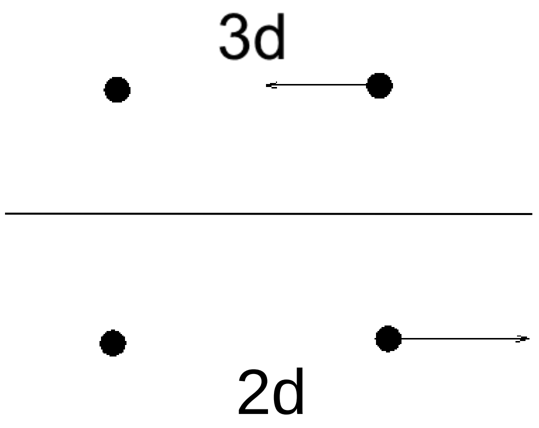
$$v_2^2 = v_1^2 \frac{k_3^2 - k_1^2}{k_3^2 - k_2^2} \approx v_1^2$$

$$v_3^2 = v_1^2 \frac{k_1^2 - k_2^2}{k_3^2 - k_2^2} \approx v_1^2 \frac{k_1^2}{k_3^2}$$



Two-point manifestation of the energy cascades

$$\nabla \langle \mathbf{u} |u|^2 \rangle = -4\epsilon$$



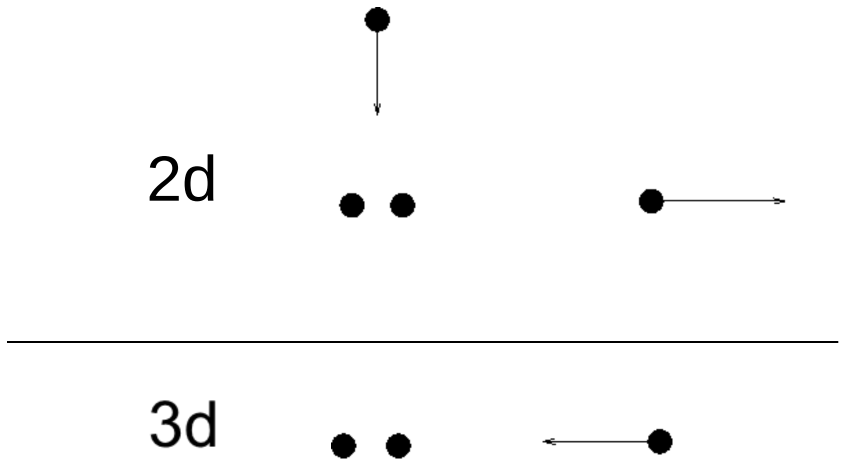
u is the velocity difference

ϵ is positive/negative for direct/inverse cascade

Three-point manifestation of the energy cascade

$$-2\langle v^i(r_1)v^i(r_2)v^k(r_3)\rangle = \langle |v(r_1) - v(r_2)|^2 v^k(r_3)\rangle$$
$$\simeq \bar{\epsilon}[r_{23}^k(4\cos^2\theta - 3) + (3d-4)r_{12}^k \cos\theta](r_{12}/r_{13})^{4/3}$$

the projection on the long direction
is proportional to $d\cos^2\theta - 1$



Operator product expansion and multi-point correlations in turbulent energy cascades
Falkovich & Zamolodchikov, *JPhysA* **48** (2015)

Two-particle manifestations of the cascade and irreversibility

Hypothetical *long-time* Richardson law

$$\langle R^2(t)\rangle \simeq |\epsilon| t^3$$

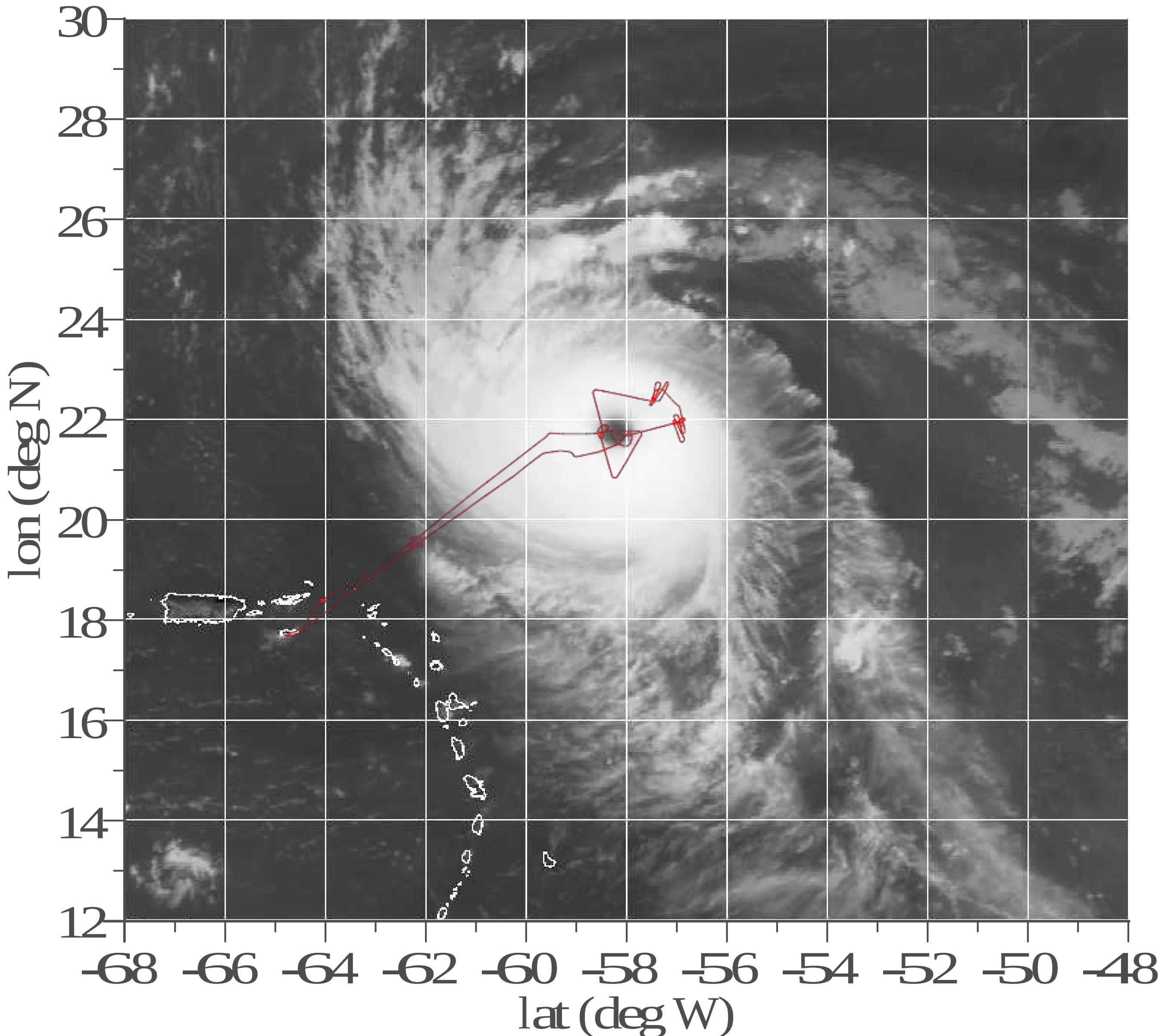
Exact *short-time* laws for 2d and 3d

Never experimentally observed

$$\left\langle \left[\frac{R_0}{R(t)}\right]^{2/3} \right\rangle - 1 = \frac{2\epsilon t^3}{27 R_0^2}$$

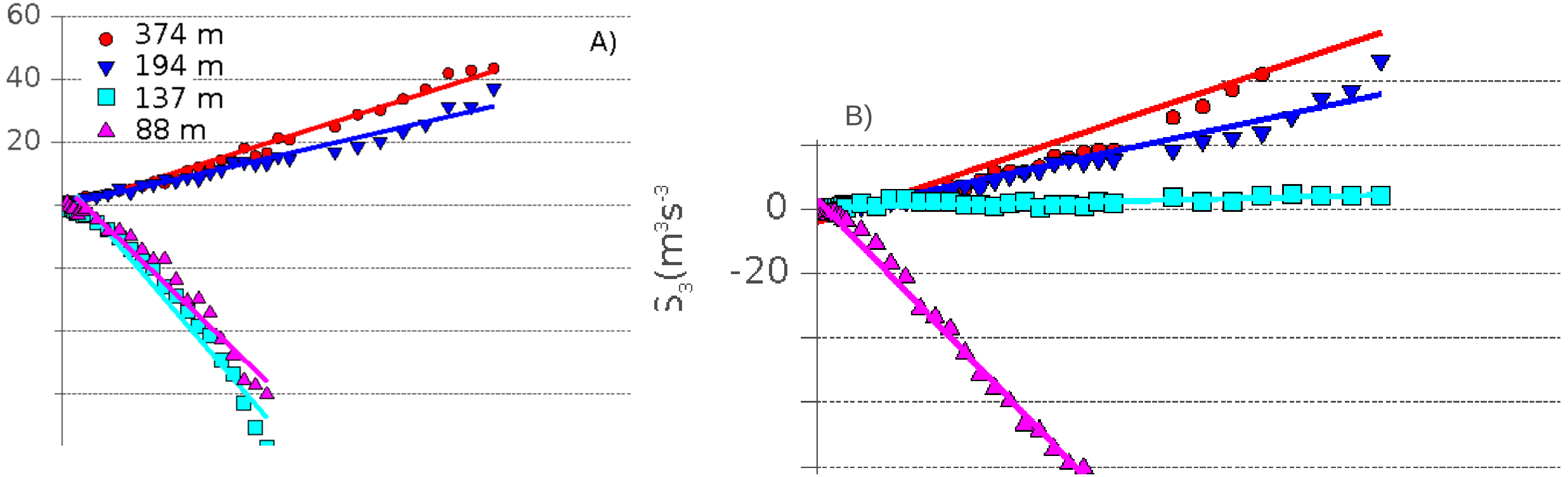
$$\left\langle \left[\frac{R_0}{R(t)}\right]^{5/3} \right\rangle - 1 = \frac{14\epsilon t^3}{81 R_0^2}$$

Fluid layers, rotating or magnetized systems – all have inverse cascades.



Three- to two-dimensional turbulence transition in the hurricane boundary layer
D. Byrne and A. Zhang, 2013

A transition from 3d to 2d turbulence from in-situ aircraft measurements in the hurricane boundary layer



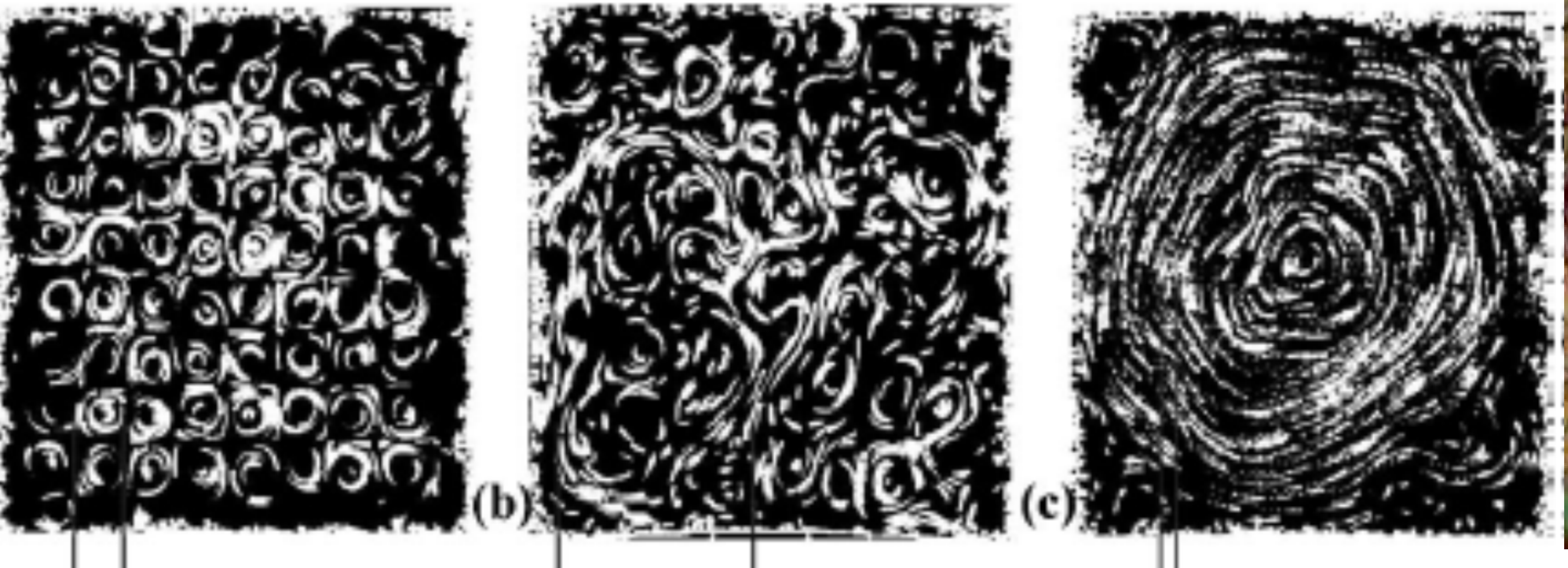
Third order structure function of horizontal velocities for different flight-leg heights in hurricane A) Isabel and B) Fabian.

These results represent the first measurement of the 2D upscale energy flux in the atmosphere and also the first to characterize the transition from 3D to 2D. It is shown that the large-scale parent vortex may gain energy directly from small-scales in tropical cyclones.

Fluid layers, rotating or magnetized systems – all have inverse cascades.

Creating larger objects out of small-scale turbulence is a form of self-organization. This is particularly visible when it creates a system-size flow (condensate).

Development of magnetic-driven turbulence in an electrolyte layer

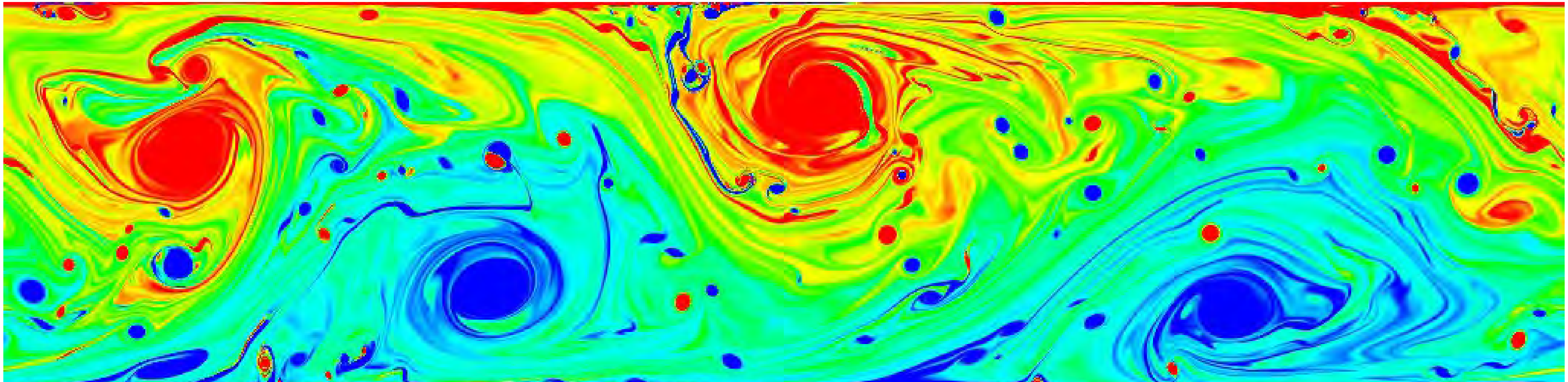


Soap-film flow

Fluid layers, rotating or magnetized systems – all have inverse cascades.

Creating larger objects out of small-scale turbulence is self-organization. This is particularly visible when it creates a system-size flow (condensate). **That lowers entropy**. Any far-from-equilibrium state must generate entropy. Balance of entropy production and absorption has not been explored experimentally.

2D pipe flow at $Re \sim 10^6$



Beyond cascades: information-theory perspective

1 How much the entropy of a turbulent state is lower that the entropy *S* of an equilibrium state with the same energy?

2 Which particular correlations are responsible for the entropy deficit. In other words: **Where the information about turbulence is encoded?**

3 Any non-equilibrium state produces *S*. To sustain turbulence, an external action must absorb *S* i.e. generate information. Where are production and extraction concentrated in k-space and r-space?

What are the **large scale flows** created by inverse cascades in different domains?

Going all the way to the **lowest available wavenumber** – is that the guiding principle? **Marginal stability** – another possible principle.

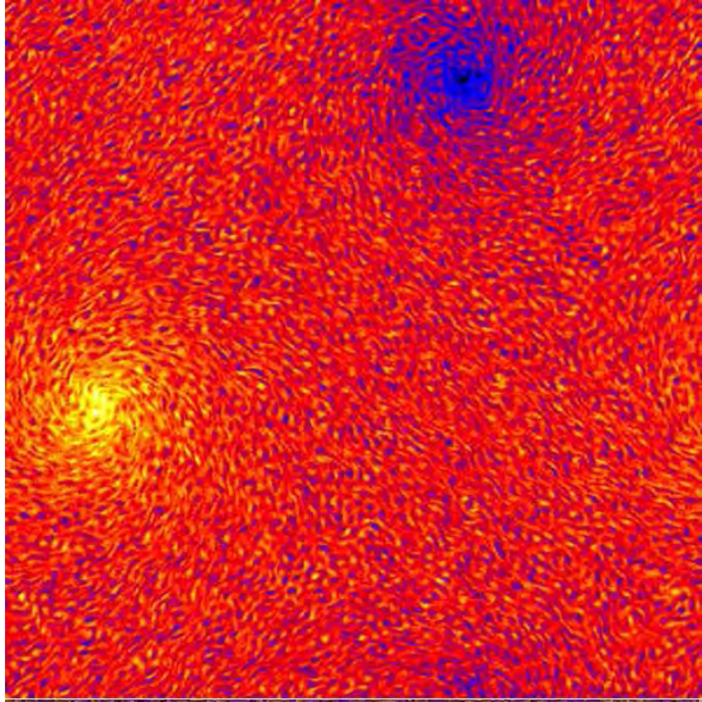
- **Box** with walls and aspect ratio r of order unity – central vortex and four counter-rotating corner vortices. What for $r \gg 1$?

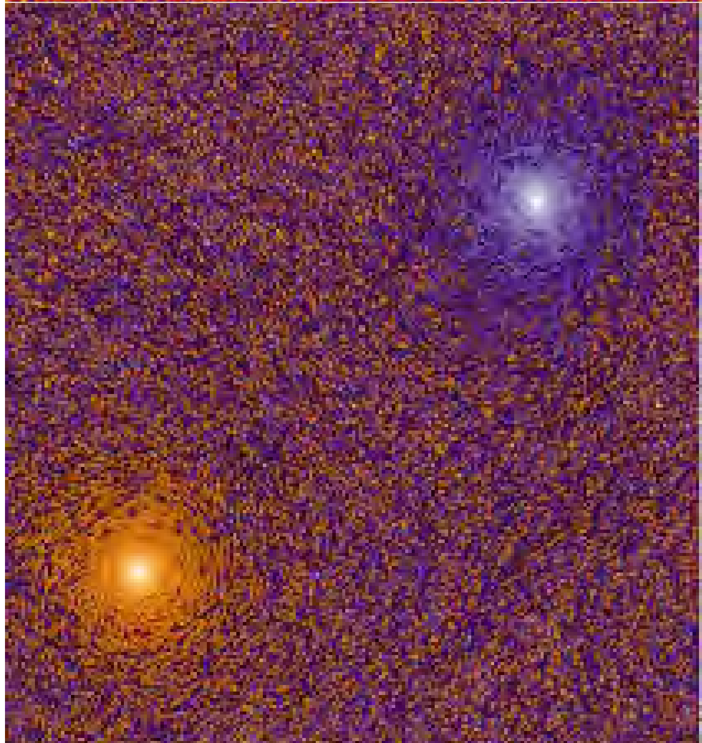
- **Sphere**, rotating and non-rotating. Expectation: a flow around a sphere. What meridional profile?

- **Torus**. Expectation: lowest wavenumber corresponds to a flow going around in short and modulated along a long direction.

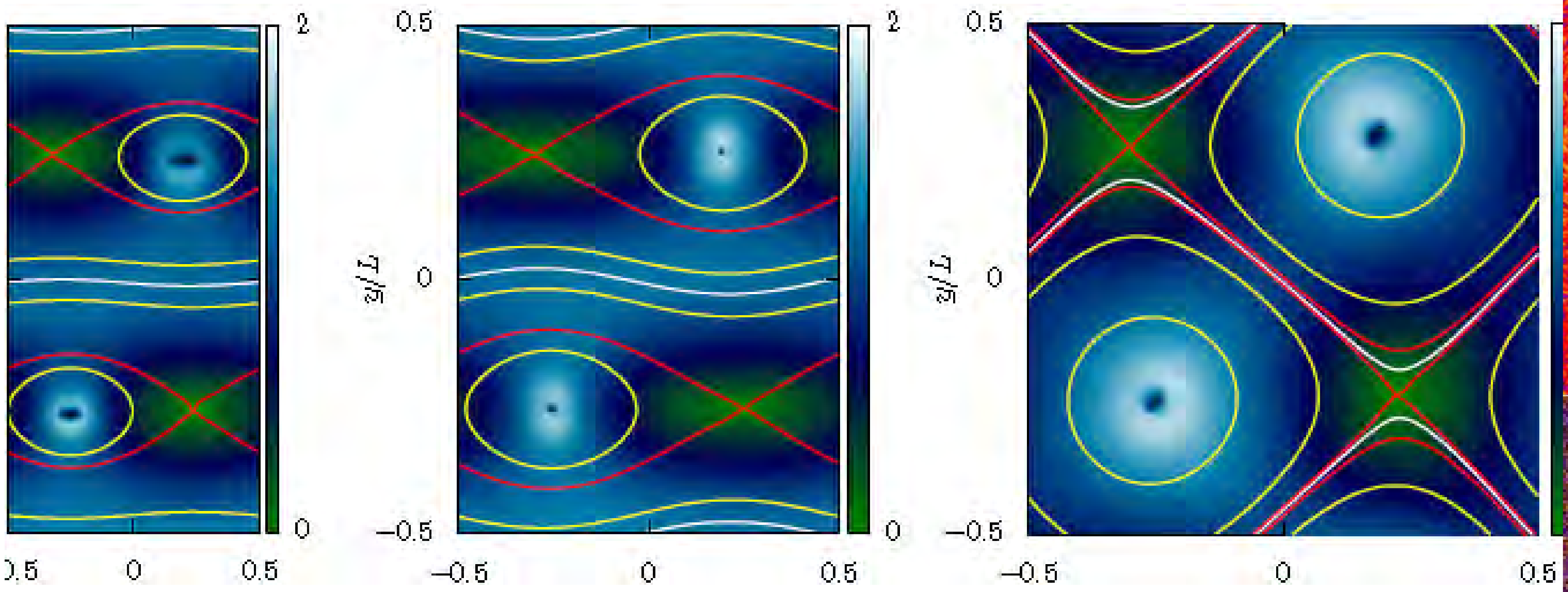
In a square box, the jets can be directed along either side and then one may expect a superposition of two sets of jets, which would look like a vortex dipole

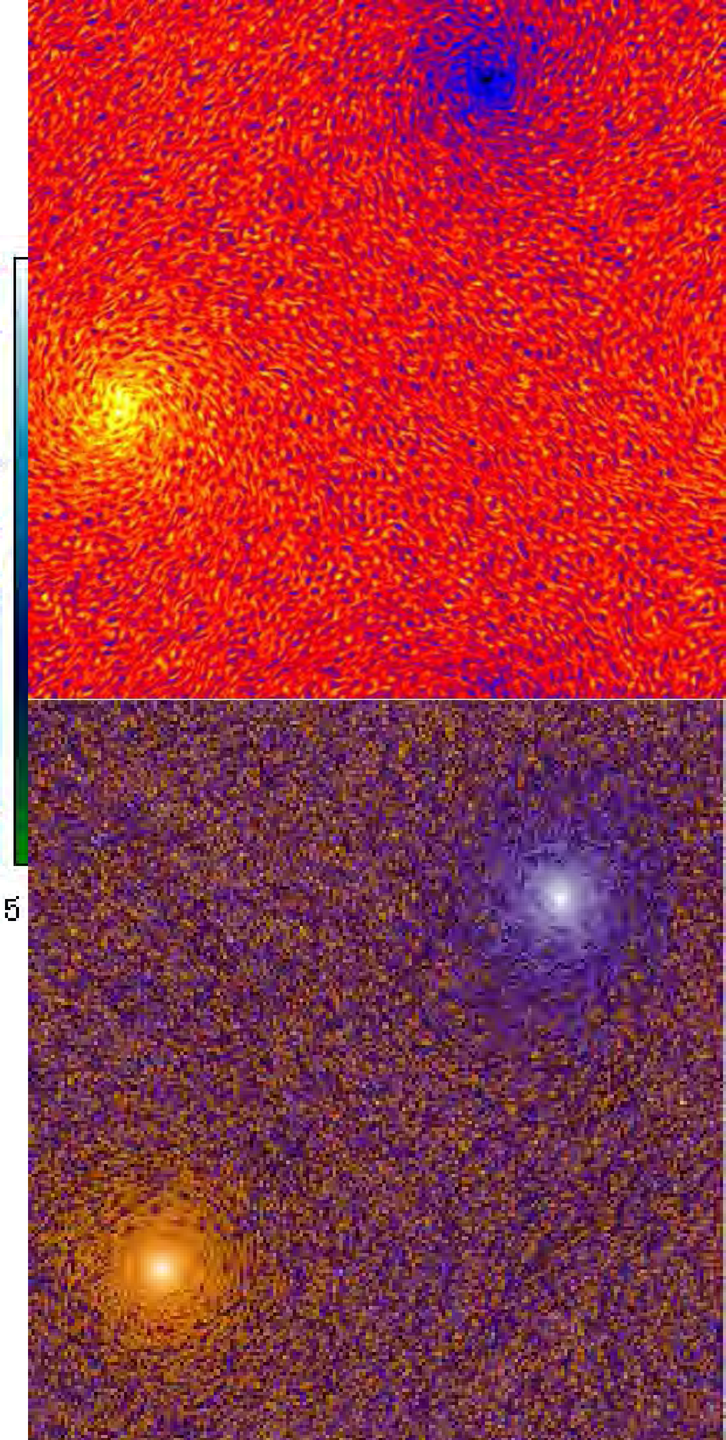
Inverse cascade on a torus





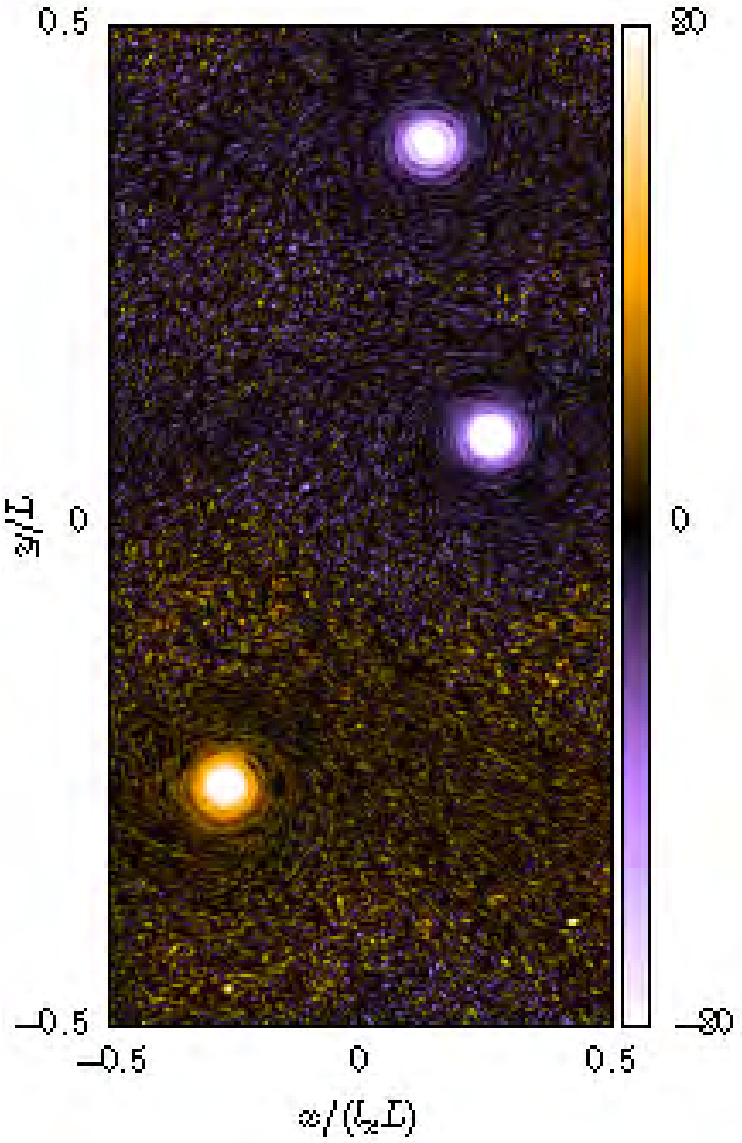
Inverse cascade on a torus: jets or vortices?

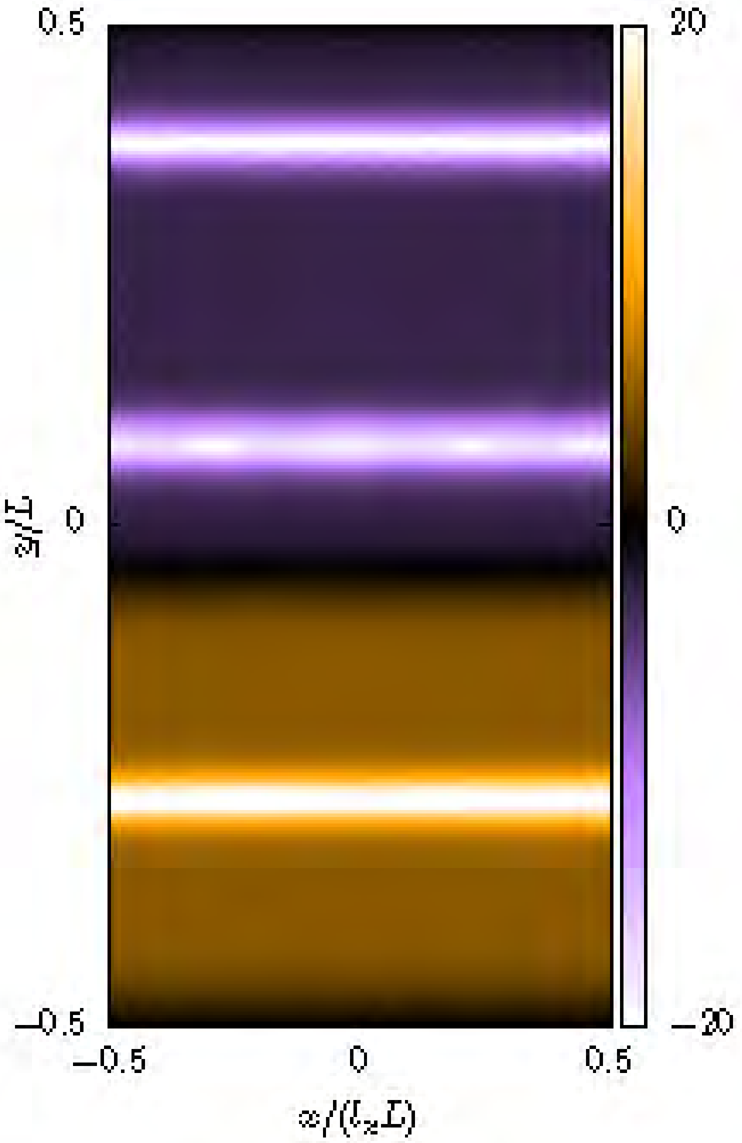




Velocity modulus averaged over short times (mean flow)
Vortices are effectively pinned. Different aspect ratios.

Even lower friction





Snapshot and long-time average

How compressibility affects the two-cascade picture?

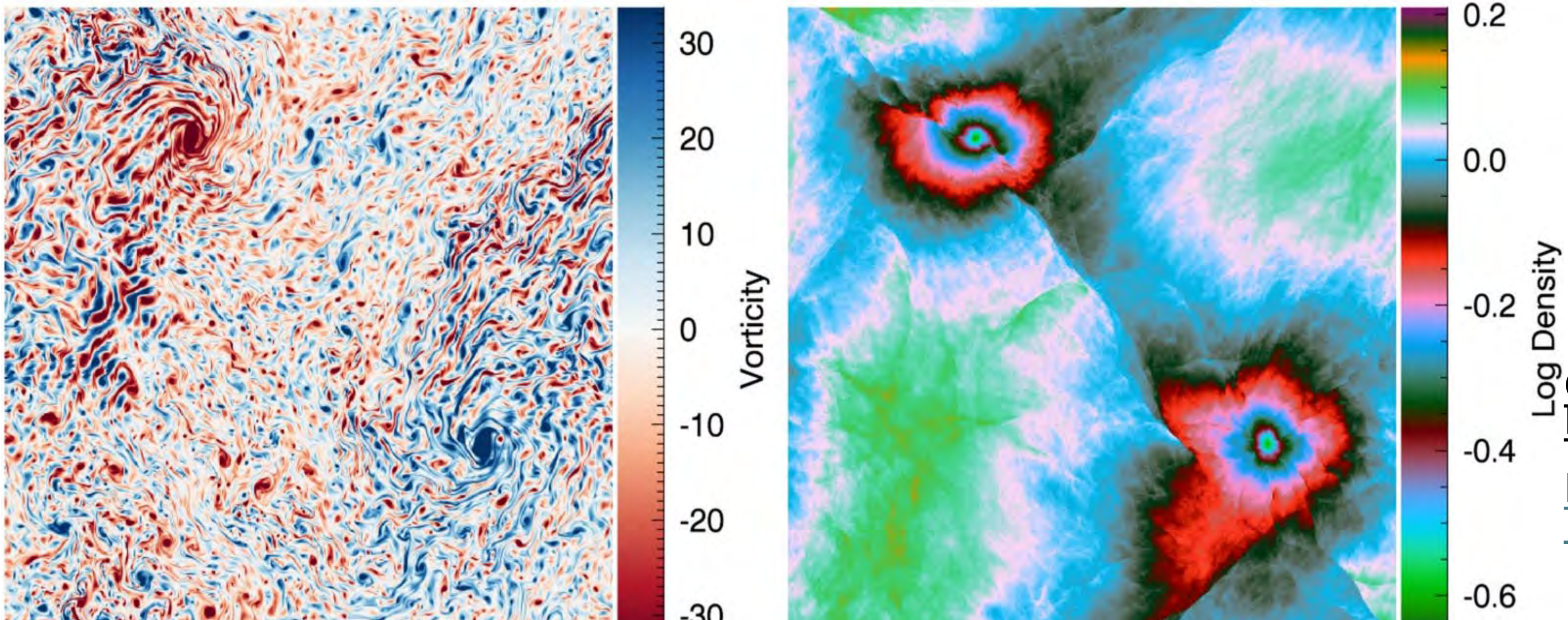
Large-scale turbulence in fluid layers consists of planar vortices and acoustic-like surface waves.

Solenoidal pumping at intermediate scales generates both direct and inverse energy cascades starting from the pumping scale.

The **inverse** energy cascade has the Kolmogorov scaling $\delta v \approx (\epsilon r)^{1/3}$.
The current Mach number $\delta v/c$ increases with the scale r.
That leads to the generation of sound which triggers a **direct** energy cascade.

Vortices and shocks provide for a flux loop in 2D compressible turbulence

Separately, acoustic turbulence produces a direct energy cascade, while solenoidal planar turbulence makes an inverse cascade. An iverse cascade of kinetic energy of vortices up to a scale *l* where typical velocity reaches the speed of sound; that creates shock waves which provide for a compensating direct cascade of potential energy. When the box is smaller than *l*, the steady state contains a system-size pair of long-living vortices connected by a system of shocks(see Figure). Turbulence in fluid layers processes energy **via a loop**: most energy first goes to large scales via vortices and then is transported by waves to small-scale dissipation.



GF, Kritsuk, Phys Rev Fluids 2017

More on flux loops: Alexakis, Biferale, Phys Rep 2018

State of the theory and challenges for experiments.
Personal view of a physicist.

General motifs.

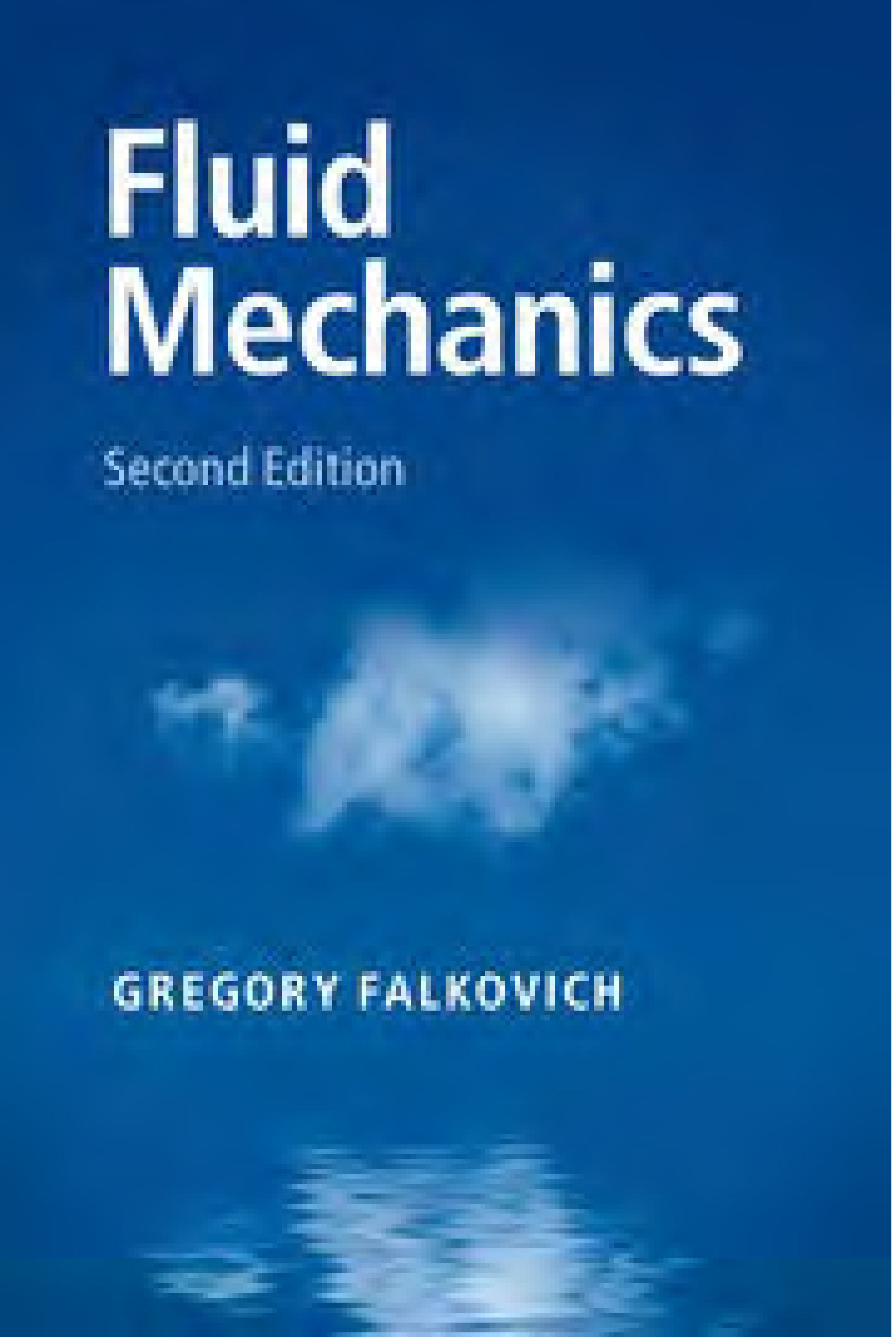
- Bird view: universality classes of developed turbulence.
- **Cascade**. Underappreciated and unexplored differences between direct and inverse cascades. Information perspective.
- Diminishing power of the flux laws.
- Beyond flux laws and spectra.

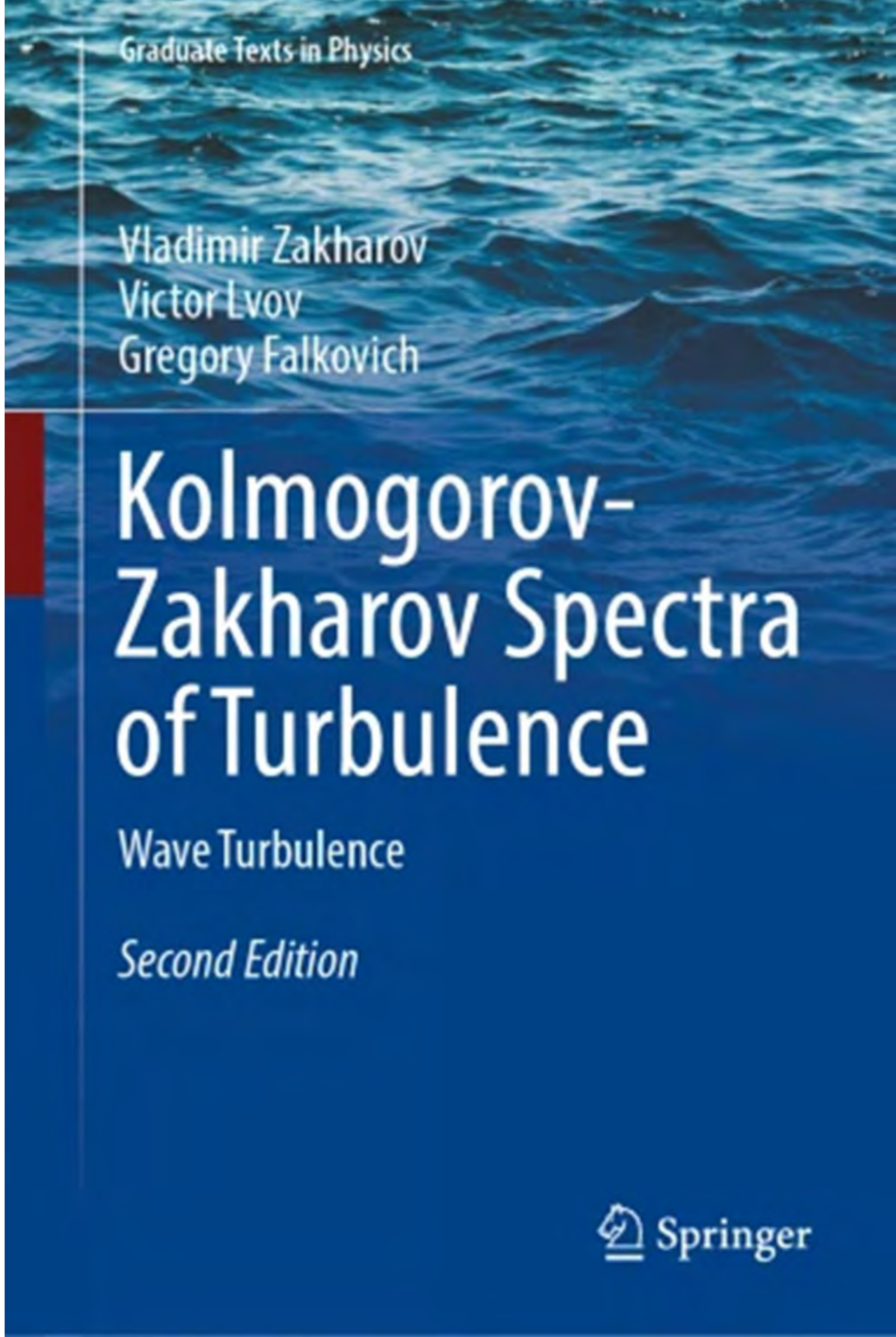
to be continued tomorrow

Turbulence of waves, particles and quasiparticles

Anything that propagates.

The law of propagation is given by $\omega(k)$


Fluid Mechanics
Second Edition
GREGORY FALKOVICH


Graduate Texts in Physics
Vladimir Zakharov
Victor Lvov
Gregory Falkovich
Kolmogorov-Zakharov Spectra of Turbulence
Wave Turbulence
Second Edition
Springer

$$H_2 = \sum_k \omega_k |a_k|^2$$

$$\mathcal{H}_3 = \int \Big[\big(V_{123} a_1^* a_2 a_3 + \text{c.c.}\big) \delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) + \\ + \big(U_{123} a_1^* a_2^* a_3^* + \text{c.c.}\big) \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \Big] \, d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3$$

$$i \frac{\partial c(\boldsymbol{k}, t)}{\partial t} - \omega_k c(\boldsymbol{k}, t) = \int \Big[\frac{1}{2} V_{k12} c_1 c_2 \delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2) \\ + V_{1k2}^* c_1 c_2^* \delta(\boldsymbol{k}_1 - \boldsymbol{k} - \boldsymbol{k}_2) \Big] \, d\boldsymbol{k}_1 d\boldsymbol{k}_2$$

Weak turbulence theory in a nutshell

$$\frac{\partial n(\boldsymbol{k},t)}{\partial t} = \mathrm{Im}\left\{ \int \left[V_{k12} J_{k12} \delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2) - 2V_{1k2} J_{1k2} \delta(\boldsymbol{k}_1-\boldsymbol{k}-\boldsymbol{k}_2) \right] d\boldsymbol{k}_1 d\boldsymbol{k}_2 \right\}$$

$$\left[i\frac{\partial}{\partial t} + (\omega_1-\omega_2-\omega_3) \right] J_{123}(t) = \int \left[-\frac{1}{2} V^*_{145} J_{4523} \delta(\boldsymbol{k}_1-\boldsymbol{k}_4-\boldsymbol{k}_5) + V^*_{425} J_{1534} \delta(\boldsymbol{k}_4-\boldsymbol{k}_2-\boldsymbol{k}_5) + V^*_{435} J_{1524} \delta(\boldsymbol{k}_4-\boldsymbol{k}_3-\boldsymbol{k}_5) \right] d\boldsymbol{k}_4 d\boldsymbol{k}_5$$

Weak turbulence theory in a nutshell: one-step closure.

$$\frac{\partial n(\boldsymbol{k},t)}{\partial t} = \mathrm{Im}\Big\{ \int \big[V_{k12} J_{k12} \delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2) - 2V_{1k2} J_{1k2} \delta(\boldsymbol{k}_1-\boldsymbol{k}-\boldsymbol{k}_2)\big]\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 \Big\}$$

$$\Big[i\frac{\partial}{\partial t} + (\omega_1-\omega_2-\omega_3)\Big] J_{123}(t) = \int \Big[-\frac{1}{2} V^*_{145} J_{4523} \delta(\boldsymbol{k}_1-\boldsymbol{k}_4-\boldsymbol{k}_5) + V^*_{425} J_{1534} \delta(\boldsymbol{k}_4-\boldsymbol{k}_2-\boldsymbol{k}_5) + V^*_{435} J_{1524} \delta(\boldsymbol{k}_4-\boldsymbol{k}_3-\boldsymbol{k}_5)\Big]\, d\boldsymbol{k}_4 d\boldsymbol{k}_5$$

$$\Big[i\frac{\partial}{\partial t} + (\omega_1-\omega_2-\omega_3)\Big] J^{(1)}_{123}(t) = V^*_{1,23}[n_1 n_3 + n_1 n_2 - n_2 n_3]$$

Weak turbulence theory in a nutshell: kinetic equation

$$\left[i\frac{\partial}{\partial t} + (\omega_1 - \omega_2 - \omega_3)\right] J^{(1)}_{123}(t)$$

$$= V^*_{1,23}[n_1 n_3 + n_1 n_2 - n_2 n_3]$$

$$J^{(1)}_{123}(t) = \frac{V^*_{123}(n_1 n_2 + n_1 n_3 - n_2 n_3)}{\omega_1 - \omega_2 - \omega_3 + i\delta}$$

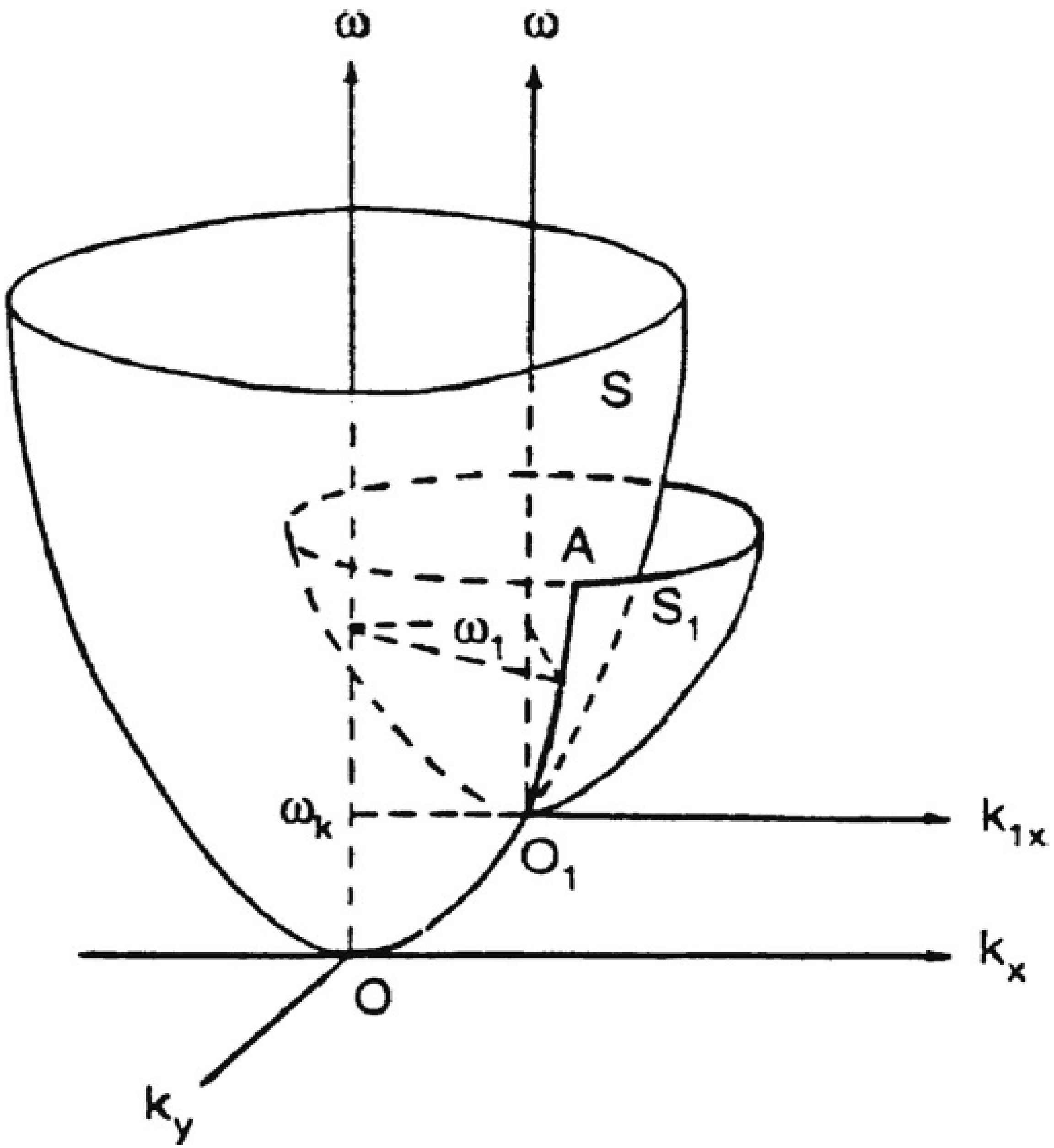
$$\frac{\partial n(\boldsymbol{k},t)}{\partial t} = \pi \int \Big[|V_{k12}|^2 f_{k12} \delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2)\delta(\omega_k - \omega_1 - \omega_2)$$

$$- 2|V_{1k2}|^2 f_{1k2} \delta(\boldsymbol{k}_1 - \boldsymbol{k} - \boldsymbol{k}_2)\delta(\omega_1 - \omega_k - \omega_2)\Big] d\boldsymbol{k}_1 d\boldsymbol{k}_2$$

$$f_{k12} = n_1 n_2 - n_k(n_1 + n_2)$$

3-wave resonance?
Concave or convex.

$$\omega(\boldsymbol{k}+\boldsymbol{k}_1)=\omega(\boldsymbol{k})+\omega(\boldsymbol{k}_1)$$



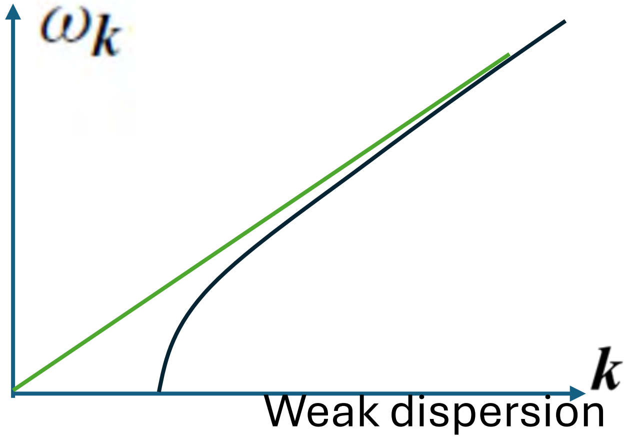
3-wave resonance? $\omega(\boldsymbol{k}+\boldsymbol{k}_1)=\omega(\boldsymbol{k})+\omega(\boldsymbol{k}_1)$

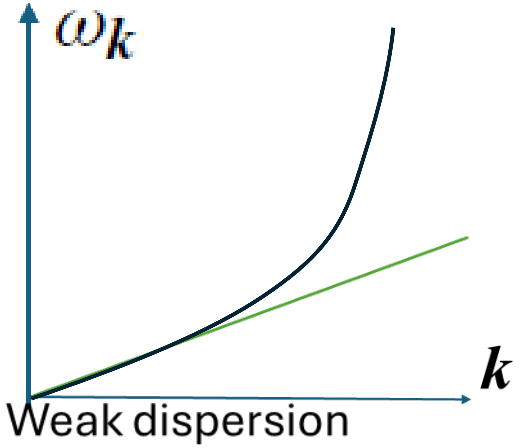
Concave or convex, long or short.

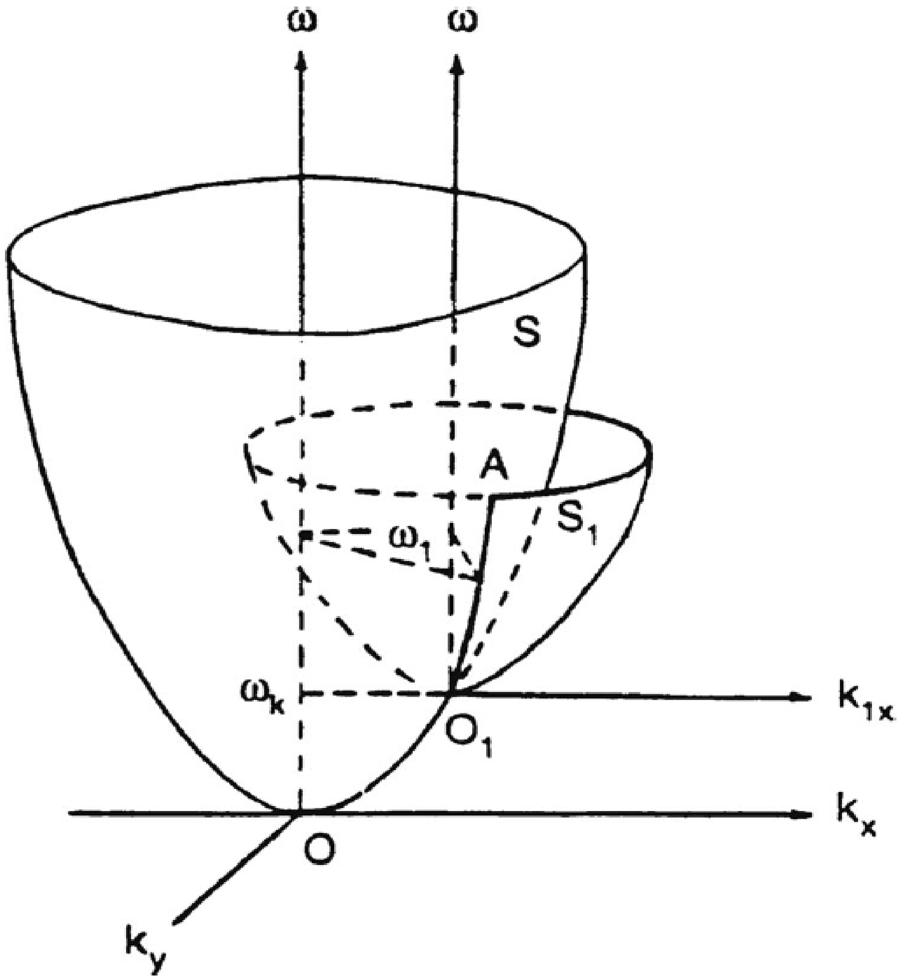
Weakly dispersive waves, short and long

$$\omega(k) = ck + \Omega(k)$$

$$\operatorname{sign}\left[\Omega(k)\frac{\partial}{\partial k}\frac{\Omega(k)}{ck}\right] > 0$$







Long waves with positive dispersion

$$\omega(k) = ck(1 + a^2k^2)$$

Three-Dimensional Acoustics with Positive Dispersion: Magnetic Sound and Phonons in Helium.

Two-Dimensional Acoustics with Positive Dispersion: Shallow-water gravity-capillary waves.

Over-condensate fluctuations, Bogolyubov spectrum

$$\omega(k) = ck\sqrt{1 + \xi^2k^2/2}$$

Acoustic Waves

$$\omega(k) = ck + \Omega(k)$$

- Weak turbulence: dispersion exceeds nonlinearity.
- Weak turbulence of dispersionless waves.
- Strong turbulence: nonlinearity exceeds dispersion.

$$i\frac{\partial c(\boldsymbol{k},t)}{\partial t} - \omega_k c(\boldsymbol{k},t) = \int \Big[\frac{1}{2}V_{k12}c_1c_2\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2) + V^*_{1k2}c_1c^*_2\delta(\boldsymbol{k}_1-\boldsymbol{k}-\boldsymbol{k}_2)\Big]d\boldsymbol{k}_1 d\boldsymbol{k}_2$$

$$\boxed{\omega(k) = ck(1+a^2k^2)}$$

$$|V(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)|^2 = bkk_1k_2$$

Kinetic equation:

$$\frac{\partial n(\boldsymbol{k},t)}{b\partial t} = \int dk_1 d\theta_1^2 d\varphi_1 kk_1^3(k-k_1)\Big\{\delta\big(3a^2kk_1(k-k_1) - \theta_1^2 kk_1/2(k-k_1)\big) \times \big[n(\boldsymbol{k}_1)n(\boldsymbol{k}-\boldsymbol{k}_1) - n(\boldsymbol{k})n(\boldsymbol{k}_1) - n(\boldsymbol{k})n(\boldsymbol{k}-\boldsymbol{k}_1)\big] - 2\delta\big(3a^2kk_1(k_1-k) - \theta_1^2 kk_1/2(k_1-k)\big) \times \big[n(\boldsymbol{k})n(\boldsymbol{k}_1-k) - n(\boldsymbol{k}_1)n(\boldsymbol{k}) - n(\boldsymbol{k}_1)n(\boldsymbol{k}_1-\boldsymbol{k})\big]\Big\}\,.$$

$$\frac{\partial n(k,t)}{\partial t} = 2^{d-1} b\pi (\sqrt{6}a)^{d-3} \Big\{ \int_0^k dk_1 [k_1(k-k_1)]^{d-1}$$

$$\times \big[n(k_1)n(k-k_1) - n(k)n(k_1) - n(k)n(k-k_1)\big]$$

$$-2\int_k^\infty dk_1 [k_1(k_1-k)]^{d-1} \big[n(k)n(k_1-k) - n(k_1)n(k)$$

$$- n(k_1)n(k_1-k)\big]\Big\} = I_d(k) \ .$$

Energy conservation

$$\varepsilon(\boldsymbol{k},t) = \omega(\boldsymbol{k})n(\boldsymbol{k},t)$$

$$\frac{\partial \varepsilon(\boldsymbol{k},t)}{\partial t} + \operatorname{div} \boldsymbol{p}(\boldsymbol{k},t) = 0$$

$$\frac{\partial n(k,t)}{\partial t} = 2^{d-1} b\pi(\sqrt{6}a)^{d-3}\Big\{ \int_0^k dk_1 [k_1(k-k_1)]^{d-1}$$

$$\times \left[n(k_1)n(k-k_1) - n(k)n(k_1) - n(k)n(k-k_1)\right]$$

$$- 2\int_k^\infty dk_1 [k_1(k_1-k)]^{d-1}\big[n(k)n(k_1-k) - n(k_1)n(k)$$

$$- n(k_1)n(k_1-k)\big]\Big\} = I_d(k)\ .$$

Energy cascade (Zakharov-Sagdeev spectrum)

$$P = P(k) = -\,2^{d-1}\pi \int_0^k \omega(k)k^{d-1} I_d(k)\, dk \ \propto\ k^{3d-2s_0}$$

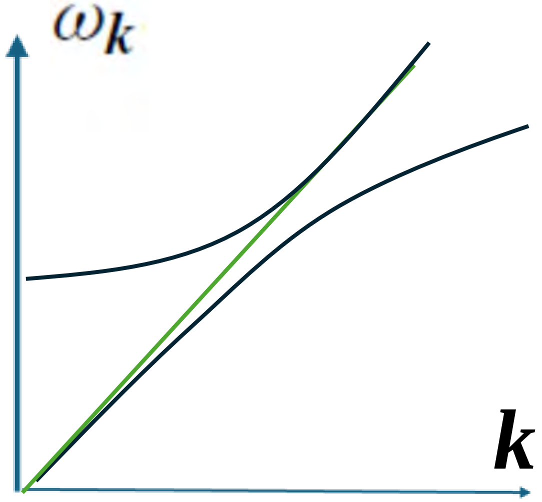
$$n(k) = \lambda P^{1/2} k^{-s_0}, \qquad s_0 = 3d/2\ .$$

Non-decay dispersive laws

$$\mathcal{H}_4 = \frac{1}{4}\int T_p c_1^* c_2^* c_3 c_4 \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\boldsymbol{k}_4$$

$$\frac{\partial n(\boldsymbol{k},t)}{\partial t} = \frac{\pi}{2}\int |T_{k123}|^2 f_{k123}\delta(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3) \times \delta(\omega_k + \omega_1 - \omega_2 - \omega_3)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 \,.$$

$$f_{k123} = n_2 n_3 (n_1 + n_k) - n_1 n_k (n_2 + n_3)$$



Two conservation laws

$$\frac{\partial \varepsilon(\boldsymbol{k},t)}{\partial t} + \operatorname{div} \boldsymbol{p}(\boldsymbol{k},t) = 0$$

$$\frac{\partial n(\boldsymbol{k},t)}{\partial t} + \operatorname{div} \boldsymbol{q}(\boldsymbol{k},t) = 0\,, \quad \operatorname{div} \boldsymbol{q}(\boldsymbol{k},t) = -I(\boldsymbol{k},t)$$

Problem with power counting

$$I(\boldsymbol{k}) = \int \left|T(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)\right|^2 \delta(\omega_k + \omega_1 - \omega_2 - \omega_3)\delta(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3)$$

$$\times n_k n_1 n_2 n_3 \left(n_k^{-1} + n_1^{-1} - n_2^{-1} - n_3^{-1}\right) d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3.$$

$$T_p = -\frac{U_{-1-212}U_{-3-434}}{\omega_3 + \omega_4 + \omega_{3+4}} + \frac{V^*_{1+212}V_{3+434}}{\omega_1 + \omega_2 - \omega_{1+2}}$$
$$- \frac{V^*_{131-3}V_{424-2}}{\omega_{4-2} + \omega_2 - \omega_4} - \frac{V^*_{242-4}V_{313-1}}{\omega_{3-1} + \omega_1 - \omega_3}$$
$$- \frac{V^*_{232-3}V_{414-1}}{\omega_{4-1} + \omega_1 - \omega_4} - \frac{V^*_{141-4}V_{323-2}}{\omega_{3-2} + \omega_2 - \omega_3}$$

$$\omega(\boldsymbol{k}) \pm \omega(\boldsymbol{k}_j) - \omega(\boldsymbol{k} \pm \boldsymbol{k}_j) \approx ck\frac{k_j\theta_j^2}{2|\boldsymbol{k} \pm \boldsymbol{k}_j|} + \Omega(k) \pm \Omega(k_j) - \Omega(k \pm k_j)$$

$\Omega\ (\lambda \boldsymbol{k}) = \lambda^{\beta}\ \Omega$

$$k_2' = \lambda_2 k = \lambda_2^2 k_2, \qquad k_1' = \lambda_2 k_3, \qquad k_3' = \lambda_2 k_1\ ,$$

$$\theta_2' = -\lambda_2^{(\beta-1)/2}\theta_2, \quad \theta_1' = \lambda_2^{(\beta-1)/2}(\theta_3 - \theta_2), \quad \theta_3' = \lambda_2^{(\beta-1)/2}(\theta_1 - \theta_2)$$

$$\begin{aligned} I(\boldsymbol{k}) = & \frac{k^{\nu}}{2} \int |T(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)|^2 \delta\Big[\Omega(k) + \Omega(k_1) - \Omega(k_2) - \Omega(k_3) \\ & + \frac{c}{2}(k_1\theta_1^2 - k_2\theta_2^2 - k_3\theta_3^2)\Big]\delta(k + k_1 - k_2 - k_3) \\ & \times \delta(k_1\theta_1\boldsymbol{\kappa}_1 - k_2\theta_2\boldsymbol{\kappa}_2 - k_3\theta_3\boldsymbol{\kappa}_3) n_k n_1 n_2 n_3 \\ & \times \left(n_k^{-1} + n_1^{-1} - n_2^{-1} - n_3^{-1}\right)\left(k^{-\nu} + k_1^{-\nu} - k_2^{-\nu} - k_3^{-\nu}\right) d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 \end{aligned}$$

$m = 3/2$

$$\nu = -3s + 4m - 3\beta - 1 - \frac{(\beta+1)(d-1)}{2} + 3\frac{(\beta-1)(d-1)}{2} + 4d$$

Two Kolmogorov-Zakharov cascade solutions

Direct energy cascade $n_k \sim P^{1/3} k^{-s_P}$ $s_P = (8 + 2d + \beta d - 4\beta)/3$

Inverse action cascade $n_k \sim Q^{1/3} k^{Q-s_Q}$ $s_Q = (7 + 2d + \beta d - 4\beta)/3$

For the long-wave limit in 3d

$$\Omega(k) = ca^2k^3, \quad \beta = 3$$

$$n_k \propto k^{-s_Q} = k^{-10/3}$$

$$n_k \propto k^{-s_P} = k^{-11/3}$$

The two-dimensional case (shallow-water gravitational waves) requires special consideration. This is because, for weakly dispersive waves propagating in a narrow angular cone, the dynamic equations may be reduced to the so-called Kadomtsev–Petviashvili (KP) equation [see ((5.63)–(5.85)) and (5.130) below, and also [29]]:

$$\frac{\partial}{\partial x}\left[\frac{\partial \eta}{\partial t} + \sqrt{gh_0}\left(\frac{\partial \eta}{\partial x} + \frac{h_0^2}{6}\frac{\partial^3 \eta}{\partial x^3} + \eta\frac{\partial \eta}{\partial x}\right)\right] = -\frac{1}{2}\sqrt{gh_0}\frac{\partial^2 \eta}{\partial y^2}$$

$$T_p = -\frac{U_{-1-212}U_{-3-434}}{\omega_3 + \omega_4 + \omega_{3+4}} + \frac{V^*_{1+212}V_{3+434}}{\omega_1 + \omega_2 - \omega_{1+2}} - \frac{V^*_{131-3}V_{424-2}}{\omega_{4-2} + \omega_2 - \omega_4} - \frac{V^*_{242-4}V_{313-1}}{\omega_{3-1} + \omega_1 - \omega_3} - \frac{V^*_{232-3}V_{414-1}}{\omega_{4-1} + \omega_1 - \omega_4} - \frac{V^*_{141-4}V_{323-2}}{\omega_{3-2} + \omega_2 - \omega_3}$$

The KP equation andt he respective kinetic equation have an infinite set of conservation laws, as well as. It results in the interaction coefficient T vanishing on the resonant surface. Thus, in the two-dimensional case, one should take into account the higher-order terms of the Hamiltonian expansion in terms of the small parameter kh_0. As a result, the coefficient of four-wave interaction will gain the h_0 factor, increasing its scaling exponent by unity. Therefore, the Kolmogorov exponents are:

$$s_P = 10/3, \quad s_Q = 3$$

Structural instability of weak turbulence.

Since waves interact resonantly only within narrow cones, what prevents an isotropic spectrum from breaking into a set of jets?

$$n(k) = \lambda\sqrt{\frac{P}{k^9}}$$

Let us discuss the stability of the isotropic spectrum with respect to anisotropic perturbations. The stability problem is nontrivial due to its proximity to the degenerate case, $\omega_k = ck$, where the interactions are resonant, $\omega(\boldsymbol{k}_1 + \boldsymbol{k}_2) = \omega(\boldsymbol{k}_1) + \omega(\boldsymbol{k}_2)$, only for waves propagating along a line, and waves at different angles do not interact. In the limit $\omega_k \to ck$, in addition to the energy $E = \int kn_{\boldsymbol{k}}\, d\boldsymbol{k}$, and the components of the momentum $\boldsymbol{\Pi}_i = \int \cos\theta_i\, kn_{\boldsymbol{k}}\, d\boldsymbol{k}$, the kinetic equation has an infinite set of integrals of motion $\int f(\zeta)kn_{\boldsymbol{k}}\, d\boldsymbol{k}$, where $f(\zeta)$ is an arbitrary function of the angular variables $\zeta = (\theta, \phi)$

Structural instability of weak turbulence.

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$$n(k) = \lambda \sqrt{\frac{P}{k^9}}$$

We model weak dispersion as $\omega_k = ck^{1+\epsilon}$ then the interaction angle is $\theta_{int} \sim \epsilon^{1/2}$

For $l \ll \epsilon^{-1/2}$,

we have integrals of motion and steady solutions of the linearized kinetic equation carrying the fluxes of these integrals

$$n(k, \theta) = \lambda P^{1/2} k^{-9/2} \left[1 + \sum_{l=1}^{L} c_l P_l(\cos\theta) k^{\epsilon l(l+1)/2} \right]$$

The higher the number l of the angular mode, the faster its contribution to the stationary spectrum increases with k . Thus, a small anisotropy of a source located at small k must lead to an essentially anisotropic spectrum at large k.

Structural instability of weak turbulence.

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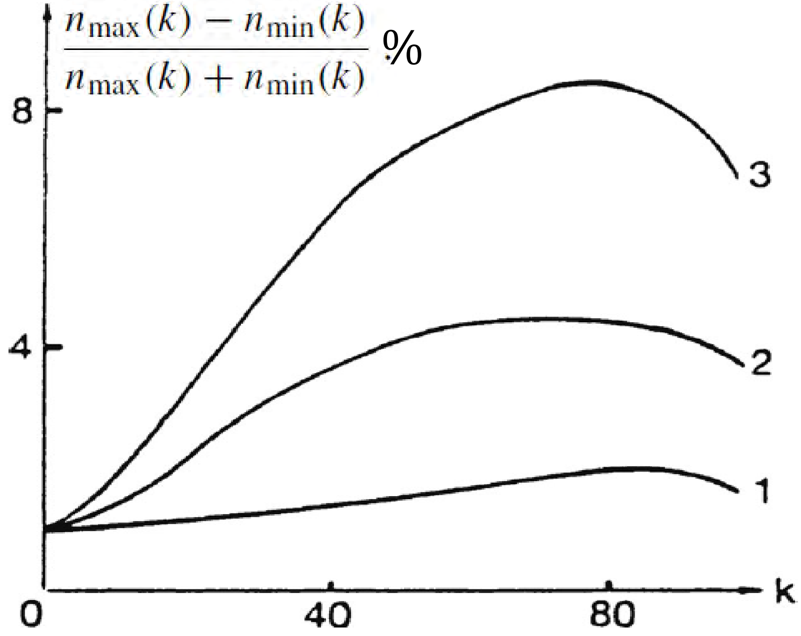
A weakly anisotropic spectrum is getting narrower in angle.
A strongly anisotropic spectrum?



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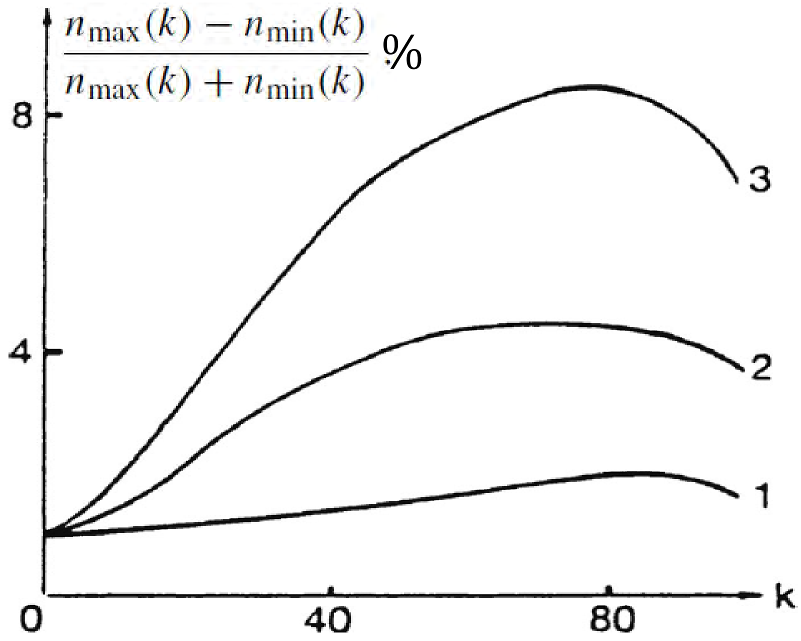
A weakly anisotropic spectrum is getting narrower in angle.
A strongly anisotropic spectrum widens along the cascade.



Far along the cascade, only the fluxes of the true (nonlinear) integrals of motion determine the spectrum

$$n_k = P^{1/2} k^{-9/2} F(y), \quad y = \frac{(\boldsymbol{R}\boldsymbol{k})\omega(k)}{Pk^2}$$

Structural instability of weak turbulence.

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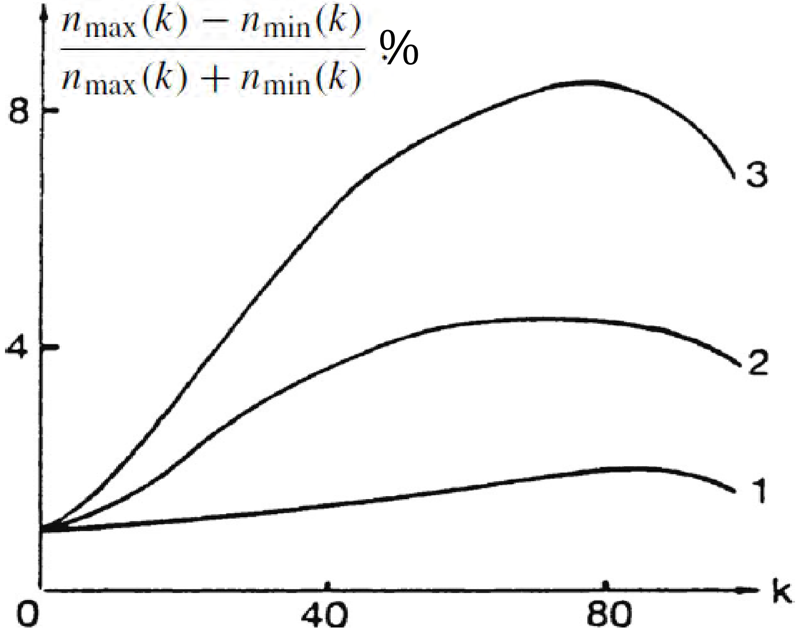

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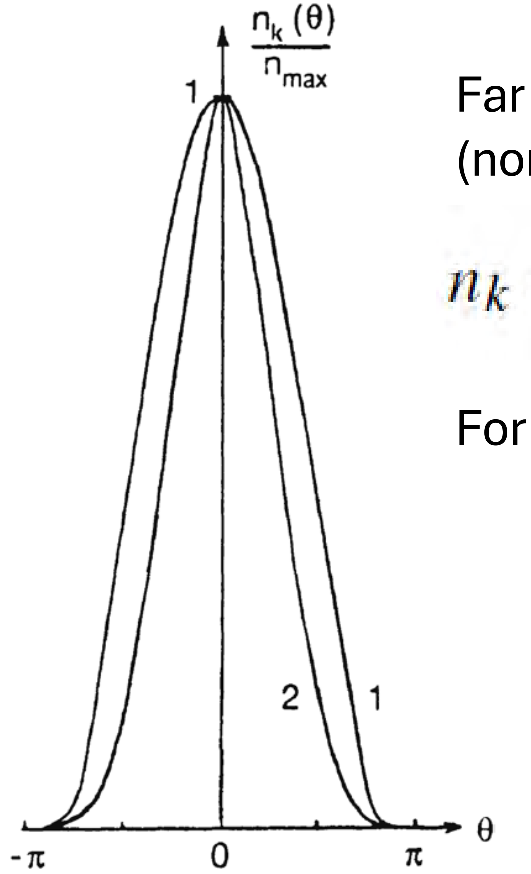


Far along the cascade, only the fluxes of the true (nonlinear) integrals of motion determine the spectrum

$$n_k = P^{1/2} k^{-9/2} F(y), \quad y = \frac{(Rk)\omega(k)}{P^{1/2}}$$

For acoustic waves one finds explicitly

$$F(y) = \begin{cases} \sqrt{C_1 y + C_2} & \text{for} \quad y > -C_2/C_1 \\ 0 & \text{for} \quad y < -C_2/C_1 \end{cases}$$

$$n_k = k^{-9/2} \left(\frac{R\omega_k \cos\theta}{k} + P \right)^{1/2}$$

$$\omega_k = ck^{1+\epsilon}$$

$$n_k = k^{-9/2} \left(\frac{R\omega_k \cos\theta}{k} - P \right)^{1/2}$$

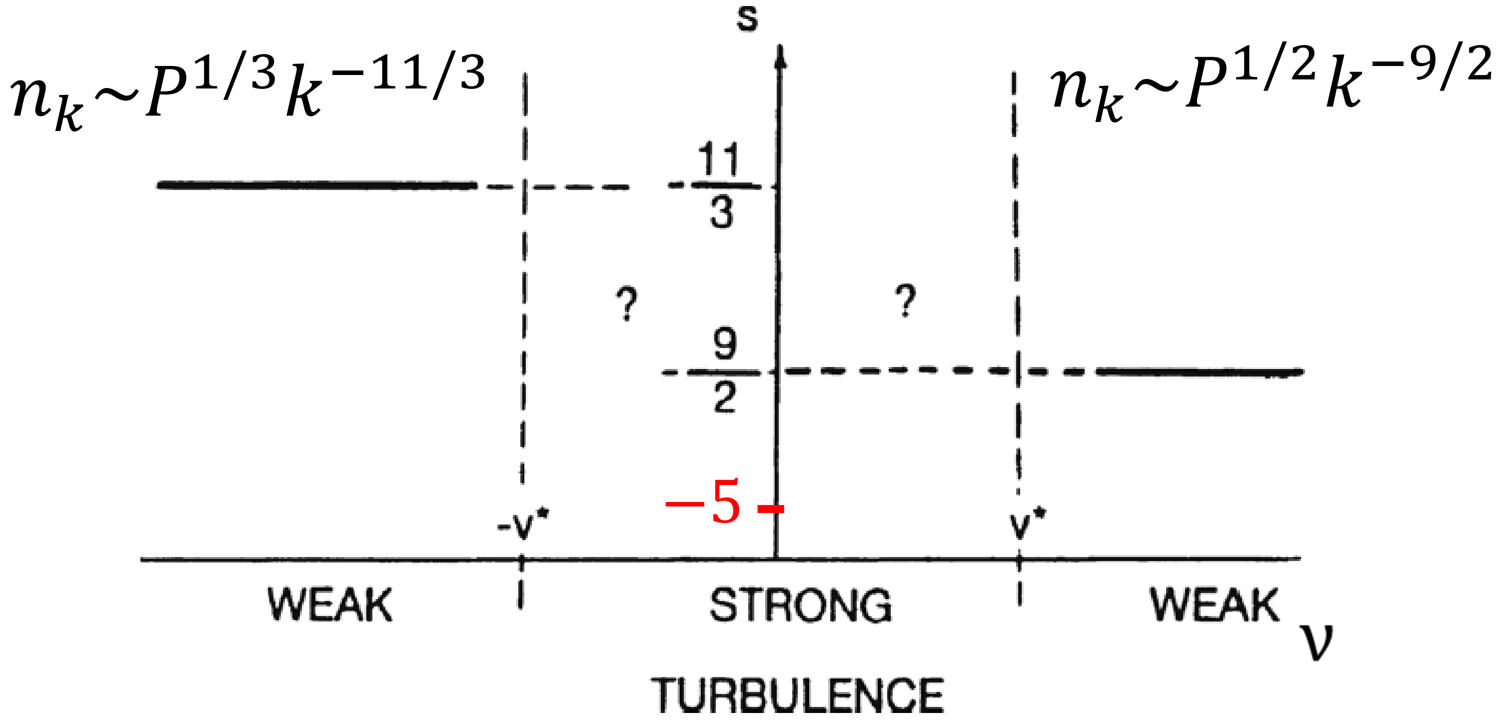
$$n_k \to k^{(\epsilon-9)/2} (R\cos\theta)^{1/2}$$

Strong turbulence of dispersionless waves

Direct energy cascades in 3d

For the long-wave limit

$$\omega_k^2 = c^2 k^2 + \nu k^4$$



Every shock front contributes $v_k \propto k^{-1}$ into the Fourier image of the velocity field, which gives the energy spectral density $|v_k|^2 \propto k^{-2}$, which gives Kadomtsev-Petviashvili spectrum

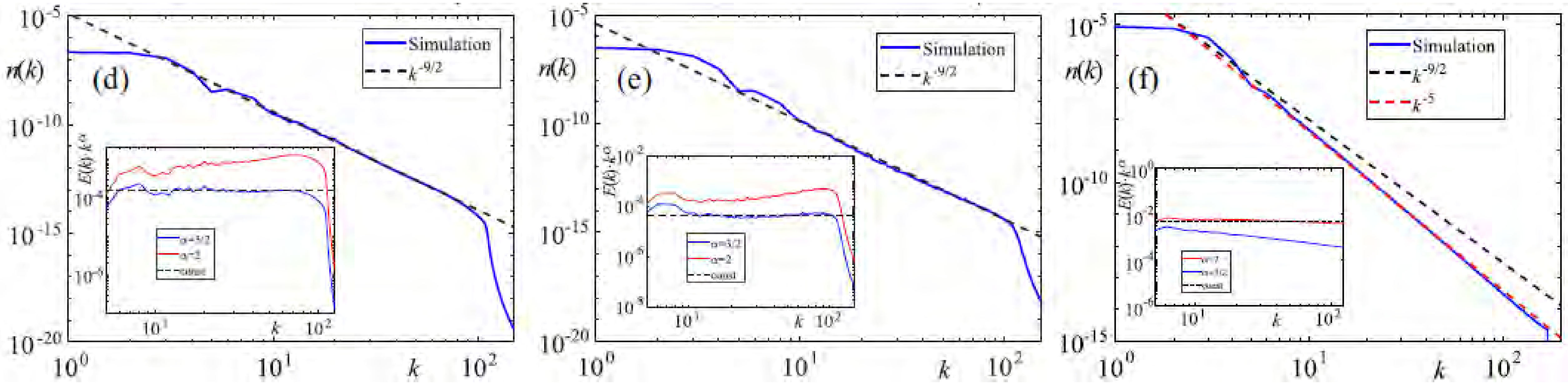
$$\boldsymbol{n_k \propto k^{-5}}.$$

Nobody ever derived this as a stationary solution of any equation.

Dispersionless weak turbulence

Kochurin, Kuznetsov, PHYSICAL REVIEW LETTERS 133, 207201 (2024)

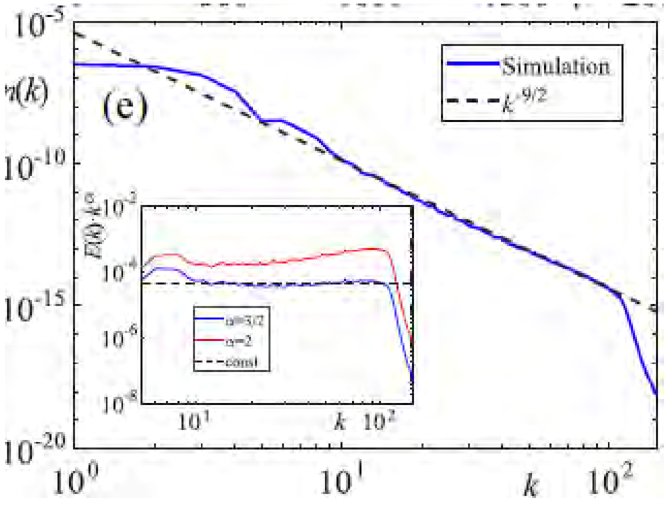
Direct numerical simulation of three-dimensional acoustic turbulence has been performed for both weak and strong regimes. Within the weak turbulence, we demonstrate the existence of the Zakharov-Sagdeev spectrum $\propto k^{-3/2}$ not only for weak dispersion but in the nondispersion (ND) case as well. Such spectra in the k space are accompanied by jets in the form of narrow cones. These distributions are realized due to small nonlinearity compared with both dispersion or diffraction. Increasing pumping in the ND case due to dominant nonlinear effects leads to the formation of shocks. As a result, the acoustic turbulence turns into an ensemble of random shocks with the Kadomtsev-Petviashvili spectrum.

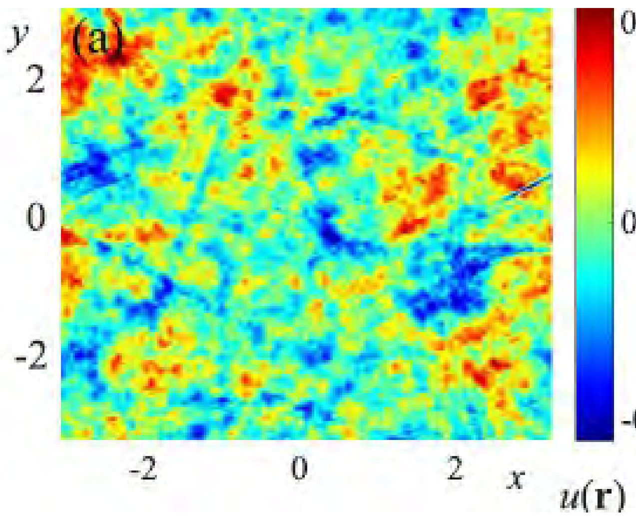


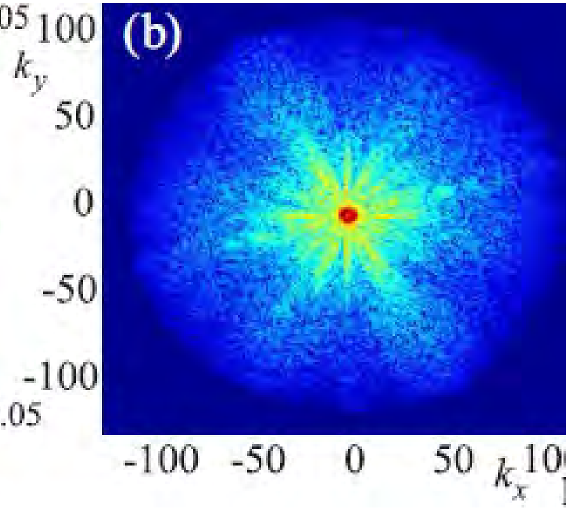
a=0.025 | *a* =0 | *a* =0

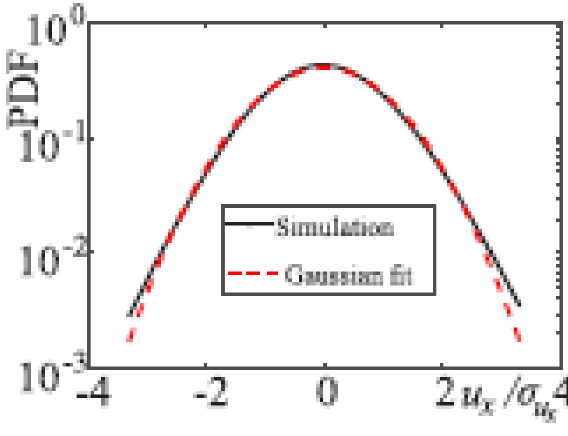
$a = 0$

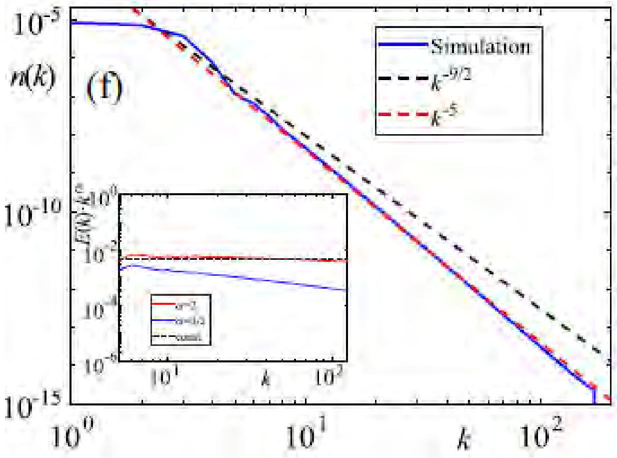
Dispersionless weak and strong turbulence

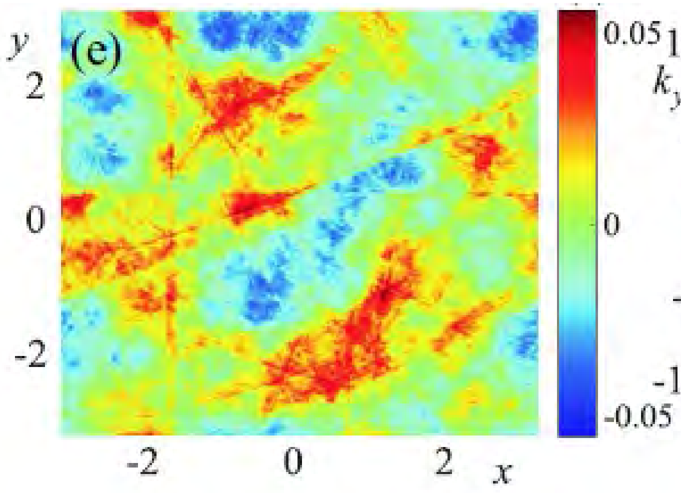


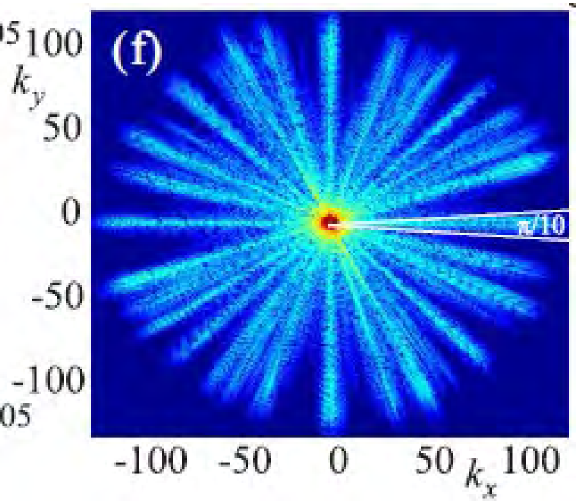


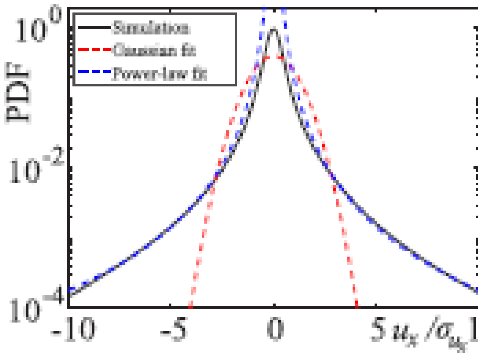












Beyond cascades: information-theory perspective

1 How much the entropy of a turbulent state is lower that the entropy *S* of an equilibrium state with the same energy?

2 Which particular correlations are responsible for the entropy deficit. In other words: **Where the information about turbulence is encoded?**

3 Any non-equilibrium state produces *S*. To sustain turbulence, an external action must absorb *S* i.e. generate information. Where in k-space production and extraction are for different cascades, particularly for two-cascade turbulence?

4 How to apply the principle **"the whole truth and nothing but the truth"** to turbulence?

Information approach to weak turbulence.

$$q\{a_k\} = \frac{1}{Z} exp\left[-\sum_k |a_k|^2/n_k\right]$$

$$S(q) = \sum_k \ln e\pi n_k$$

$$\frac{dn_k}{dt} = I_k$$

$$\frac{dS}{dt} = -\sum_k I_k \log n_k$$

Information approach to weak turbulence.

$$\frac{dS}{dt} = -\sum_k I_k \log n_k$$

For energy cascade $\qquad \omega_k \overline{I}_k k^d \simeq P = \mathrm{const}$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \int \frac{\overline{I}_k}{\overline{N}_k}\,\mathrm{d}\boldsymbol{k} \simeq \int \frac{P}{\overline{N}_k \omega_k k^d}\,\mathrm{d}\boldsymbol{k} \simeq P \max_k (\overline{N}_k \omega_k)^{-1}.$$

For action cascade

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \int \frac{\overline{I}_k}{\overline{N}_k}\,\mathrm{d}\boldsymbol{k} \simeq \int \frac{Q}{\overline{N}_k k^d}\,\mathrm{d}\boldsymbol{k} \simeq Q \max_k \overline{N}_k^{-1}.$$

The main contribution is always due to largest *k* which is in the direct cascade

Information approach to weak turbulence.

$$q\{a_k\} = \frac{1}{Z} exp\left[-\sum_k |a_k|^2/n_k\right]$$

But the true distribution $\rho\{a_k\}$ is generally non-Gaussian

The difference between distributions can be measured by the **relative entropy** (the price of nonoptimal coding):

$D(\rho|q) = \langle \log(\rho/q) \rangle = \sum_k \int da_k \rho \log(\rho/q).$

$$D(\rho|q) = \langle \log(\rho/q) \rangle = \sum_k \int da_k \rho \log(\rho/q).$$

$S(q) = \sum_{\mathbf{k}} \ln(e\pi n_k) = \sum_{\mathbf{k}} S_k$. The relative entropy then coincides with the multimode mutual information $D(\rho|q) = \sum_{\mathbf{k}} S_k - S(\rho) := I(\{a_k, a_k^*\})$. We keep in mind that the mutual information is defined for any subsystems, A and B, via their entropies: $I(A,B) = S(A) + S(B) - S(A,B)$. For example, the mutual information between two parts of the message measures how much of the future part we can predict given the part already received.

For the 3-wave case,

$$\rho = \frac{1}{Z}\exp\left[-\sum_{\mathbf{k}}\alpha_k|a_k|^2 + \sum_{\mathbf{kpq}}F_{kpq}a_k^* a_p a_q + \text{c.c.}\right]$$

$$F_{123} = \frac{J_{123}^*}{2n_1 n_2 n_3}, \qquad \frac{1}{\alpha_i} = n_i - \sum_{\mathbf{k}_1\mathbf{k}_2}\frac{|J_{i12}|^2 + 2|J_{12i}|^2}{2n_1^2 n_2^2}$$

the relative entropy is the sum of the mutual information, $I_{\mathbf{k}_i+\mathbf{k}_j,\mathbf{k}_i,\mathbf{k}_j}$, of all resonant triads,

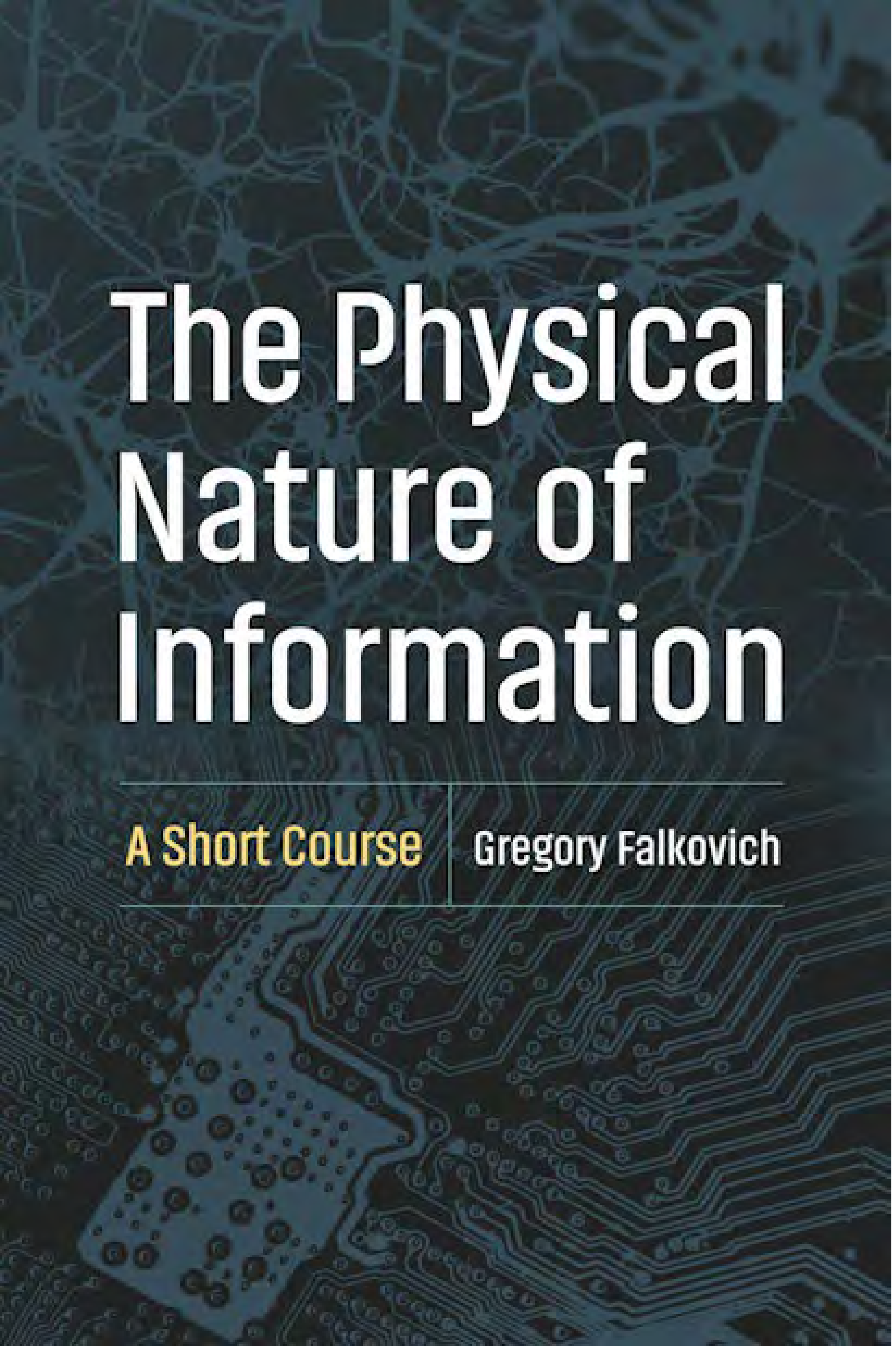
$$D(\rho|q) = \sum_{\mathbf{k}_i,\mathbf{k}_j} I_{\mathbf{k}_i+\mathbf{k}_j,\mathbf{k}_i,\mathbf{k}_j} = \sum_{\mathbf{k}_i\mathbf{k}_j}\frac{|J_{i+jij}|^2}{2n_i n_j n_{i+j}}$$

This quantity could be large for large Re even in weak turbulence.
Measuring or computing it is a challenge.

Weak turbulence is only good for predicting occupations numbers. How much and where the rest of the information about turbulence is encoded ?

There is a vast literature devoted to cumulant anomalies away from equilibrium and to the singular (fractal) distributions in dynamical chaos, that is for systems with few degrees of freedom (SRB measures).Not much is known about turbulence.

If we learn where the information is encoded we may discover a new fundamental variational principles for far-from-equilibrium states.



A unified introduction to information theory for scientists.

Published by Princeton University Press in March 2025.

- Provides a panoramic approach to information theory
- Draws on examples from physics, engineering, biology, economics, and linguistics
- Applications range from thermodynamics and statistical mechanics to dynamical chaos, information and communication theories, and quantum information
- Includes materials for lectures and tutorials along with exercises with detailed solutions
- Can be used to design a one-semester introductory course

Renormalizing weak turbulence into strong

$$H = \sum_{p} \omega_p a_p^* a_p + \sum_{p_1,p_2,p_3,p_4} \lambda_{p_1 p_2 p_3 p_4} \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) a_{p_1}^* a_{p_2}^* a_{p_3} a_{p_4} \; .$$

$$H = \sum_p \omega_p a_p^* a_p + \sum_{p_1,p_2,p_3,p_4} \lambda_{p_1p_2p_3p_4} \delta(\mathbf{p_1}+\mathbf{p_2}-\mathbf{p_3}-\mathbf{p_4}) a_{p_1}^* a_{p_2}^* a_{p_3} a_{p_4} \ .$$

$$i\frac{da_k}{dt} = \frac{dH}{da_k^*} = \omega_k a_k + \sum_{123} \lambda_{k123} a_1^* a_2 a_3$$

$$\frac{d < |a_k|^2 >}{dt} = \frac{dn_k}{dt} = Im \sum_{123} \lambda_{k123} < a_k^* a_1^* a_2 a_3 >$$

$$\mathcal{P}\{a_k\} \propto \exp[-\textstyle\sum_k |a_k|^2/n_k]$$

The first-order in λ correction to Gaussian gives the standard kinetic equation

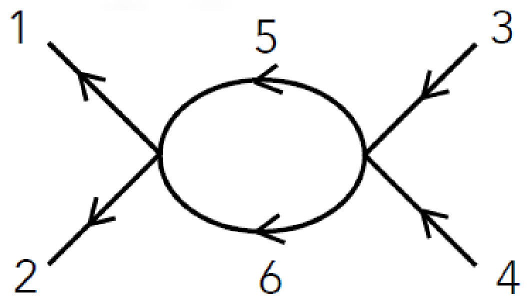
$$\frac{\partial n_k(t)}{\partial t} = 16 \sum_{p_1,\dots,p_4} \delta_{k,p_1} \pi \delta(\omega_{p_1p_2;p_3p_4}) \lambda^2_{1234} n_1 n_2 n_3 n_4 \left(\frac{1}{n_1}+\frac{1}{n_2}-\frac{1}{n_3}-\frac{1}{n_4}\right) = -\frac{\partial Q_k}{\partial \vec{k}}$$

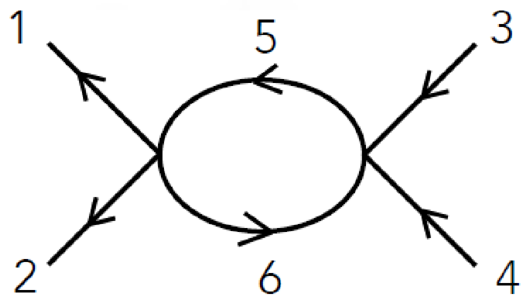
Writing KE as continuity equation $\frac{\partial n_k}{\partial t} = -\frac{\partial Q_k}{\partial \vec{k}}$, assuming $\omega_k \propto k^\alpha$, $\lambda_{kkkk} \sim \lambda k^m$ and requiring constant flux, $Q_k = Q \sim \lambda^2 k^{2\beta-\alpha+3d} {n_k}^3$, we obtain Kolmogorov-Zakharov cascade spectrum:

$$n_k \sim (Q/\lambda^2)^{1/3} k^{-d-\frac{2m}{3}+\frac{\alpha}{3}}$$

Next-order corrections to the fourth moment and to the collision integral of the kinetic equation

$$\frac{\partial n_k(t)}{\partial t} = I_k = 4 \sum_{p_1,\dots,p_4} \delta_{kp_1} \lambda_{p_3p_4p_1p_2} \mathrm{Im}\langle a^*_{p_1} a^*_{p_2} a_{p_3} a_{p_4} \rangle$$





$$\omega_{p_1p_2;p_3p_4} \equiv \omega_{p_1} + \omega_{p_2} - \omega_{p_3} - \omega_{p_4}$$

$$64\pi \sum_{p_1,\dots,p_6} \delta_{k,p_1} \delta(\omega_{p_1p_2;p_3p_4}) \prod_{i=1}^{6} n_i \frac{\lambda_{3412}\lambda_{1256}\lambda_{5634}}{\omega_{p_1p_2;p_5p_6}} \left(\frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4}\right)\left(\frac{1}{n_5} + \frac{1}{n_6}\right) +$$

$$256\pi \sum_{p_1,\dots,p_6} \delta_{k,p_1} \delta(\omega_{p_1p_2;p_3p_4}) \prod_{i=1}^{6} n_i \frac{\lambda_{3412}\lambda_{1635}\lambda_{2546}}{\omega_{p_4p_6;p_2p_5}} \left(\frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4}\right)\left(\frac{1}{n_5} - \frac{1}{n_6}\right).$$

Vertex renormalization

$$\delta\lambda_{1234} = \lambda_{1234}\left(\mathcal{L}_a + \mathcal{L}_b\right)$$

$$\mathcal{L}_a = 4\sum_{p_5,p_6}(n_5+n_6)\lambda_{k256}\lambda_{5634}/\lambda_{k234}\omega_{k256}$$
$$\mathcal{L}_b = 16\sum_{p_5,p_6}(n_6-n_5)\lambda_{k635}\lambda_{2546}/\lambda_{k234}\omega_{4625}$$

$$\omega_{p_1p_2;p_3p_4} \equiv \omega_{p_1}+\omega_{p_2}-\omega_{p_3}-\omega_{p_4}$$

The sign changes when passing through the resonances.
What is the overall effect, enhancement or suppression?

The simplest four-wave interaction:

Nonlinear Schrodinger equation

$$\imath \Psi_t = -\frac{\delta \mathcal{H}}{\delta \Psi^*}\,,\ \mathcal{H} = \int d\mathbf{r}\, \left[|\nabla \Psi|^2 + \lambda |\Psi|^4\right]$$

Two integrals of motion – two cascades

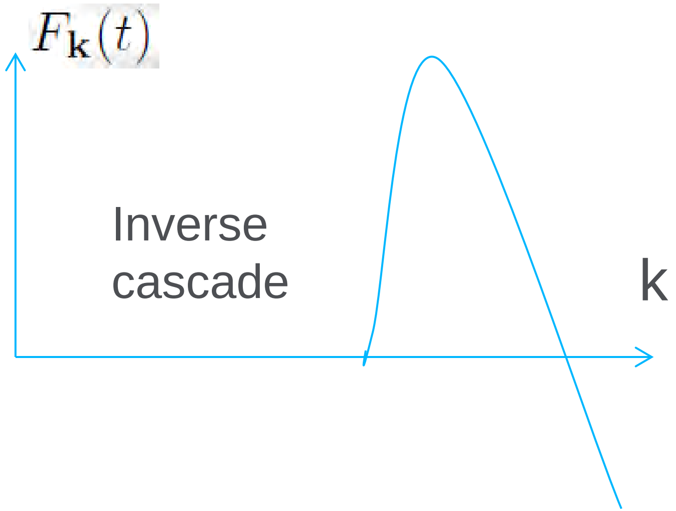
$$n_k = k^{-d+2/3}(Q/\lambda^2)^{1/3}$$

adding to the rhs of the Fourier-transformed NSE $F_{\mathbf{k}}(t)$

With cold atoms one needs to supply constantly high-momentum atoms and simultaneously cool the system to preserve the energy .

Kolmogorov-Zakharov weakly-turbulence action cascade

$$n_k = k^{-d+2/3}(Q/\lambda^2)^{1/3}$$

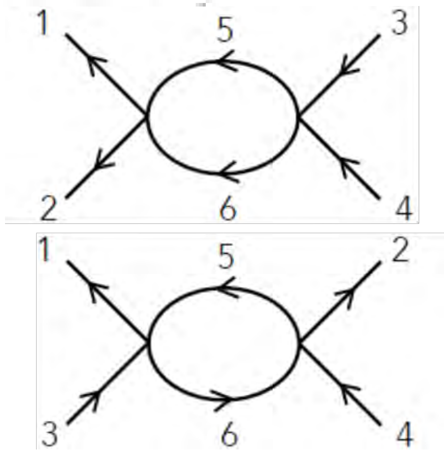


Kinetic equation with one-loop correction

$$\frac{\partial n_1}{\partial t} = 16\pi\lambda^2 \,\mathrm{Re} \int d\vec{p}_2 d\vec{p}_3 d\vec{p}_4 \, n_1 n_2 n_3 n_4 \Big(\frac{1}{n_1}+\frac{1}{n_2}-\frac{1}{n_3}-\frac{1}{n_4}\Big)\left[1+2\mathcal{L}_+ + 8\mathcal{L}_-\right] \delta(\omega_{p_1p_2;p_3p_4})\delta(\vec{p}_{12;34})$$

$$\mathcal{L}_+ \;=\; 4\lambda \int d^d q \frac{n_q}{\omega_+ - q^2 - (\vec{p}_+ - \vec{q})^2 + i\epsilon} \,,$$

$$\mathcal{L}_- \;=\; -2\lambda \int d^d q \, n_q \left[\frac{1}{\omega_- - q^2 + (\vec{p}_- - \vec{q})^2 + i\epsilon} - \frac{1}{\omega_- + q^2 - (\vec{p}_- + \vec{q})^2 + i\epsilon}\right] \,,$$



For weak-turbulence spectrum

$$\mathrm{Re}\,\mathcal{L}_+ \;=\; -\frac{4\pi^2 2^{\frac{1}{3}}\sqrt{3}\lambda}{|\vec{p}_1+\vec{p}_2|}\left(\tilde{p}_+^{-1/3} + \tilde{p}_- |\tilde{p}_-|^{-4/3}\right), \qquad \tilde{p}_\pm \equiv |\vec{p}_1+\vec{p}_2| \pm |\vec{p}_1-\vec{p}_2|$$

$$\mathrm{Re}\,\mathcal{L}_- \;=\; -\frac{\pi^2 2\sqrt{3}\lambda}{|\vec{p}_2-\vec{p}_4|^{2/3}}\left(\frac{p_4^2-\vec{p}_2\cdot\vec{p}_4}{|p_4^2-\vec{p}_2\cdot\vec{p}_4|^{4/3}} + \frac{p_2^2-\vec{p}_2\cdot\vec{p}_4}{|p_2^2-\vec{p}_2\cdot\vec{p}_4|^{4/3}}\right).$$

$$Sign \int \mathcal{L}_\pm d\Omega = -\lambda$$

Enhancement or suppression of interaction?
Zero charge or confinement?

$$Sign \int \mathcal{L}_{\pm} d\Omega = -\lambda$$

Enhancement or suppression of interaction?

Zero charge or confinement?

$$Sign \int \mathcal{L}_{\pm} d\Omega = -\lambda \qquad\qquad Sign\ \overline{\delta\lambda} = -\lambda\omega''$$

Repulsion (defocusing) is suppressed.

Attraction (focusing is enhanced.

On average.

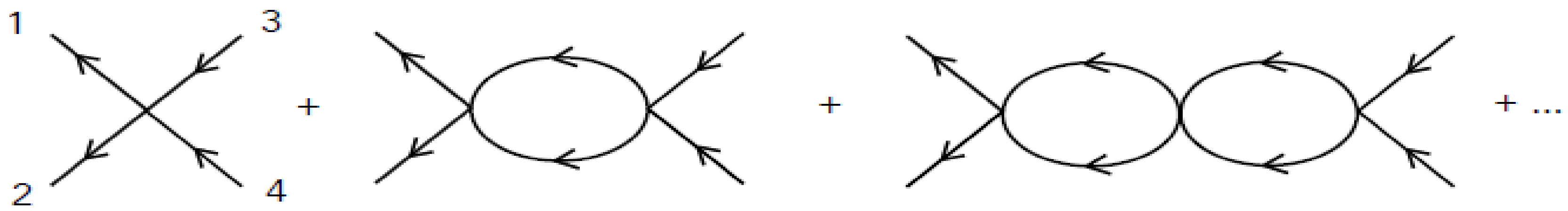
wave) of amplitude $|a_k|$ and wavenumber k, the Bogolyubov spectrum of perturbations with the wavenumber $q \ll k$ has the form $\Omega = \sqrt{\lambda_{kkkk}\omega''|a_k|^2 q^2 + \omega''^2 q^4/4}$, which shows that $\lambda\omega'' < 0$ is the (Lighthill) condition for modulational instability. Since $\lambda = \partial^2\widetilde{\omega}/\partial|a_k|^2$, we see that the instability of a monochromatic wave takes place when the second derivatives of the frequency with respect to the amplitude and to the wavenumber have opposite signs, that is, $\widetilde{\omega}(k, |a_k|^2)$ has a saddle at zero.

What will be the ultimate result of the suppression/enhancement of interaction in the domain of strong turbulence?

Nonlinear Schrodinger equation, vector ψ with $\mathscr{N}$>>1

$$\mathcal{H} = \int d\mathbf{r}\left[|\nabla\vec{\Psi}|^2 + \frac{\lambda}{N}(\vec{\Psi}^*\cdot\vec{\Psi})^2\right] \, ,$$

$$\varepsilon_k = \lambda n_k k^{d-2}$$



$$\frac{\partial n_1}{\partial t} = \frac{8\pi}{N}\,\mathrm{Re}\int d\vec{p}_2 d\vec{p}_3 d\vec{p}_4\, n_1 n_2 n_3 n_4 |\Lambda_{1234}|^2\left(\frac{1}{n_1}+\frac{1}{n_2}-\frac{1}{n_3}-\frac{1}{n_4}\right)\delta(\omega_{12;34})\delta(\vec{p}_{12;34})$$

$$|\Lambda_{1234}|^2 = \frac{\lambda^2}{|1-\mathcal{L}_-|^2}$$

$$Sign\int \mathcal{L}_{\pm} d\Omega = -\lambda$$

$$\mathcal{L}_- = \frac{2\pi\lambda}{p_-}\int_0^\infty dq\, q\, n_q \log\left|\frac{((p_- - q)^2 - q^2)^2 - \omega_-^2}{((p_- + q)^2 - q^2)^2 - \omega_-^2}\right| - i\frac{2\pi^2\lambda}{p_-}\int_{\frac{|p_-^2 - \omega_-|}{2p_-}}^{\frac{|p_-^2 + \omega_-|}{2p_-}} dq\, q\, n_q$$

Nonlinear Schrodinger equation, defocusing case

Consider strong nonlinearity $\mathcal{L} > 1$ In this case, λ disappears from the equation.

$$|\Lambda_{1234}|^2 = \frac{\lambda^2}{|1 - \mathcal{L}_-|^2} \longrightarrow |\Lambda_{1234}|^2 \approx \frac{\lambda^2}{|\mathcal{L}_-|^2}$$

$$\mathcal{L} \simeq \lambda N p_-^2 / |p_-^4 - \omega_-^2|.$$

Forced case.

In the strong turbulence regime (with the renormalized vertex)
the new stationary action-cascade solution is

$$n_k \simeq Q^{1/3} N^{2/3} k^{-d-2/3} \simeq Q k_0^{-4/3} k^{-d-2/3}.$$

Nonlinear Schrodinger equation, focusing case
Critical spectral balance regime

$$\frac{\partial n_{p_1}}{\partial t} = \int d\vec{p}_2 d\vec{p}_3 d\vec{p}_4 \delta(\omega_{p_1 p_2; p_3 p_4}) \delta(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)$$
$$\times\, n_1 n_2 n_3 n_4 \left(\frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4} \right) \frac{\lambda^2}{|1 - \mathcal{L}|^2} = \tilde{I}_k \,, \qquad (6)$$

$$\frac{\mathcal{L}}{\pi\lambda} = \int_0^\infty \frac{dq}{p_-}\, q^{d-1}\, n_q \log \left| \frac{(p_-^2 - 2p_- q)^2 - \omega_-^2}{(p_-^2 + 2p_- q)^2 - \omega_-^2} \right| + i\pi \int_{q_-}^{q_+} \frac{dq}{p_-} q^{d-1}\, n_q \,.$$

$$\varepsilon_k = \lambda n_k k^{d-2}$$

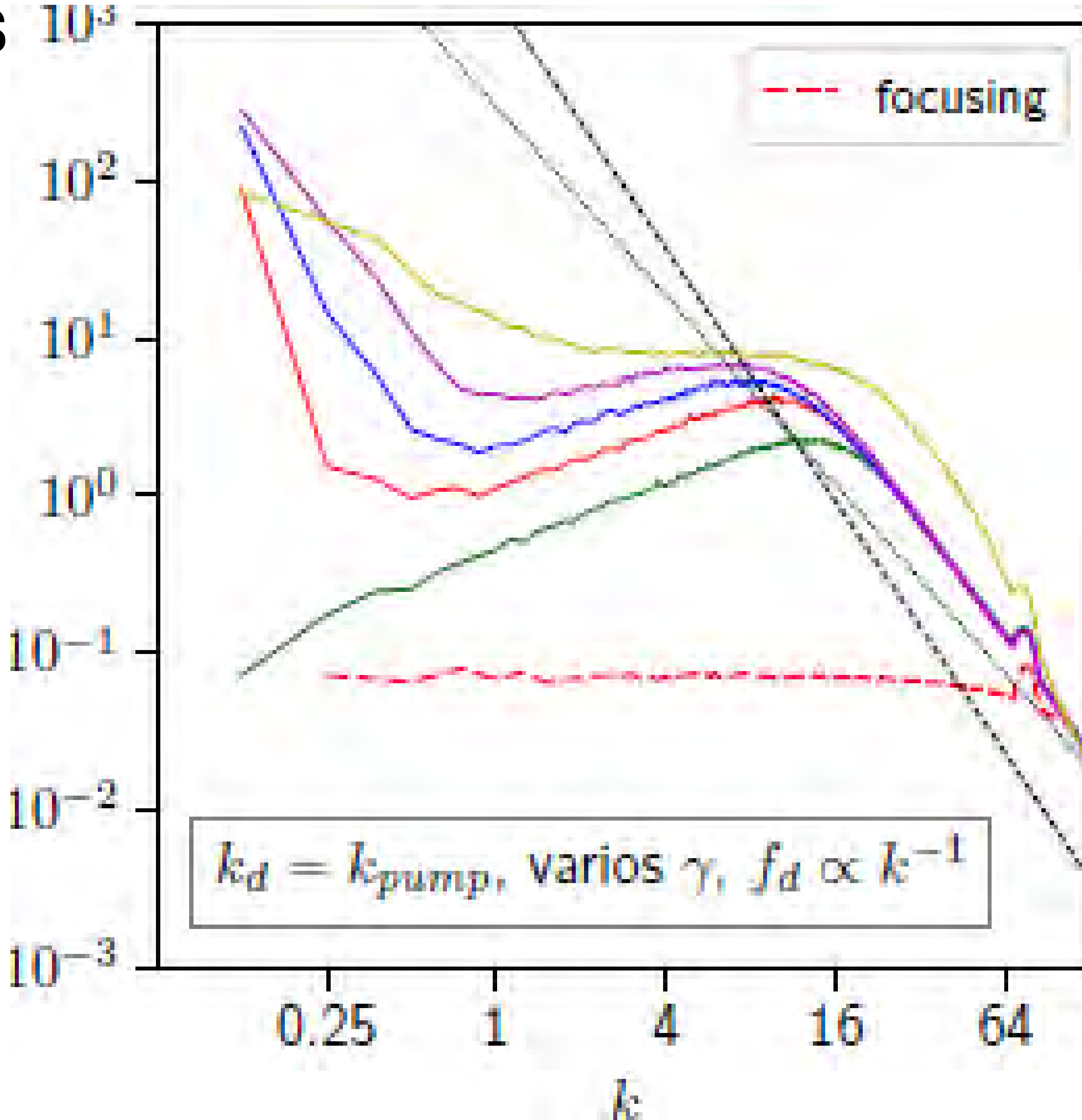
$\mathrm{Re}\,\mathcal{L} \simeq \lambda n_k k^{d-2} \simeq 1$ means that nonlinear and linear times are of the same order for all wavenumbers. Wave collapses (light self-focusing), which happen when interaction is comparable to dispersion, restrict the growth of $\mathrm{Re}\,\mathcal{L}$ with the decrease of k and give a universal flux-independent spectrum at $q \ll k_* = (\lambda Q)^{1/4}$:

$$\tilde{n}_q = \frac{q^{2-d}}{2\pi\lambda \log(k_*/q)} \,. \qquad (12)$$

Phillips, *J. Fluid Mech.* (1958)
Goldreich, Sridhar, *Astrophys. J.* (1995)
Kuznetsov, *JETP Letters*, (2004)
Nazarenko, Schekochihin, *J. Fluid Mech* (2011)

Nonlinear Schrodinger equation focusing and defocusing cases

Vladimirova and Falkovich,
in preparation.



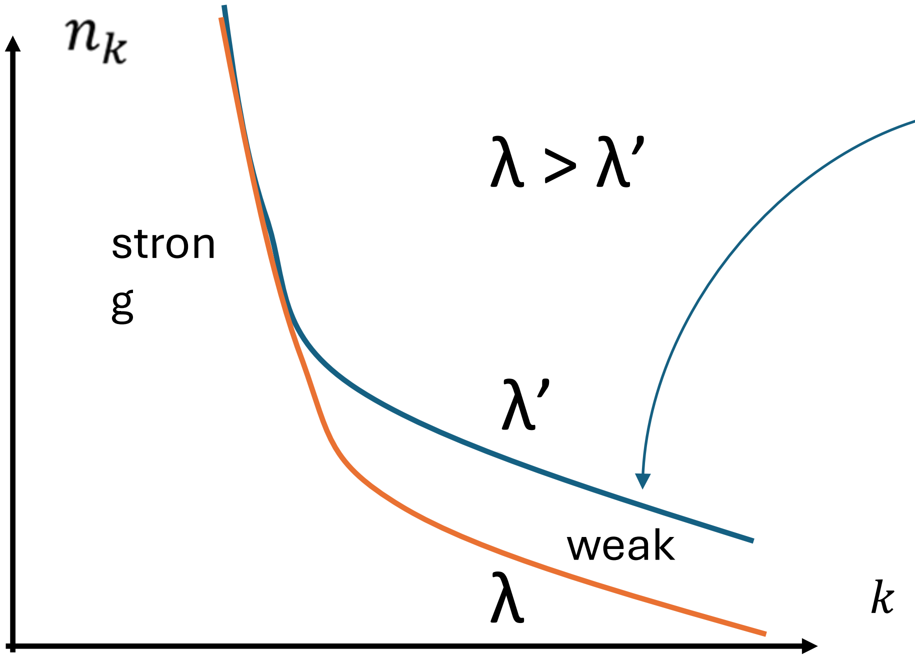
Two types of universality of strong turbulence

Kolmogorov-Zakharov weak-turbulence spectrum

$$n_k = k^{-d+2/3}(Q/\lambda^2)^{1/3}$$

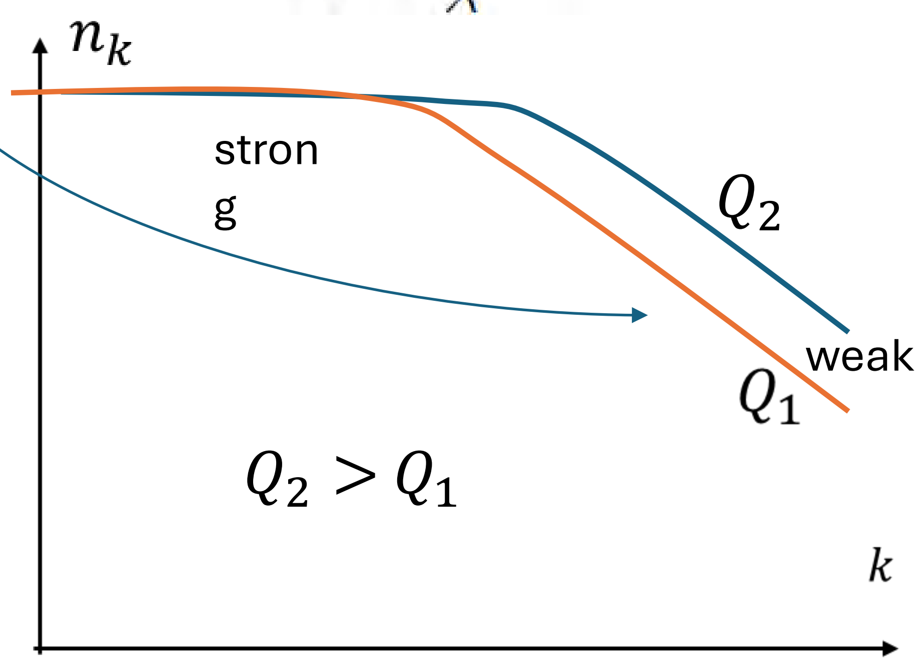
Defocusing: λ-independent

$$n_k \simeq Q^{1/3} N^{2/3} k^{-d-2/3} \simeq Q k_0^{-4/3} k^{-d-2/3}$$



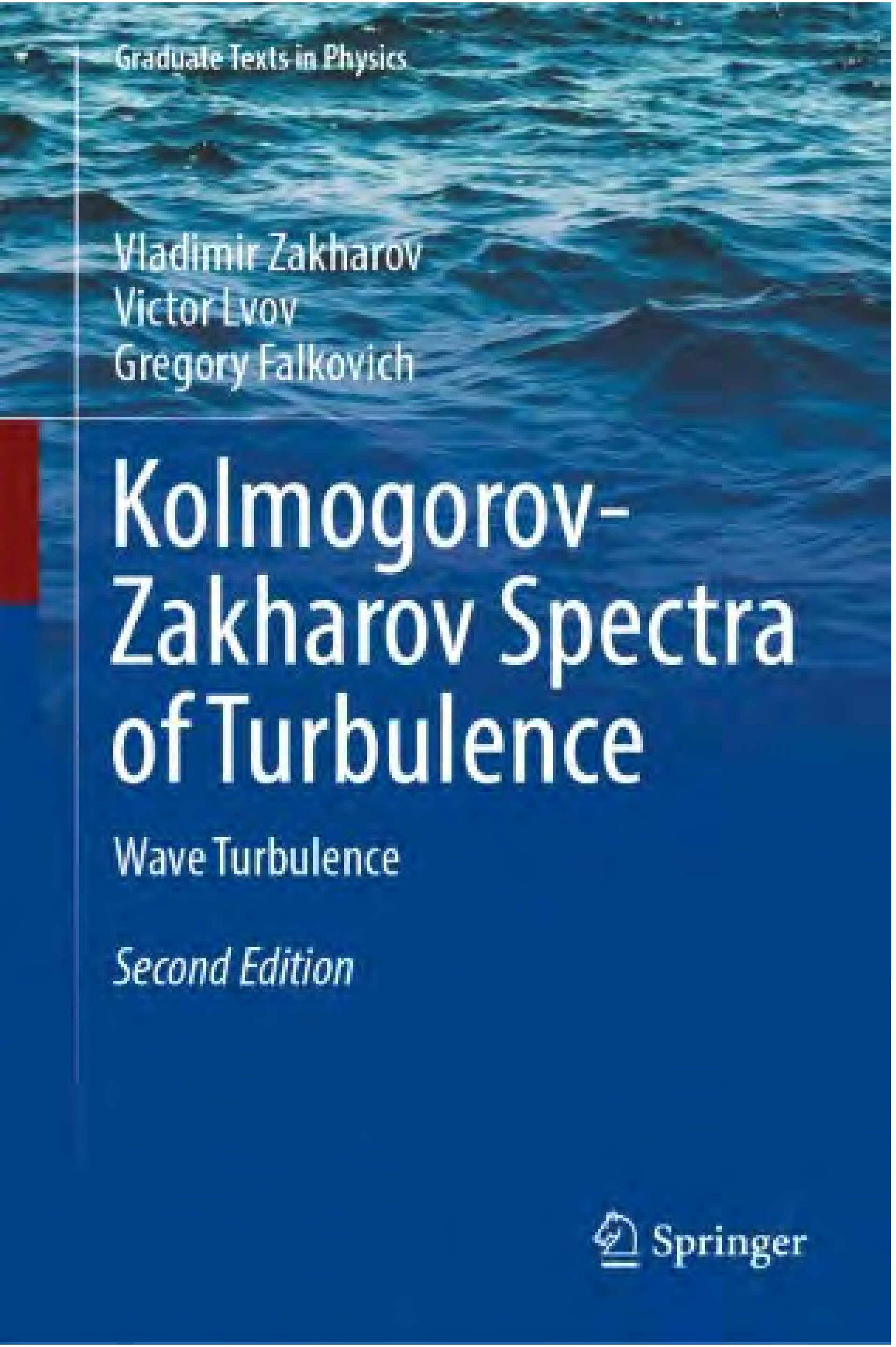
Focusing: Q-independent

$$n_k \sim \frac{1}{\lambda} k^{2-d}$$



Conclusion on optical turbulence

While weak turbulence is the same for focusing and defocusing Nonlinear Schrödinger Equation, strong turbulence spectra differ dramatically. Here we show that renormalization due to multi-mode interactions suppresses repulsion (analog of screening in quantum electrodynamics), which leads to a steeper spectrum in the defocusing case. On the contrary, the attraction enhancement (like in chromodynamics) makes strong-turbulence spectum less steep in the focusing case. To describe strong turbulence, we consider a vector model in the limit of large number of components, where the vertex renormalization is given by a geometric series of the bubble diagrams. The renormalized kinetic equation has an inverse-cascade solution whose two asymptotics (at high and low wavenumbers) describe weak and strong turbulence respectively. We find the universality of two kinds appearing in strong turbulence: independence of the flux magnitude in the focusing media and of the bare coupling constant in the defocusing media (where dependence on the largest scale appears instead).



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Compressible turbulence
Theory review

Acoustic waves, vortices and their interaction

Basic idea – cascade. Works well for quadratic conservation laws. But there is more to turbulence that one moment.
Two wys to turbulence, via space and time. Do al roads lead to the same developed turbulence at least for incompressible

1. Incompressible vortex turbulence, 3d and 2d, Anomalous scaling as non-universality, Lagrangian conservation laws.
2. For experiment what is really universal in higher and lower moments, n->0 dzeta/dn=0.36?

2. Sound with dispersion.
3. Low-Mach acoustic turbulence – hand-waving reduction to Burgers. Flux law and other moments from shocks.
4. Cascades of non-quadratic invariants FFO and GB flux laws, density-weighted velocity.
5. Cascade loops (2d compressible).