Modeling astrophysical turbulence

recommended review articles:

"Astrophysical turbulence modeling" by Brandenburg & Nordlund, Reports on Progress in Physics 74:046901 (2011)

"Turbulence modelling in neutron star merger simulations" by Radice & Hawke, Living Reviews in Computational Astrophysics 10:1 (2024)

Matthew Duez, WSU, support from NSF, NASA
INT "Compressible Turbulence", June 25, 2025

Astro turbulence: distinctive features

- very high Re
- relativity
- background variation (e.g. solar convective zone: $\rho_b/\rho_t \sim 10^6, T_b/T_t \sim 10^2$)
- high temp $\Rightarrow$ high conductivity $\sigma \Rightarrow$ MHD

Magnetohydrodynamics

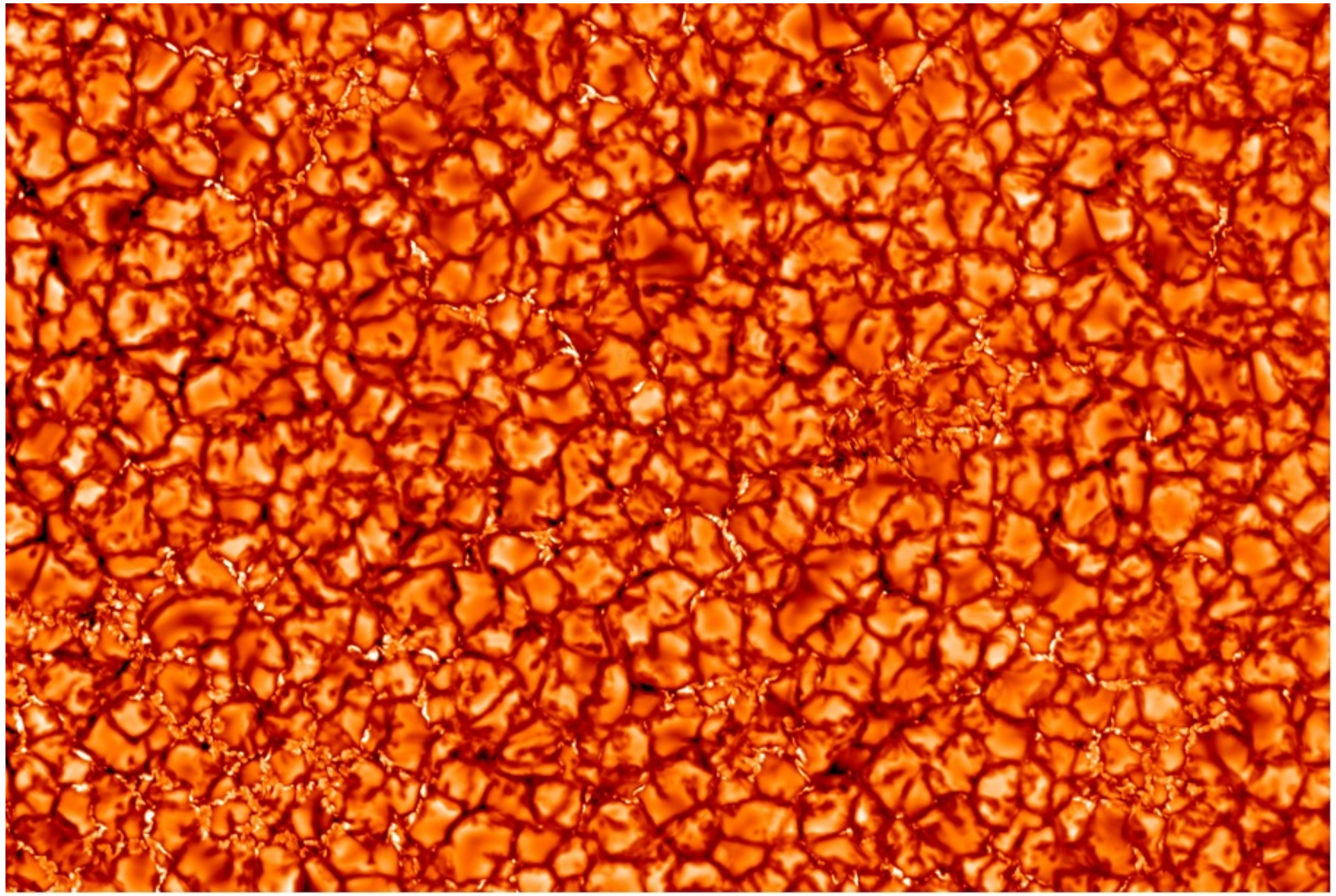
- magnetic field lines advect with fluid
- exert $\mathbf{J} \times \mathbf{B}$ force on fluid: field lines have tension, pressure
- new instabilities
- magnetosphere/jet regions where $P_B/P_{\rm gas} \gg 1$
- conserved magnetic helicity: $\int \mathbf{A} \cdot \mathbf{B} \, dV$

Cases: interstellar medium

- stirred by supernovae
- supersonic turbulence
- trigger for star formation

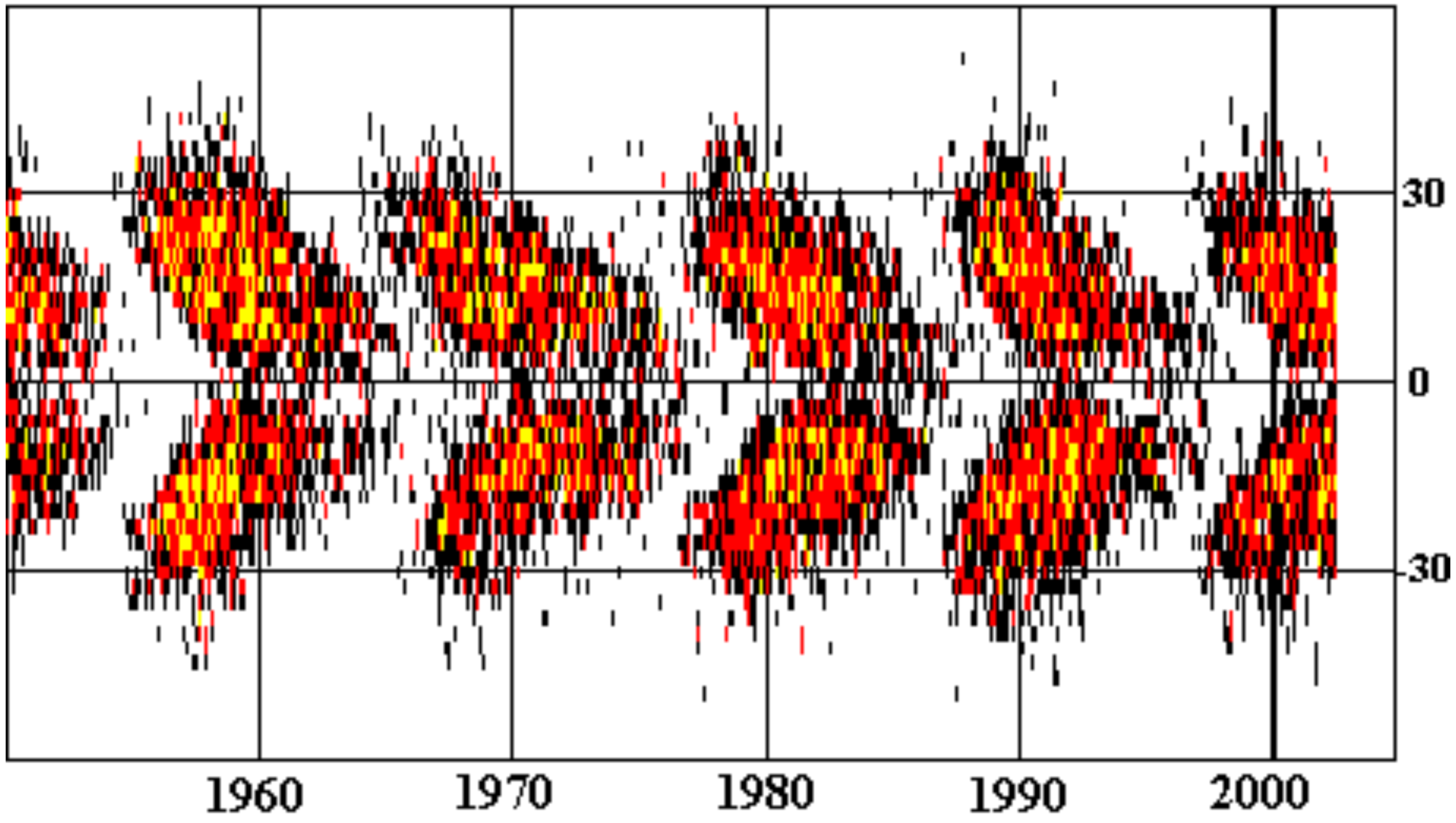
Cases: convection in stars

- e.g. granular suface of sun:
- energy transfers from high-temp core
- If high opacity, convective instability ($dS/dr < 0$)
- transfers energy, evens composition in convective regions



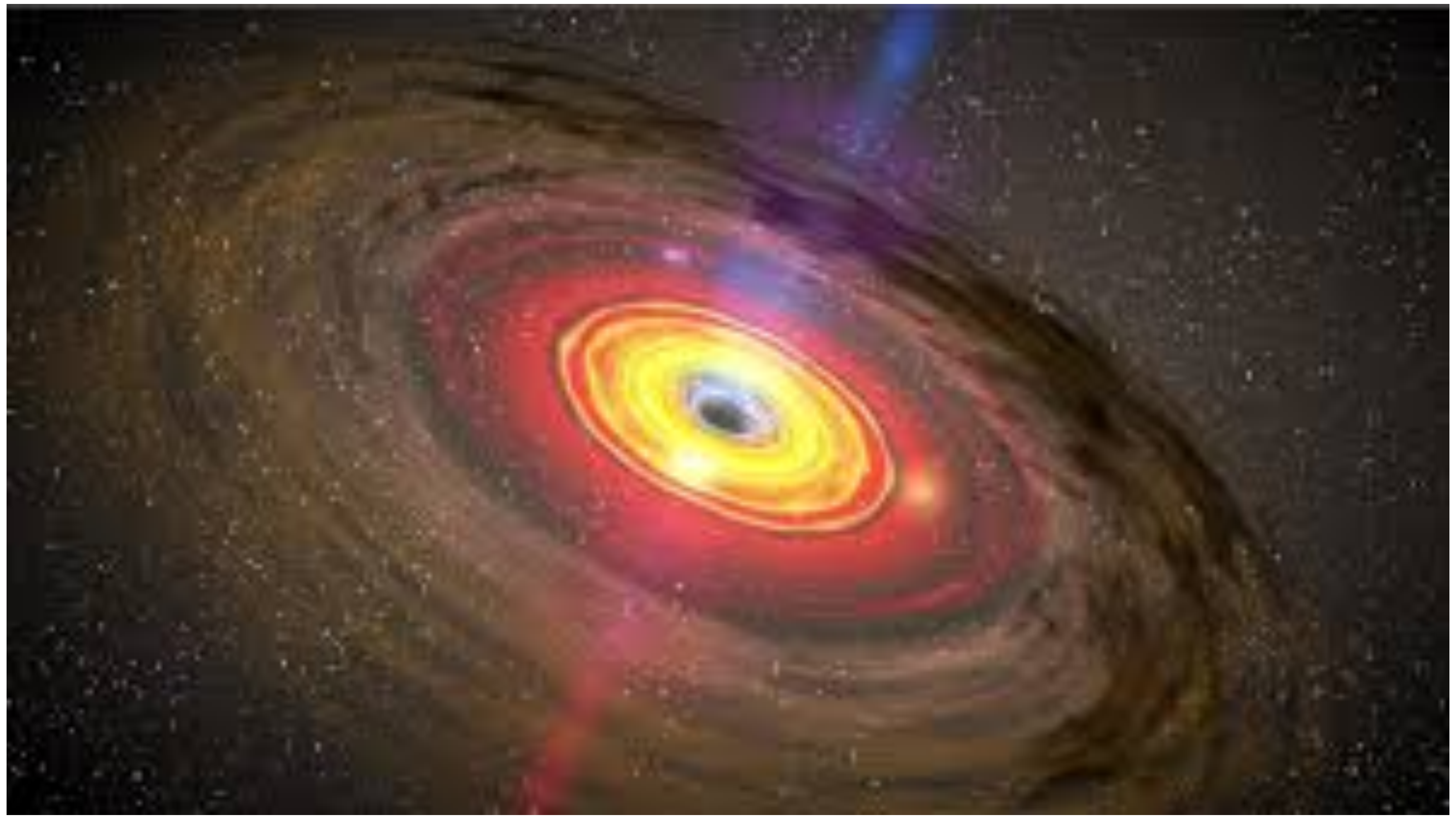
Cases: stellar dynamo

- e.g. 22 year solar cycle:
- large scale $\mathbf{B}$ generated by differential rotation and convection
- small scale $\rightarrow$ large scale



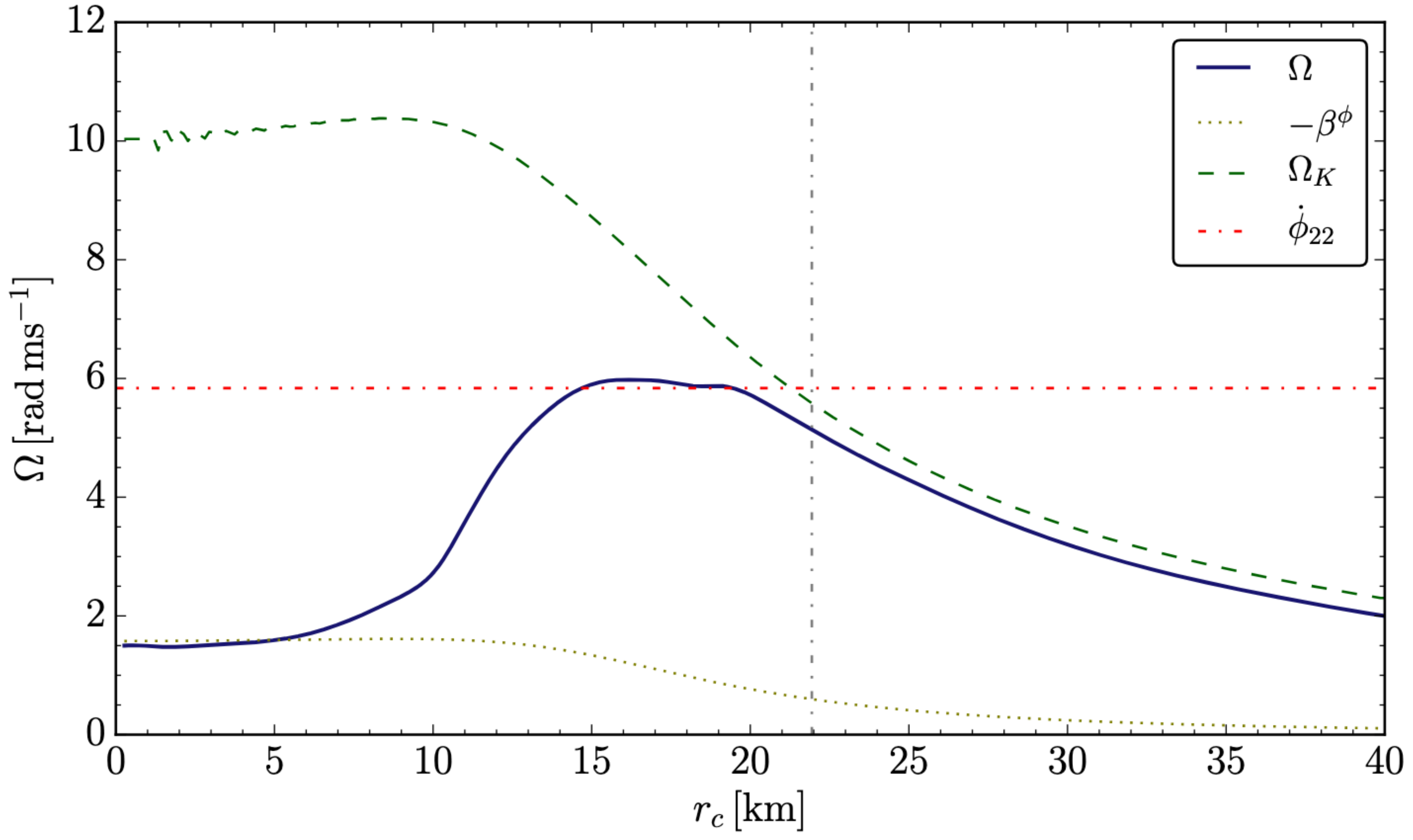
Cases: accretion disks

- gas orbiting compact object
- magnetorotational instability (MRI): high σ + weak $\mathbf{B}$ + $d\Omega^2/dr < 0$ $\Rightarrow$ unstable $\Rightarrow$ turbulence
- turbulent angular momentum transport makes accretion possible



Cases: hot, differentially rotating neutron stars

- protoneutron star formed from supernova or accretion-induced collapse
- remnant from binary neutron star merger
- subject to convection, MRI, Kelvin-Helmhotz instability (KHI), Tayler-Spruit instability (TSI) $\Rightarrow$ turbulence



from Kastaun, Ciolfi, and Giacomazzo 2016

General setup

- A slowly varying axisymmetric background $\mathscr{B}$
  + 3D, rapidly varying turbulence $\mathscr{T}$.
- only care about $\mathscr{B}$, but $\mathscr{T}$ affects $\mathscr{B}$:
  - dynamo, jet (cf. 2D anti-dynamo theorem)
  - transport of energy, momentum, composition
  - enables heating
  - turbulent pressure
- goal of model: evolve only $\mathscr{B}$ in 1D or 2D with $\mathscr{T}$ effects

Subtleties

- How to separate $\mathscr{B}$ from $\mathscr{T}$?
  - average: azimuthal, time, high-k
- Modeling all turbulence or only sub grid turbulence (MFT vs LES)?
  - e.g. should $\nu_{\mathrm{turb}} \propto \Delta x$?
- What do fields on 2D grids represent?
  - az. averages or representative cuts?
  - az. averaging tends to filter out most of $\mathbf{u}$, $\mathbf{B}$. Do we need new variables like $\overline{\mathbf{u}^2}$, $\overline{\mathbf{B}^2}$ to capture turbulent effects?

The closure problem

- $f = \bar{f} + \Delta f, \quad \overline{\Delta f} = 0, \quad f \in \{\rho, \mathbf{B}, \mathbf{u}, Y, \dots\}$
- $\qquad \partial_t(\rho u_i) = -\partial_j \left[\rho u_i u^j + \delta_i^j (P + B^2/2) - B_i B^j\right]$
- $\Rightarrow \ \partial_t(\overline{\rho u_i}) = -\partial_j \left[\overline{\rho u_i u^j} + \delta_i^j (\overline{P} + \overline{B^2}) - \overline{B_i B^j}\right]$
- $\overline{\rho f} = \overline{\rho}\bar{f}$ (Favre average, or ignoring compressibility in turbulence)
- $\overline{\rho u_i u_j} = \overline{\rho}\,\overline{u_i}\overline{u_j} + \overline{\rho}\overline{\Delta u_i \Delta u_j}$
- (need to supply 2nd-order correlation term, a turbulent stress)
- $\overline{T}^{\mu\nu}(g_{\mu\nu} + \overline{u}_\mu \overline{u}_\nu) \neq 3P(\overline{\rho}, \overline{T})$ ($\mathscr{T}$ pressure) [Also, $\overline{\gamma} \neq 1/\sqrt{1 - \overline{\mathbf{u}^2}}$.]

The closure problem

- $f = \bar{f} + \Delta f, \quad \overline{\Delta f} = 0, \quad f \in \{\rho, \mathbf{B}, \mathbf{u}, Y, \ldots\}$
- similarly
- $\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B})$
- $\Rightarrow \partial_t \overline{\mathbf{B}} = \nabla \times (\overline{\mathbf{u} \times \mathbf{B}}) = \nabla \times (\overline{\mathbf{u}} \times \overline{\mathbf{B}} + \overline{\boldsymbol{\Delta}\mathbf{u} \times \boldsymbol{\Delta}\mathbf{B}})$
- $\partial_t(\rho Y) + \nabla \cdot (\rho Y \mathbf{u}) = 0$
- $\Rightarrow \partial_t(\overline{\rho}\overline{Y}) + \nabla \cdot (\overline{\rho}\overline{Y}\overline{\mathbf{u}} + \overline{\rho}\overline{\Delta Y \boldsymbol{\Delta}\mathbf{u}}) = 0$
- must specify $\Delta \mathbb{T}$ ($\overline{\boldsymbol{\Delta}\mathbf{u} \otimes \boldsymbol{\Delta}\mathbf{u}}$ for pure hydro), $\overline{\boldsymbol{\Delta}\mathbf{u} \times \boldsymbol{\Delta}\mathbf{B}}$, $\overline{\Delta Y \boldsymbol{\Delta}\mathbf{u}}$

2nd-correlation closures

- $\Delta T_{ij} = -2\overline{\rho}\nu_T\sigma_{ij}(\overline{\mathbf{u}})$
  - "turbulent viscosity"
- $\left(\overline{\Delta u \times \Delta B}\right)_i = \alpha_i^j \overline{B}_j + \eta_i^{jk}\partial_j \overline{B}_k$
  - "turbulent EMF" = "alpha effect" + "turbulent diffusivity"
- $\overline{\Delta Y \Delta u_i} = D_T \partial_i Y$
  - "turbulent particle diffusion"

3rd-correlation closures

- can find evolution equations for the 2nd-order correlations, e.g.
- $\partial_t(\overline{\mathbf{\Delta u} \times \mathbf{\Delta B}}) = \overline{\partial_t \mathbf{\Delta u} \times \mathbf{\Delta B}} + \overline{\mathbf{\Delta u} \times \partial_t \mathbf{\Delta B}}$
- $= \cdots \ + \text{triple correlations} \quad \overline{u, B}_3$
- must specify $\overline{u, B}_3$
- e.g. τ approximation: $\overline{u, B}_3 = -\overline{\mathbf{\Delta u} \times \mathbf{\Delta B}}/\tau$ (damping)
- $k - \epsilon$ models: $\nu_T \propto k^2$ where $k =$ specific turbulent kinetic energy
- + evolution equation for k

Inferring coefficients: test field method

- use 3D turbulent MHD simulations
- test field method idea:
  - redo evolution with different *specified* $\overline{\mathbf{B}}$, same $\mathbf{u}$, evolve $\mathbf{\Delta B}$
  - average over results
  - get α_{ij}, η_{ijk}
- lack of $\mathbf{\Delta B} \to \mathbf{\Delta u}$ feedback

Inferring coefficients: SVD method

- i.e. least squares fitting
- one evolution of $\mathbf{u}, \mathbf{B}$
- average to get $\overline{\mathbf{u}}$, $\overline{\mathbf{B}}$, $\overline{\mathbf{u} \times \mathbf{B}} - \overline{\mathbf{u}} \times \overline{\mathbf{B}}$
- fit for α_{ij}, η_{ijk}
- can check for quality of fit
- inside accretion disks, it's not good ($\overline{\mathbf{B}}$ is a small residual, not a good source for EMF) (Dhang et al 2020, Duez et al 2025)

Challenges in binary neutron stars: relevant MHD instabilities

- MRI: $\ell \sim v_A/\Omega$, difficult but possible. Central region is MRI stable ($d\Omega/dr > 0$)
- KH: $\mathbf{B}$ grows in vortices, faster for higher k, growth cannot be resolved even for $\Delta x \approx 17\,\mathrm{m}$ (Kiuchi et al 2015) unless LES subgrid closure terms are added (Palenzuela et al 2022) (LES terms calibrated to separate MHD turbulence simulations, but still uncertain).
- Tayler-Spruit instability (TSI): predicted for $N \gg \Omega \gg v_A/r$, $\ell_r \sim v_A/N$
  - stress uncertain by orders of magnitude, might be very fast (Margalit et al 2022)
  - not clear what to do in transition to weak stratification

Challenges in binary neutron stars: coefficient fitting

- A couple of high resolution simulations with SACRA code have been used to extract effective viscous stress, dynamo alpha coefficients (e.g. Kiuchi et al 2018). Effective mixing length fit as a function of density by Radice (2020). (Accounts for MRI stable center via density dependence.)

- Problem: fitting might not be general enough for other systems or other parts of evolution

- TSI expected to be present (Reboul-Salze et al 2024) but effects cannot yet be resolved.

- High resolution MHD simulations often lack detailed microphysics.

- How to do relativistic momentum transport? 3D? Israel-Stewart? BDNK?