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Radiative decay of the resonant K^* and the $\gamma K \to K\pi$ amplitude from lattice QCD

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(for the Hadron Spectrum Collaboration)

$\gamma K \to \pi K$ and the K^* resonance from lattice QCD

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current induced transitions to hadron-hadron resonances

for example

pion photoproduction, $\gamma N \to \pi N$ in which the Δ resonance appears

meson resonance production in semileptonic heavy-flavor decays, e.g. $B \to \ell\ell\, K^* \to \ell\ell\, K\pi$

or things not easily measurable but of theoretical interest, $\gamma\{\omega,\phi\} \to \{\pi\pi, K\bar{K}\}$

$f_0(980)$ flavor content & spatial size ?

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can compute with lattice QCD – **finite-volume** matrix elements from three-point functions

"large" finite-volume corrections controlled by the hadron-hadron scattering amplitude

complication of presence of multiple J^P owing to cubic boundary

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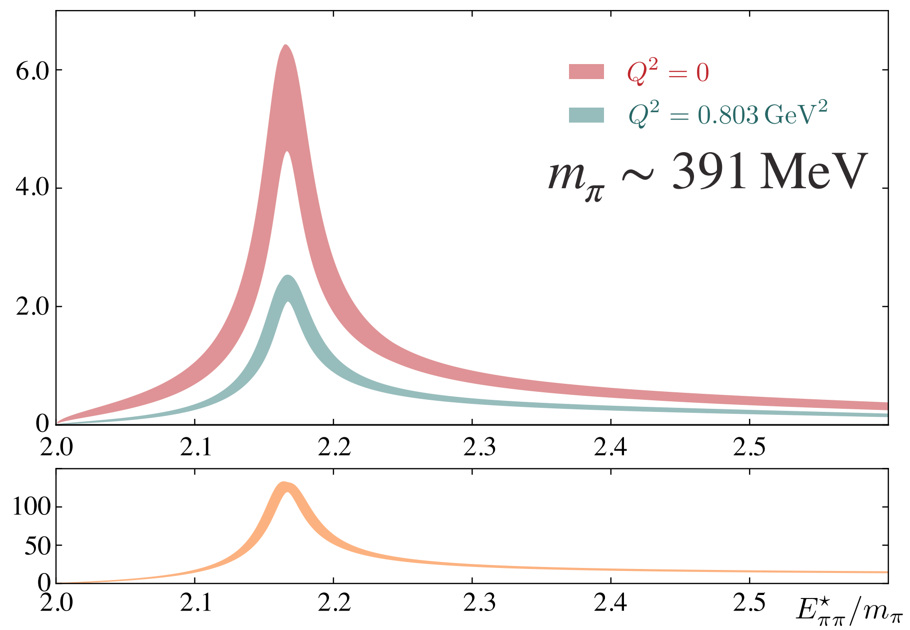
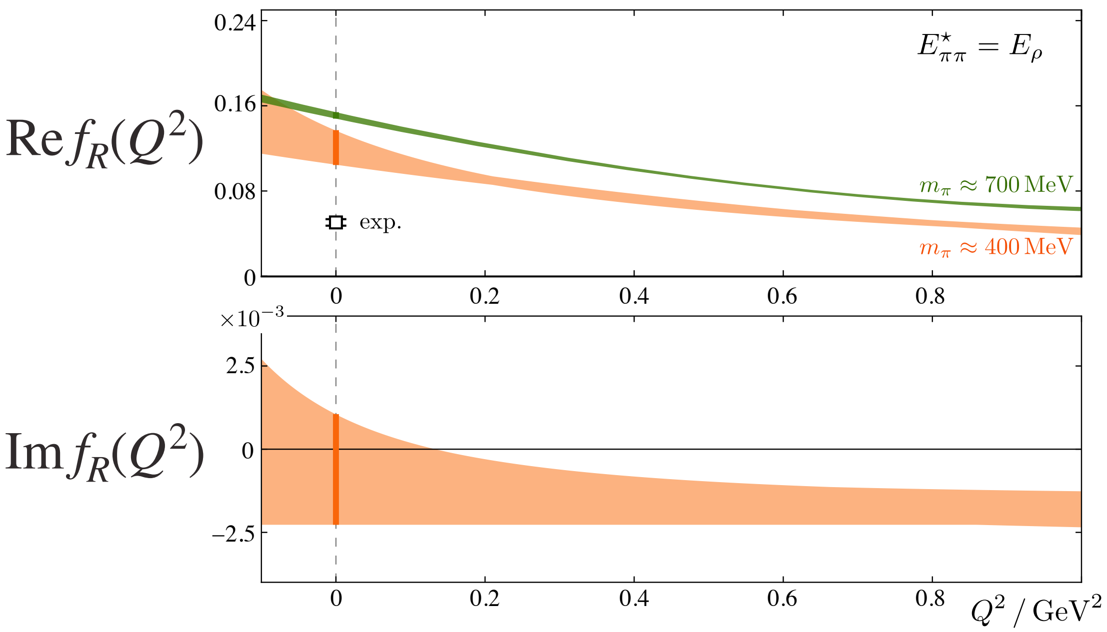
to date, only concrete application to $\gamma\pi \to \pi\pi$



analytic continuation to the ρ pole



but $\pi\pi$ is "special", no $J^P = 0^+$ with isospin=1, so $J^P = 1^-$ is always lowest partial wave

current induced transitions to hadron-hadron resonances

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complication of presence of multiple J^P owing to cubic boundary

next simplest case $\gamma K \to \pi K$

πK with isospin=½ : 0^+ ("κ"), 1^- (K^*), …

no amplitude $\gamma K \to (\pi K)_{0^+}$ but still an effect from 0^+ in finite-volume …

resonance transition form-factors

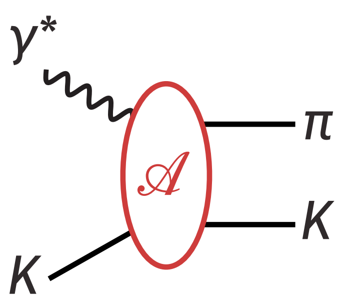
the process of interest is

current + stable hadron → resonance → hadron–hadron pair

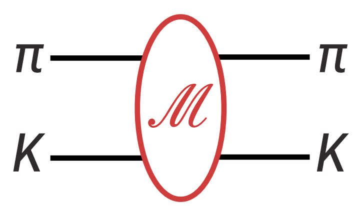
actually don't really need there to be a resonance

e.g. $\gamma K \to \pi K$ in a P-wave

after the current produces $K\pi$ …



… $K\pi$ strongly rescatters



$$\mathscr{H}(Q^2, E^\star_{K\pi}) \equiv \langle K \,|\, j \,|\, K\pi; E^\star_{K\pi} \rangle$$

suspressing kinematic variables, helicity and lorentz indices

$$= \mathscr{A}\big(Q^2, E^\star_{K\pi}\big) \cdot \frac{1}{k^\star_{K\pi}} \cdot \mathscr{M}^{\ell=1}(E^\star_{K\pi})$$

removing an 'excess' P-wave threshold factor

unitarity insists that production amplitude, $\mathscr{A}$, is **real** in the region of interest

(free of singularities, polynomial in $(E^\star_{K\pi})^2$)

Omnès function also an option here

★ means cm-frame

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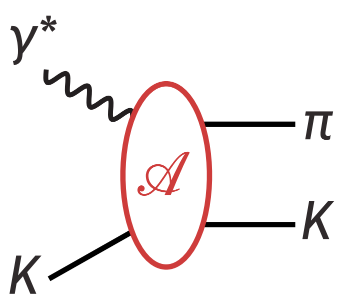

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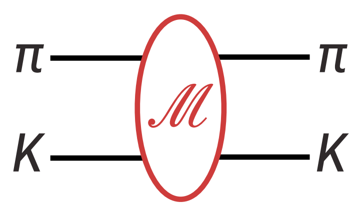

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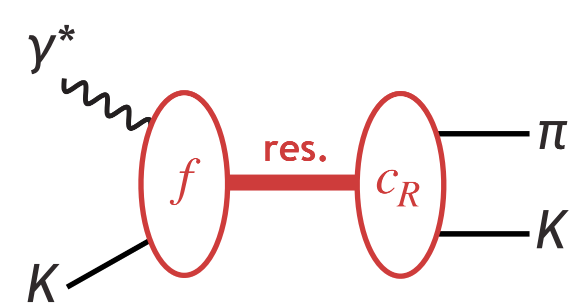
$$= \mathscr{A}\big(Q^2, E^\star_{K\pi}\big) \cdot \frac{1}{k^\star_{K\pi}} \cdot \mathscr{M}^{\ell=1}(E^\star_{K\pi})$$

strong scattering amplitude, $\mathscr{M}$, can have resonance poles $\quad \mathscr{M}^{\ell=1}(s) \sim \dfrac{c_R^2}{s_0 - s} \quad \sqrt{s_0} = m_R - i\tfrac{1}{2}\Gamma_R$

hence $\quad \mathscr{H}(Q^2, s) \sim \dfrac{c_R\, f(Q^2)}{s_0 - s}$

residue at the complex pole



lattice QCD means a finite-volume

infinite volume

continuum of scattering states

$\mathscr{M}(E^\star)$

finite volume

discrete spectrum of states

$E_n(L)$

$E_n(L)$ are solutions of

$$\det\left[F^{-1}(E^\star;L) + \mathscr{M}(E^\star)\right] = 0$$

kinematic finite-volume functions

spectra obtained from two-point correlation functions $C_{ij}(t) = \langle 0 | \mathscr{O}_i(t)\, \mathscr{O}_j^\dagger(0) | 0 \rangle$

evaluate with a large basis of operators to form a matrix

and diagonalize $\mathbf{C}(t)\, v_n = \lambda_n(t,t_0)\, \mathbf{C}(t_0)\, v_n$

eigenvalues given energies

$$\lambda_n(t,t_0) \sim e^{-E_n(t-t_0)}$$

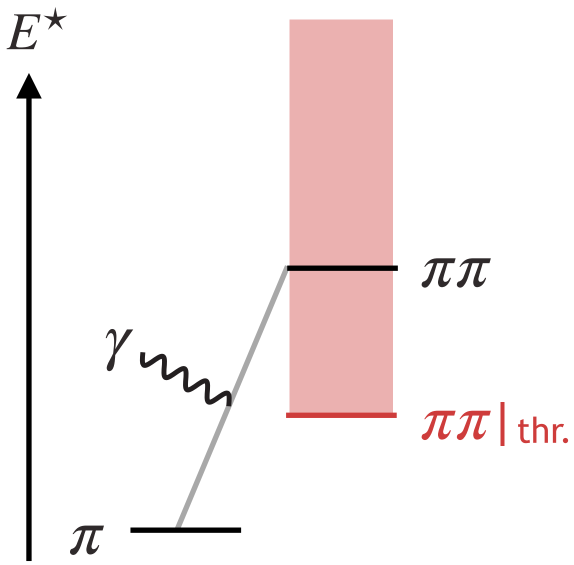
eigenvectors give **optimal operators**

$$\Omega_n \sim \sum_i (v_n)_i\, \mathscr{O}_i$$

produce just one state in the 'tower'

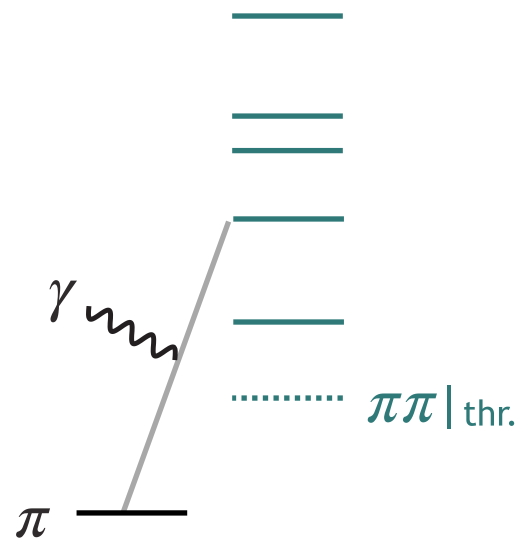
current matrix-elements in a finite-volume – cartoon

infinite volume



can transition
to any energy
in the $\pi\pi$ continuum

finite volume



can only transition
to one of the discrete
f.v. eigenstates

finite-volume matrix element

$${}_L\langle \pi | j | \pi\pi; E_n^\star \rangle_L$$

single hadron state

$$|\pi\rangle_L \sim |\pi\rangle_\infty + \mathcal{O}(e^{-m_\pi L})$$

hadron-hadron state

$$|\pi\pi; E_n^\star\rangle_L \sim \sqrt{\tilde{R}_n}\, |\pi\pi; E_{\pi\pi}^\star = E_n^\star\rangle_\infty$$

effective f.v. normalization

c.f. "Lellouch-Lüscher" factor

$$\tilde{R}_n(L) \equiv 2E_n \cdot \lim_{E \to E_n} (E - E_n)\Big(F^{-1}(E^\star; L) + \mathcal{M}(E^\star)\Big)^{-1}$$

effective f.v. normalization
depends on the scattering
amplitude

what's different in $\gamma K \to \pi K$?

cubic nature of lattice puts spectra in **irreducible representations** of a **reduced group of rotations**

in $\pi\pi$ case, this has limited impact because even and odd ℓ are in different isospins

consequence of Bose symmetry

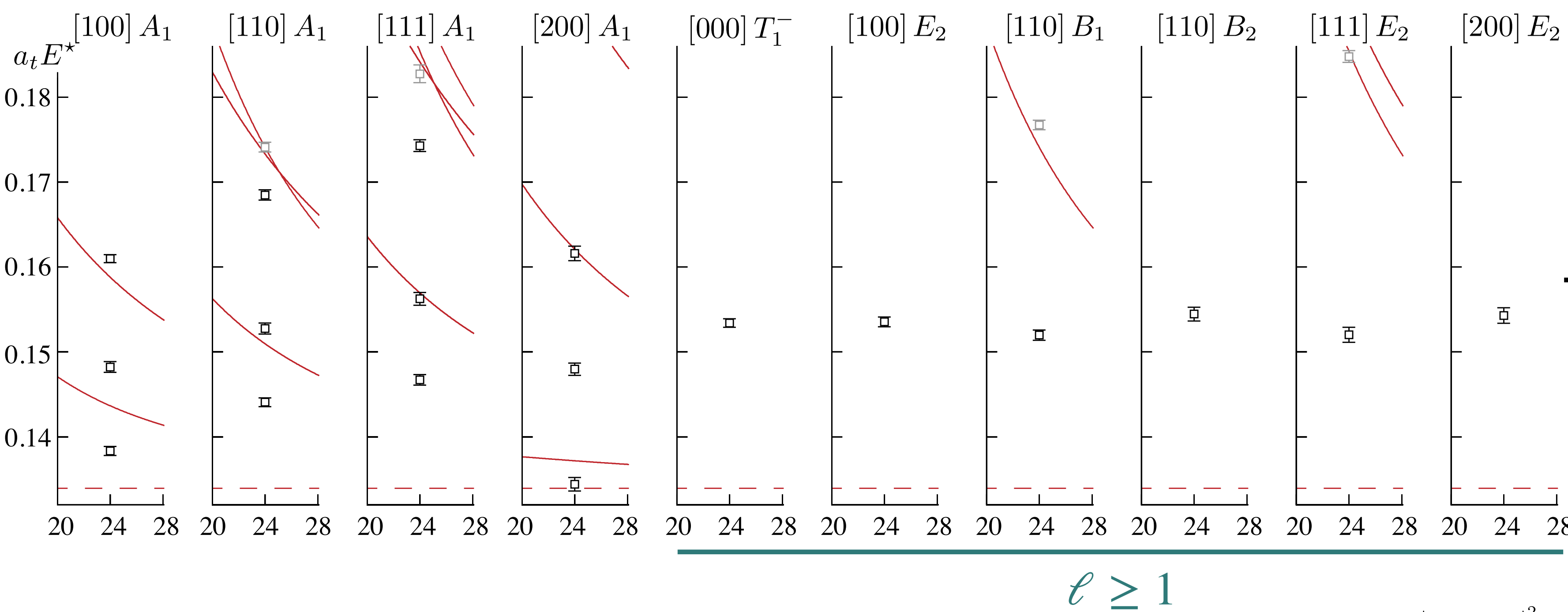
in πK case, there is no Bose symmetry

$\mathbf{p}_{K\pi}\ \Lambda$	$[000]A_1^+$	$[000]T_1^-$	$[100]A_1$	$[100]E_2$	$[110]A_1$	$[110]B_1$	$[110]B_1$	$[111]A_1$	$[111]E_2$	$[200]A_1$
$\ell \leq 2$	0	1	$0,1,2$	$1,2$	$0,1,2$	$1,2$	$1,2$	$0,1,2$	$1,2$	$0,1,2$

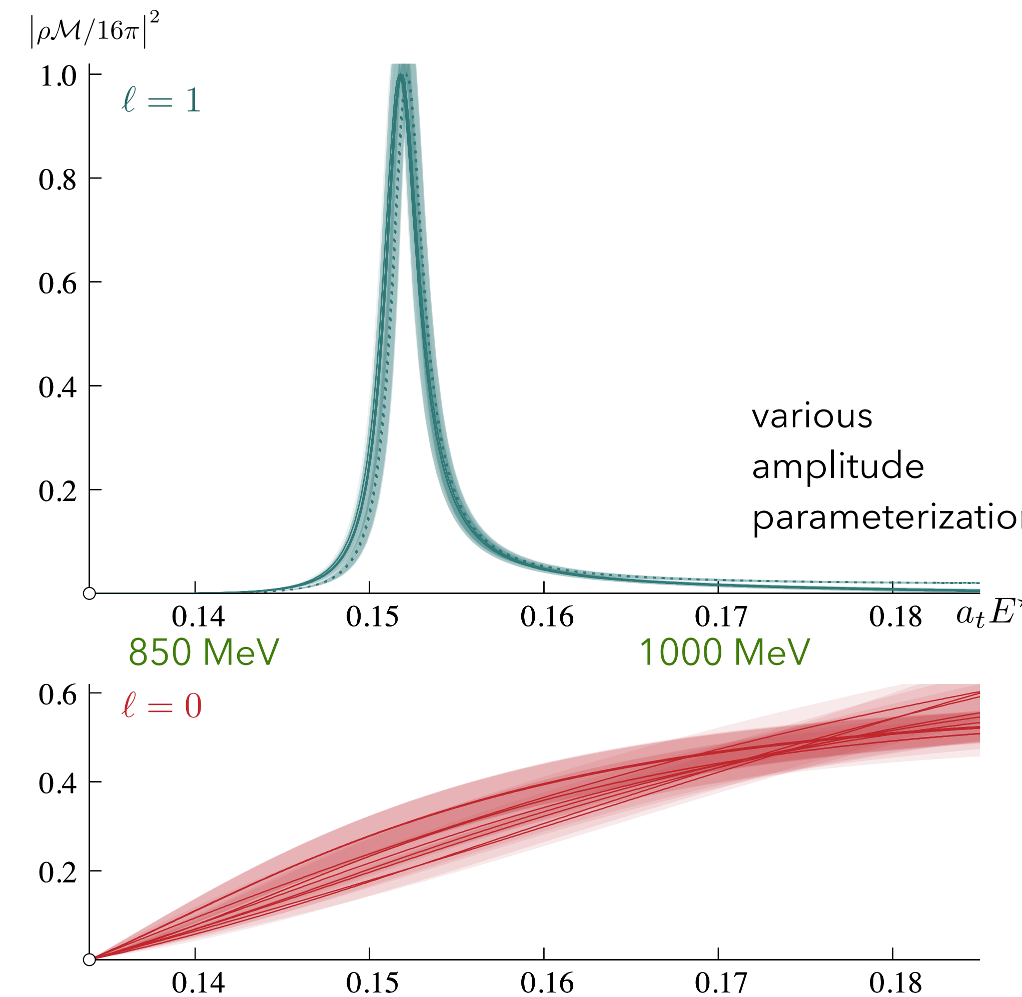
spectrum in some irreps sensitive to scattering in both $\ell = 0, \ell = 1$

finite-volume spectrum → S,P-wave amplitudes

$m_\pi \sim 284\,\mathrm{MeV}$



$$\det\left[F^{-1}(E^\star; L) + \mathcal{M}(E^\star)\right] = 0$$



various amplitude parameterizations

$\ell = 2$ found to be negligible in this energy region relation between finite-volume matrix element, and infinite-volume matrix element, $\mathscr{H}$

$$\left| {}_L\langle K | j | K\pi \rangle_L \right| \propto \left(\mathscr{H} \cdot \tilde{R}_n \cdot \mathscr{H} \right)^{1/2}$$

where the residue of the finite-volume hadron-hadron propagator appears

$$\tilde{R}_n(L) \equiv 2E_n \cdot \lim_{E \to E_n} \left(E - E_n \right) \Big(\underbrace{F^{-1}(E^\star; L)}_{\text{matrix in } \ell = 0,1} + \underbrace{\mathscr{M}(E^\star)}_{\text{diagonal matrix in } \ell = 0,1} \Big)^{-1}$$

$E_n(L)$ are solutions of

$$\det \left[F^{-1}(E^\star; L) + \mathscr{M}(E^\star) \right] = 0$$

what's different in $\gamma K \to \pi K$?

relation between finite-volume matrix element, and infinite-volume matrix element, $\mathscr{H}$

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using an eigen-decomposition $F + \mathscr{M}^{-1} = \sum_i \mu_i \, \mathbf{w}_i \mathbf{w}_i^{\intercal}$ $\qquad \mathbf{w}_i = \begin{pmatrix} \mathbf{w}_i^{\ell=0} \\ \mathbf{w}_i^{\ell=1} \end{pmatrix}$

the residue factorizes $\tilde{R}_n = \left(-\frac{2E_n^\star}{\underbrace{\mu_0^{\star\prime}}_{\text{slope of zero crossing eigenvalue}}} \right) \mathscr{M}^{-1} \mathbf{w}_0 \, \underbrace{\mathbf{w}_0^{\intercal}}_{\text{zero crossing eigenvector}} \mathscr{M}^{-1}$

relation between finite-volume matrix element, and infinite-volume matrix element, $\mathscr{H}$

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the residue factorizes $\tilde{R}_n = \left(-\frac{2E_n^\star}{\mu_0^{\star\prime}} \right) \mathscr{M}^{-1} \mathbf{w}_0 \, \mathbf{w}_0^\intercal \mathscr{M}^{-1}$

($\mu_0^{\star\prime}$: slope of zero crossing eigenvalue; $\mathbf{w}_0$: zero crossing eigenvector)

and the net finite-volume correction is $F(Q^2, E_{K\pi}^\star = E_n^\star) = \frac{1}{\tilde{r}_n(L)} F_L(Q^2, E_n^\star)$

remember, no $\gamma K \to (K\pi)_{\ell=0}$ amplitude

$$\mathscr{H} = \mathscr{A} \cdot \frac{1}{k_{K\pi}^\star} \cdot \mathscr{M}^{\ell=1}$$

$$\mathscr{A} = \underbrace{K}_{\text{kinematic factor}} \cdot \underbrace{F}_{\text{form-factor}}$$

where $\tilde{r}_n(L) = \sqrt{-\frac{2E_n^\star}{\mu_0^{\star\prime}}} \left| \mathbf{w}_0^{\ell=1} \right| \frac{1}{k_{K\pi}^\star}$

evaluating $\gamma K \to \pi K$?

$$F(Q^2, E^\star_{K\pi} = E^\star_n) = \frac{1}{\tilde{r}_n(L)} F_L(Q^2, E^\star_n)$$

extract finite-volume form-factor, $F_L(Q^2, E^\star_n)$, from lattice QCD computed three-point functions

compute the finite-volume corrections, $\tilde{r}_n(L)$, using lattice QCD obtained scattering amplitudes

$$\tilde{r}_n(L) = \sqrt{-\frac{2E^\star_n}{\mu_0^{\star\,\prime}}} \left| \mathbf{w}_0^{\ell=1} \right| \frac{1}{k^\star_{K\pi}}$$

$$\langle 0 \,|\, \Omega_K(\mathbf{p}_K, \Delta t)\, j(\mathbf{q}, t)\, \Omega^\dagger_{K\pi}(\mathbf{p}_{K\pi}, 0) \,|\, 0 \rangle = e^{-E_K(\Delta t - t)}\, e^{-E_n t} \cdot K \cdot F_L(Q^2, E_n^\star) + \ldots\,,$$

just a single $\Delta t = 32\, a_t$

a range of kaon and current three-momenta
for each kaon-pion discrete energy level

three-point functions – our kinematical coverage

$$\langle 0 | \Omega_K(\mathbf{p}_K, \Delta t)\, j(\mathbf{q}, t)\, \Omega^\dagger_{K\pi}(\mathbf{p}_{K\pi}, 0) | 0 \rangle = e^{-E_K(\Delta t - t)}\, e^{-E_n t} \cdot K \cdot F_L(Q^2, E^\star_n) + \dots ,$$

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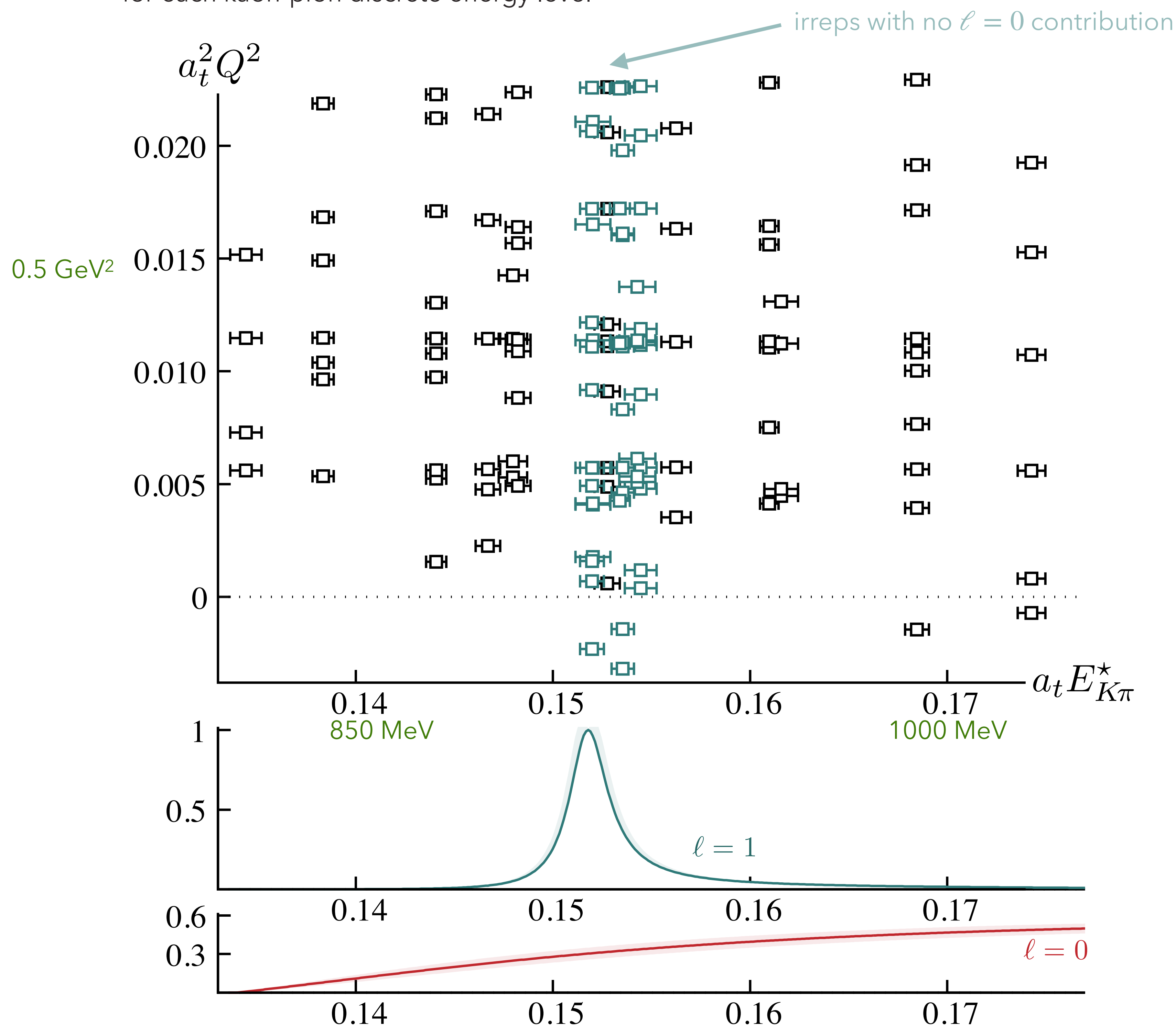
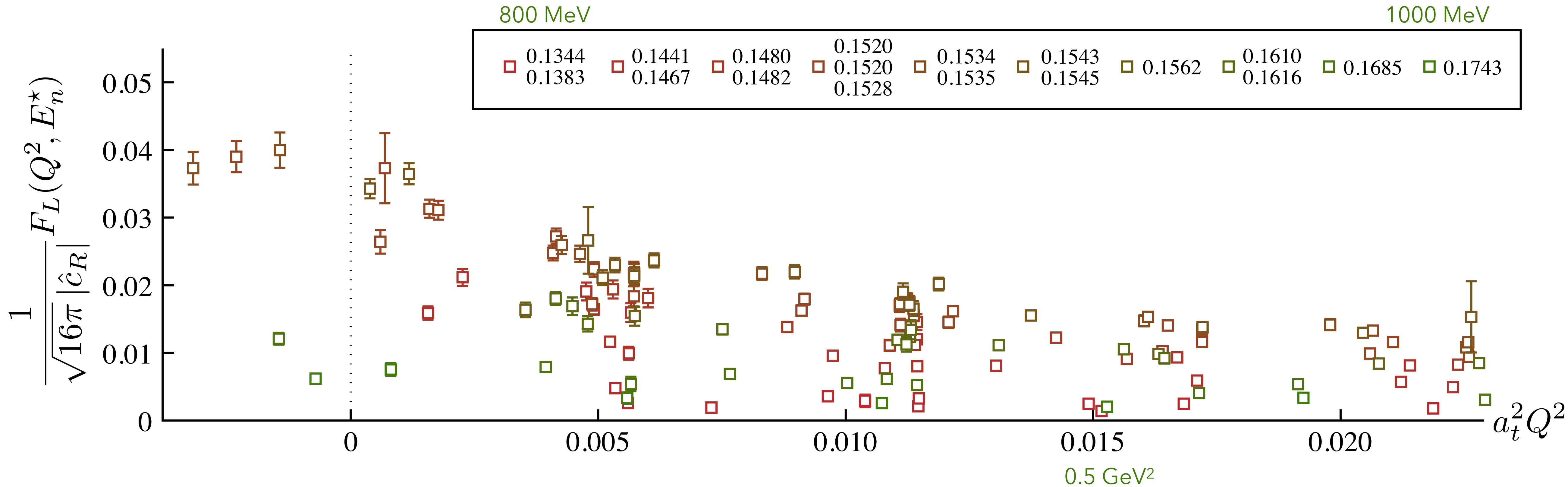


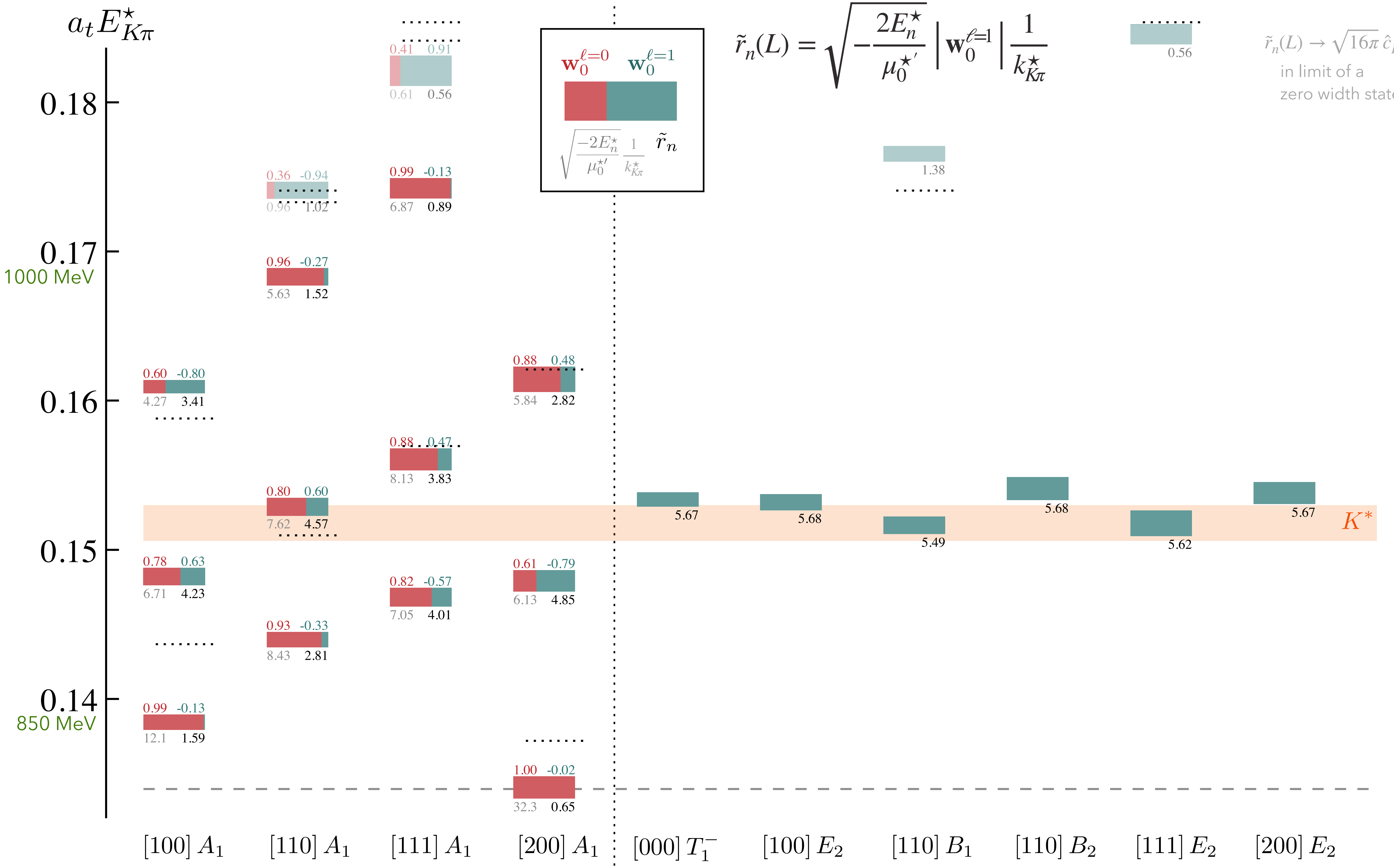
finite-volume form-factor



finite-volume correction factors

$$F(Q^2, E^\star_{K\pi} = E^\star_n) = \frac{1}{\tilde{r}_n(L)} F_L(Q^2, E^\star_n)$$

$$\tilde{r}_n(L) = \sqrt{-\frac{2E^\star_n}{\mu_0^{\star\prime}}}\left|\mathbf{w}_0^{\ell=1}\right|\frac{1}{k^\star_{K\pi}}$$



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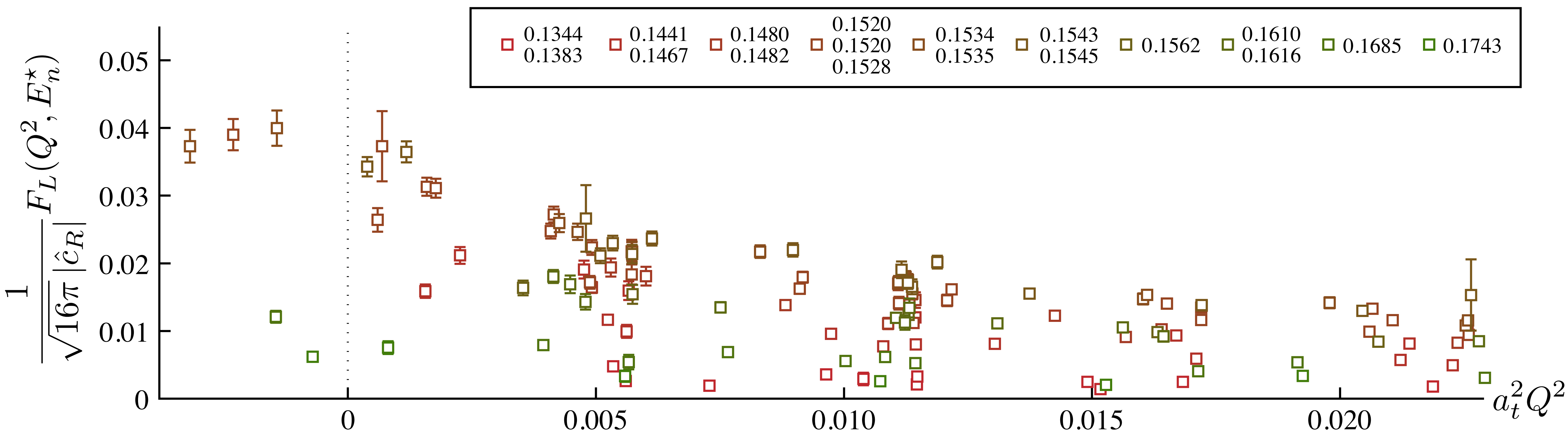
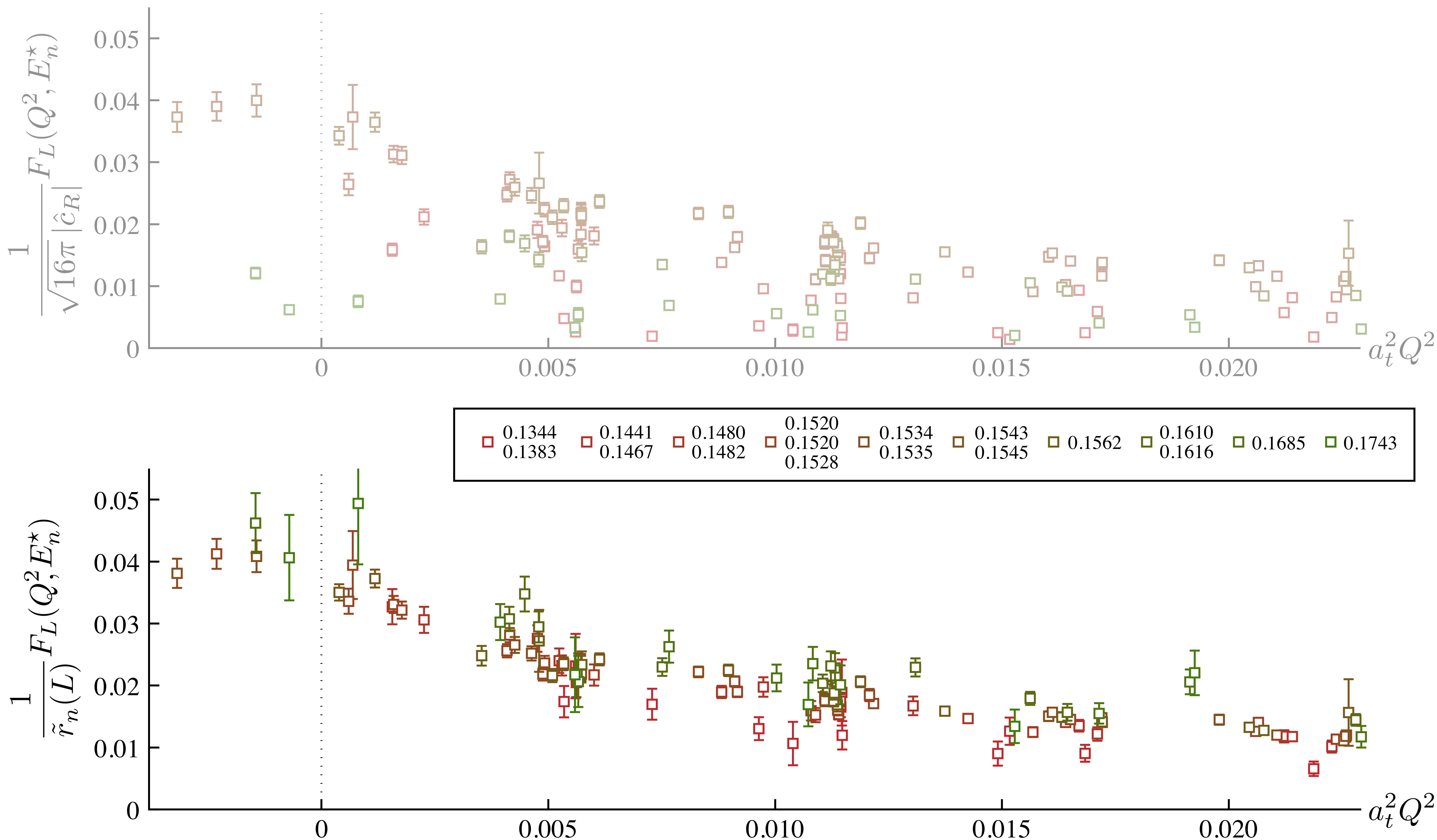
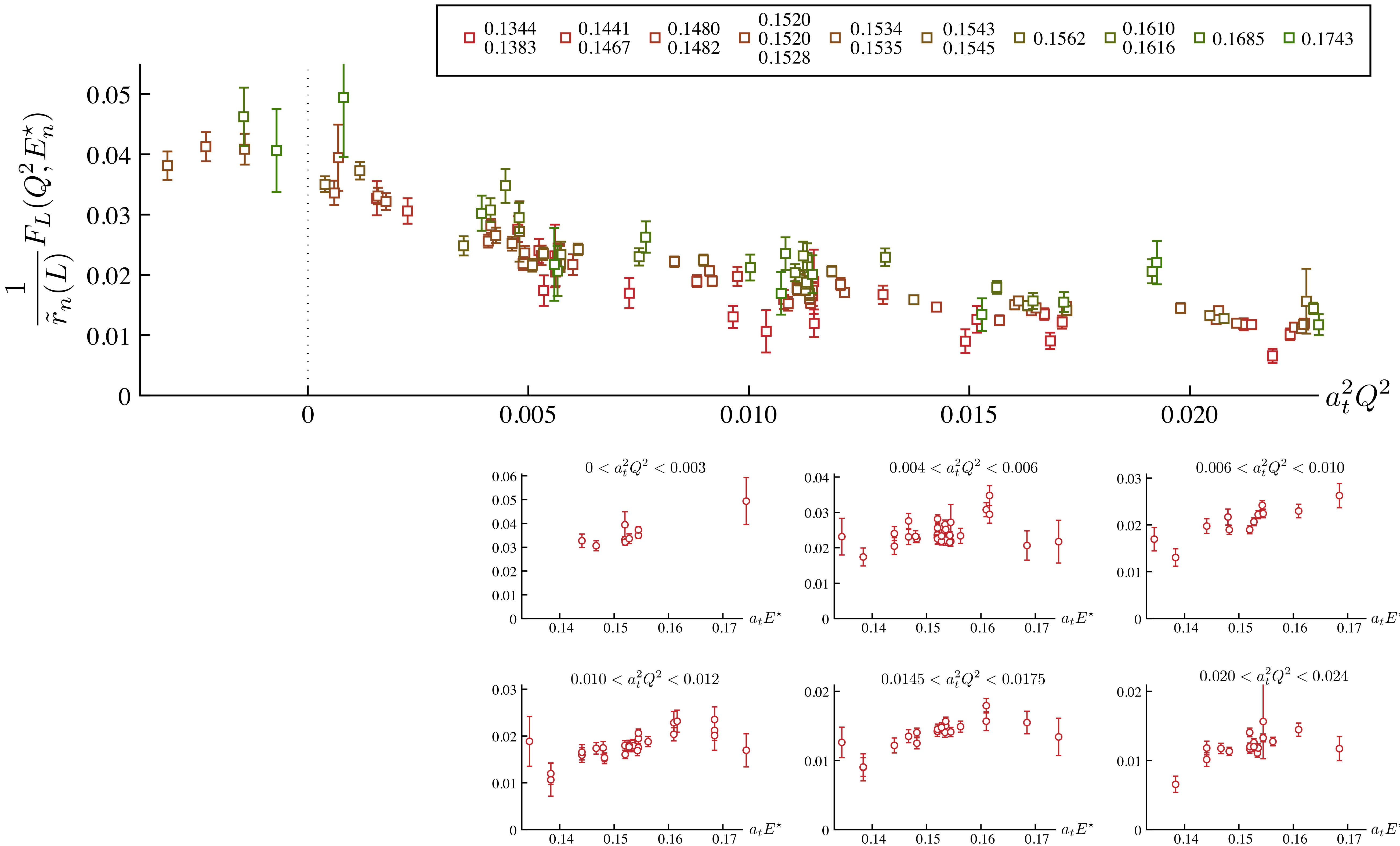


infinite-volume form-factor



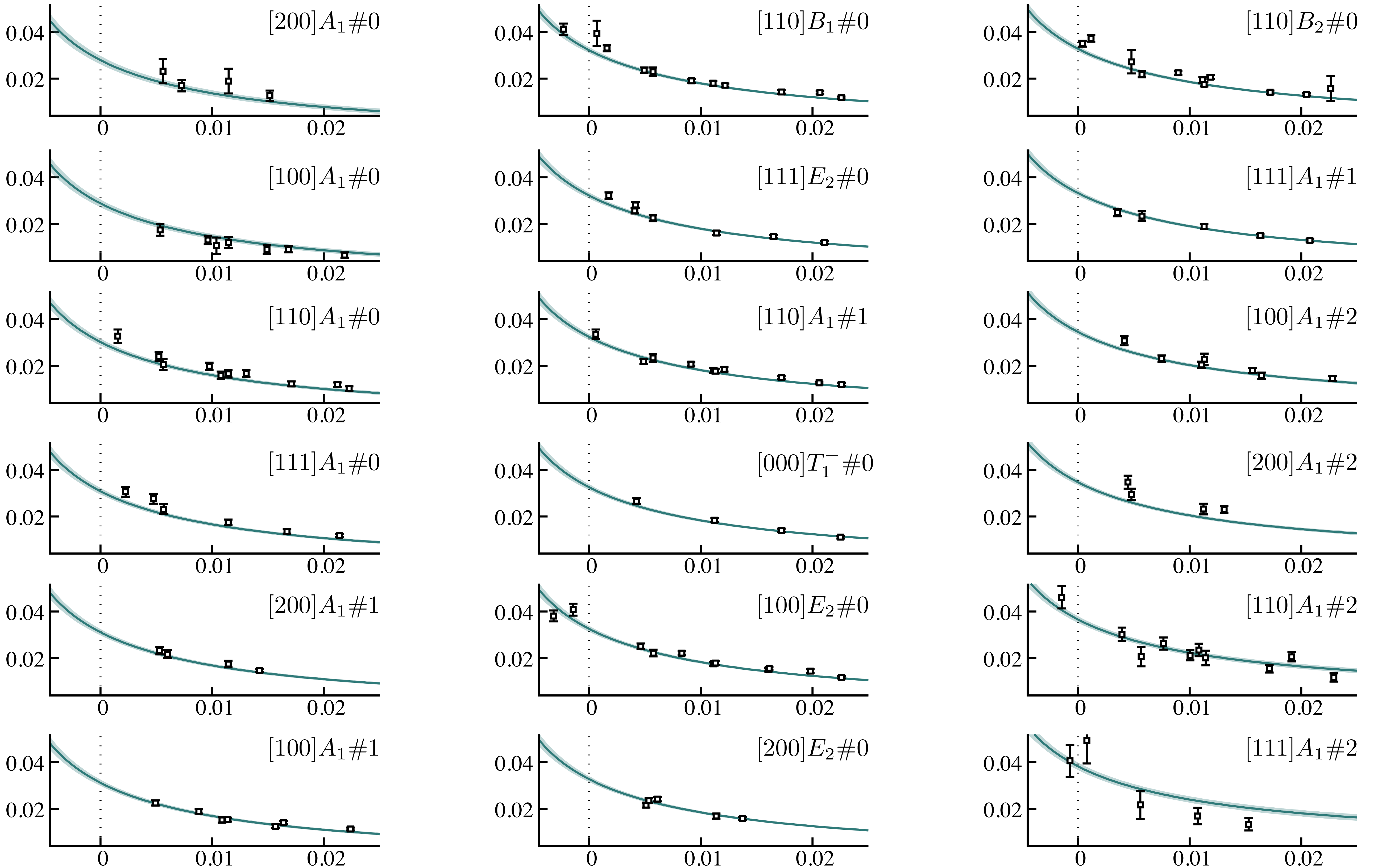
infinite-volume form-factor



modest energy dependence as expected

global fitting of all the infinite-volume form-factor data

energy dependent conformal mapping fit here $F(Q^2, s) = \left(b_{0,0} + b_{0,1}\frac{s-s_0}{s_0}\right) + b_{1,0}\cdot\left(z(Q^2)-z(0)\right) + b_{2,0}\cdot\left(z(Q^2)-z(0)\right)^2$ 128 data points, 4 free params



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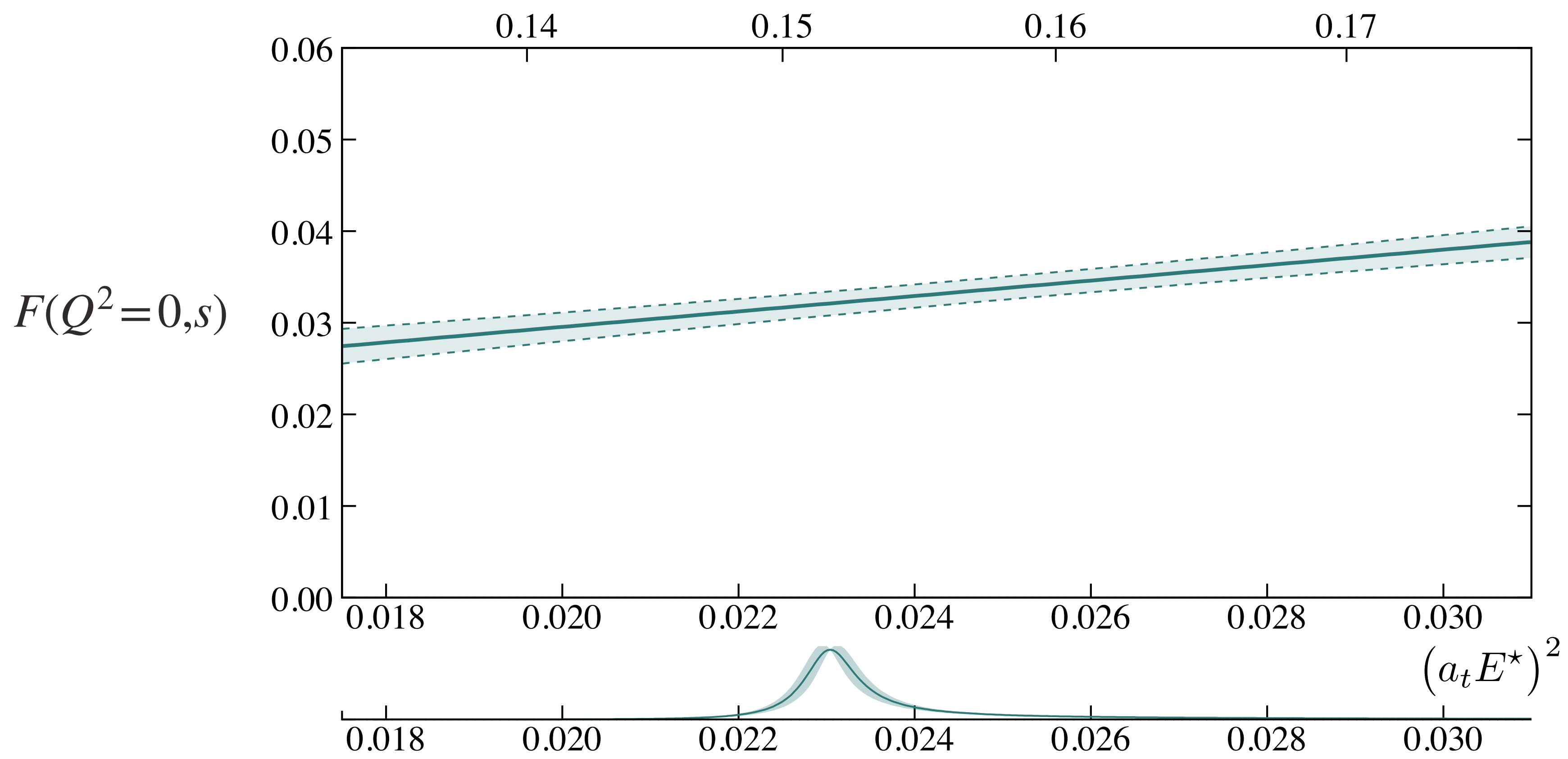

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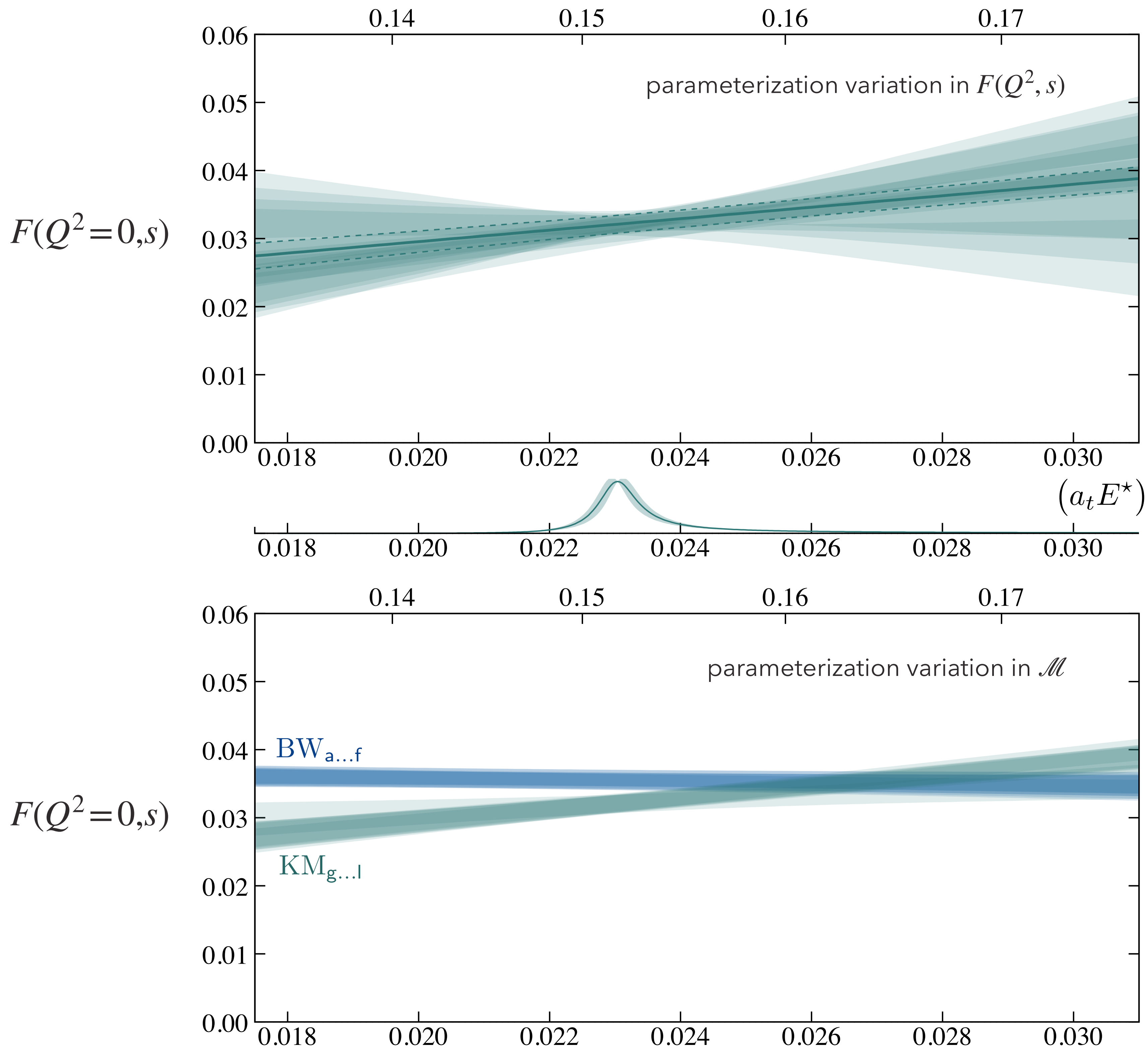
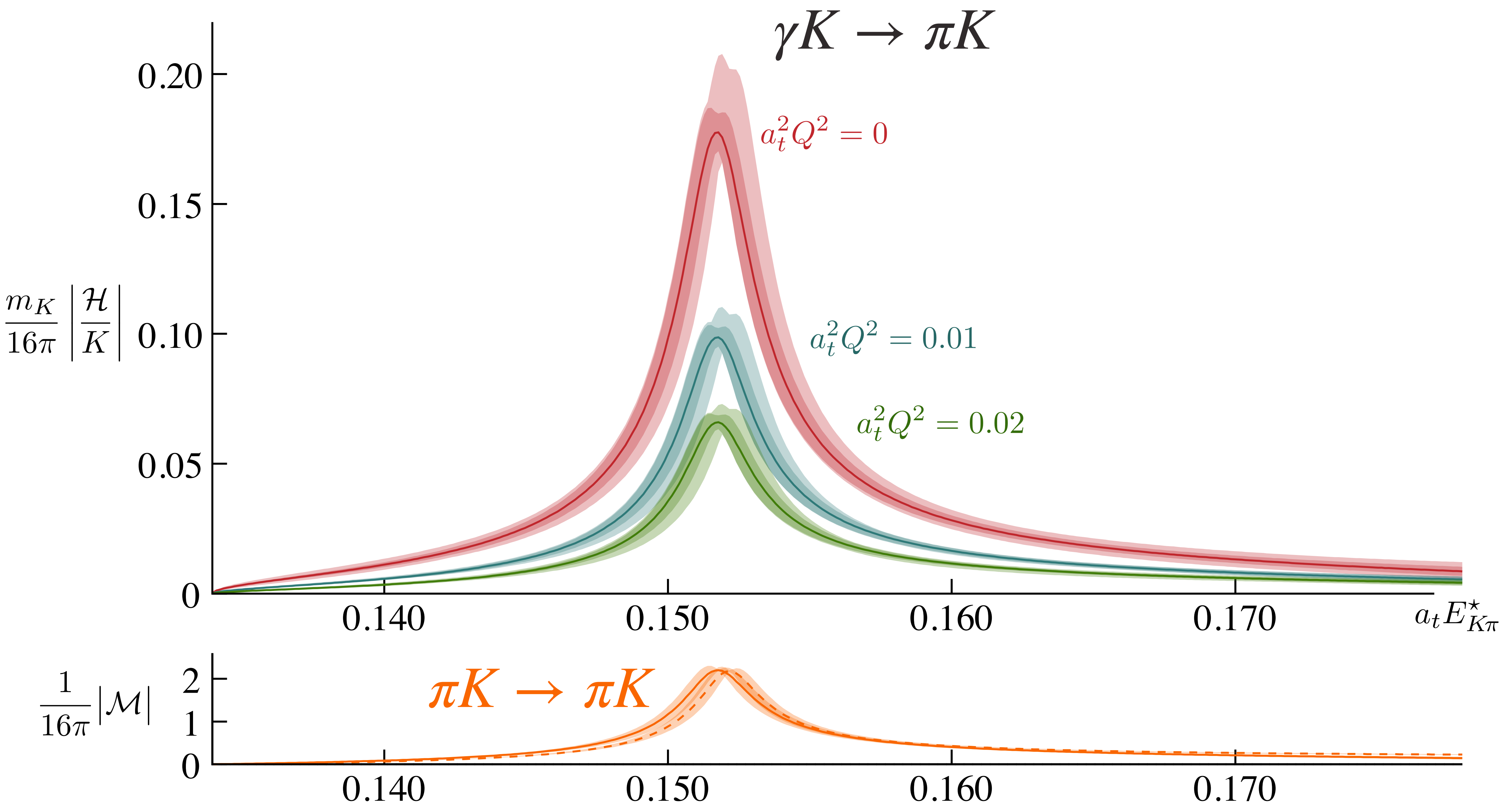
parameterization variation



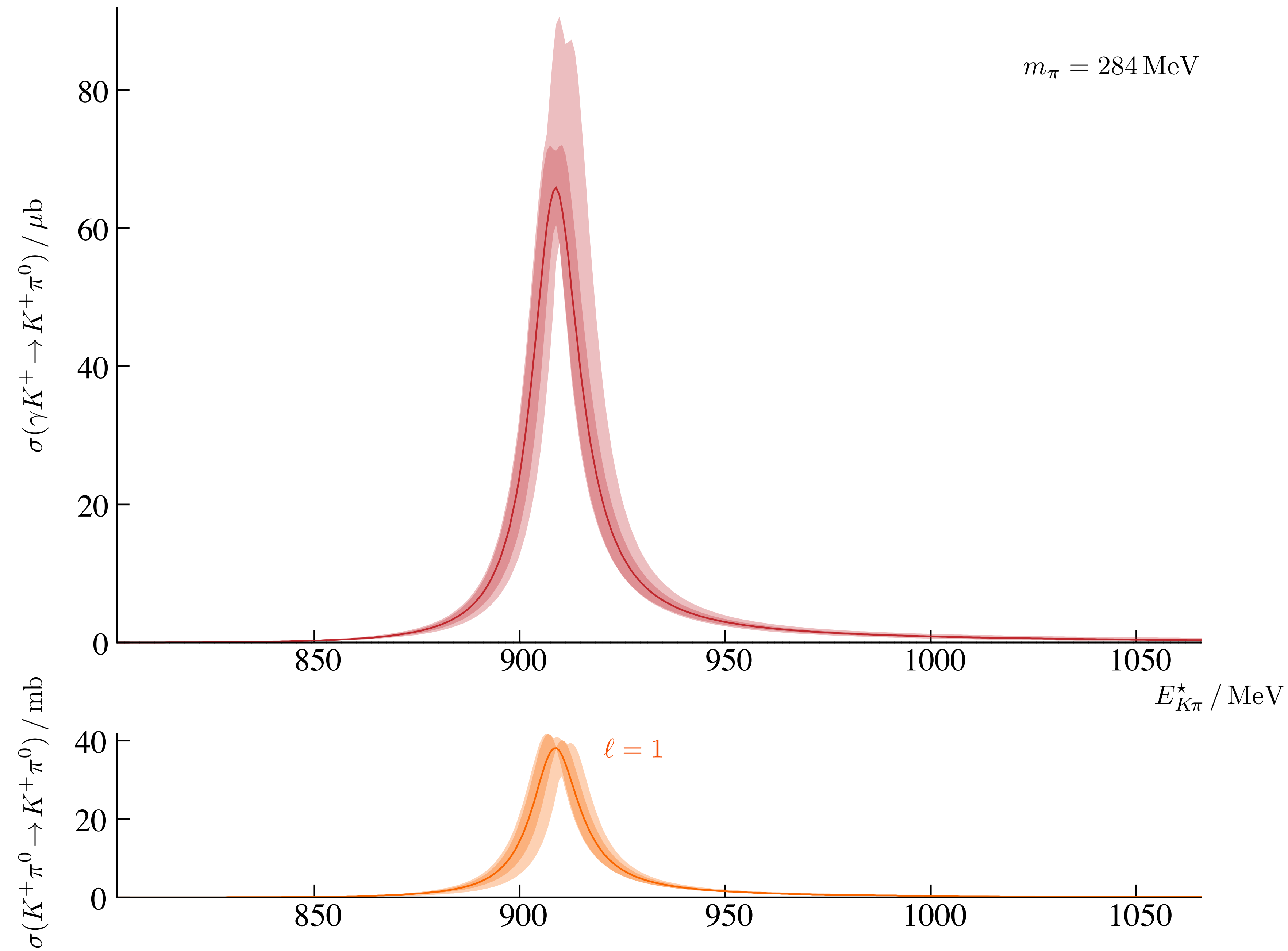
the transition amplitude



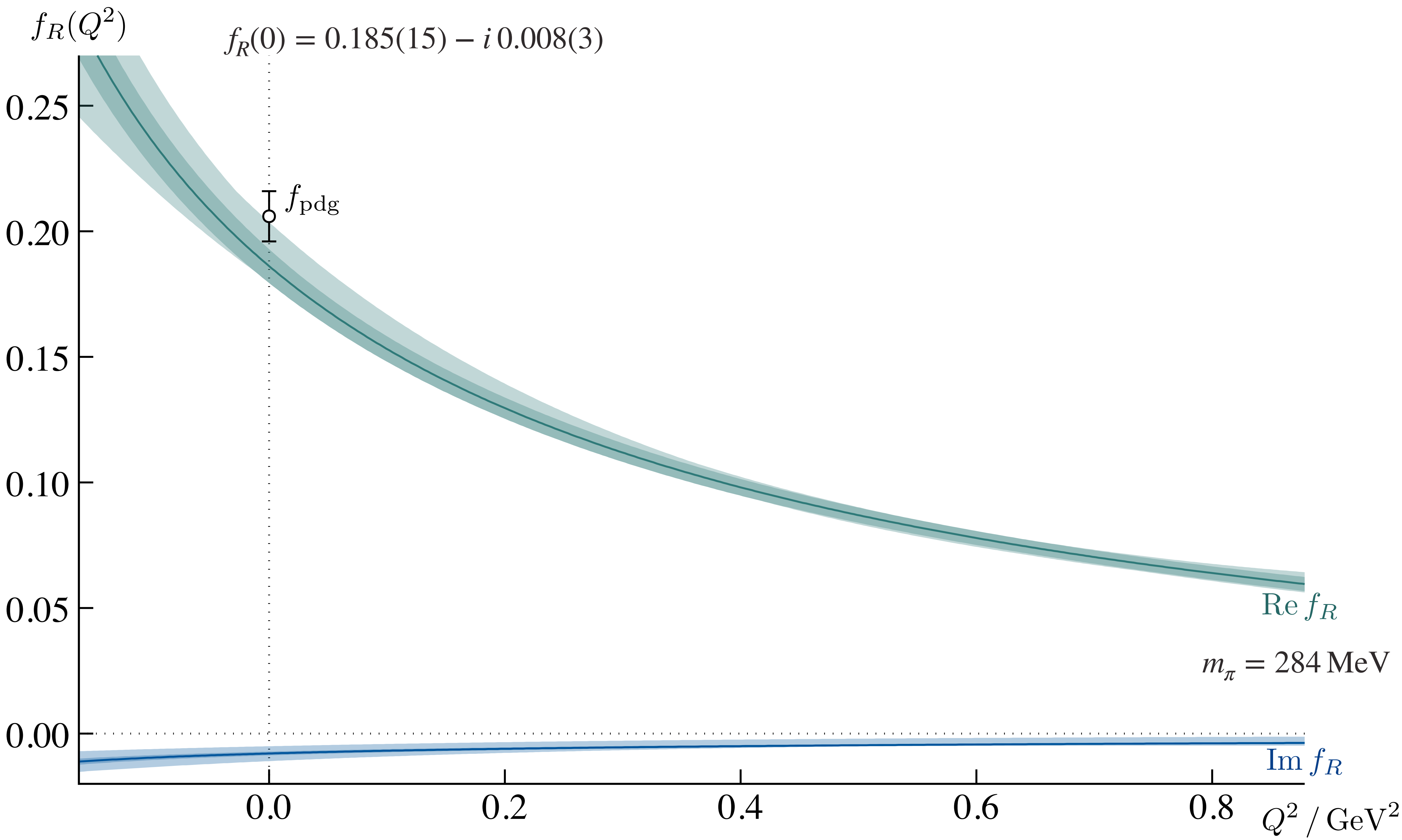
real photon cross-section

$$\left|\mathscr{H}\left(\gamma K^+ \to K^+\pi^0\right)\right| = \frac{1}{\sqrt{3}}\left|\mathscr{H}\left(\gamma K^+ \to (K\pi)_{1/2,+1/2}\right)\right|.$$

$$\sigma\left(\gamma K^+ \to K^+\pi^0\right) = \frac{1}{3}\alpha\frac{k^\star_{K\gamma}}{k^\star_{K\pi}}\frac{1}{m_K^2}\left|F\mathscr{M}\right|^2$$

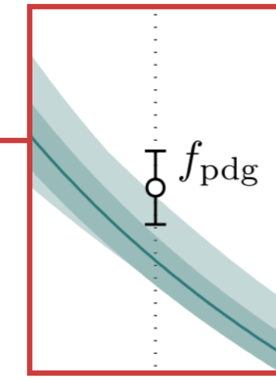


resonance transition form-factor

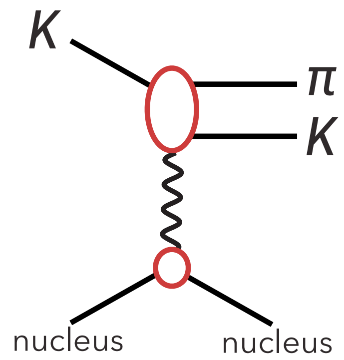


$$\mathcal{H}(Q^2, s) \sim \frac{c_R\, f(Q^2)}{s_0 - s}$$

experimental determination



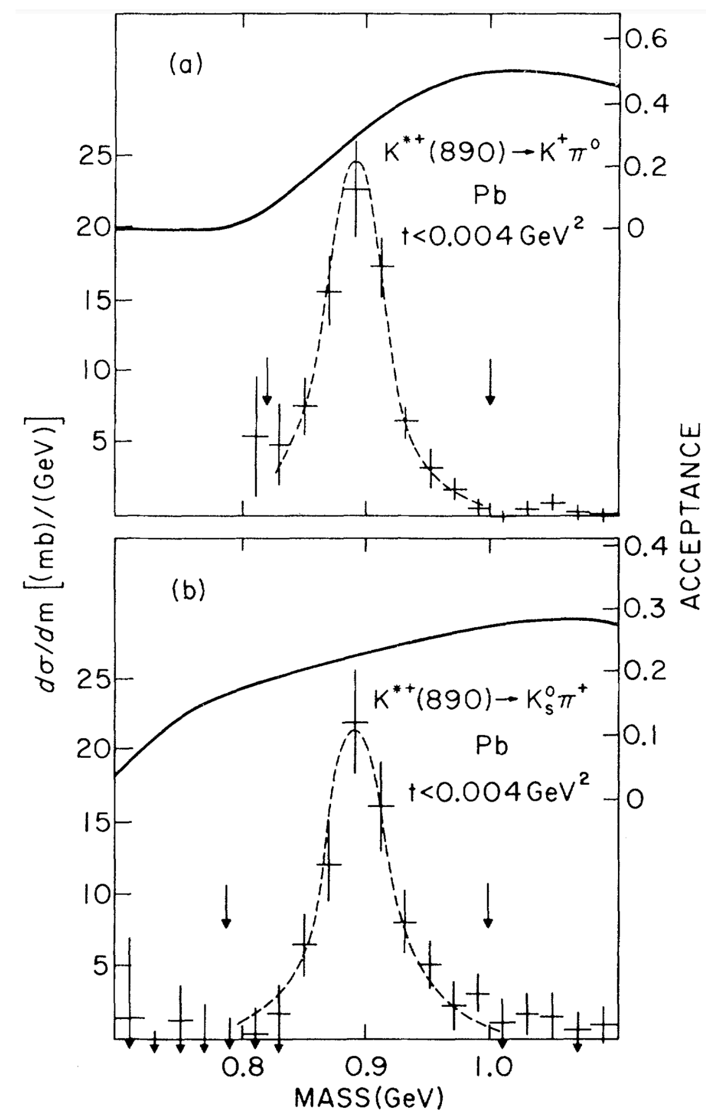
handful of Primakoff experiments in the 70s, 80s



(very forward production of πK using $K^{\pm}, K_L^0$ beams on nuclear targets)

pdg average of a couple of experiments $\Gamma(K^{*\pm} \to K^{\pm}\gamma) = 50(5)\,\text{keV}$

$\Gamma(K^{*0} \to K^{0}\gamma) = 116(10)\,\text{keV}$



very simplistic analysis scheme

$$\frac{d\sigma}{dt\,dm} = 3\pi\alpha Z^2 \frac{\Gamma_o}{k_o^3} \frac{t - t_{min,o}}{t^2} \,|f_{C_o}|^2 BW(m);$$

$$BW(m) = \frac{1}{\pi} \frac{m^2 \Gamma^{tot}}{[m^2 - m_o^2] + [m_o \Gamma^{tot}]^2} \left| \frac{g(k)}{g(k_o)} \right|^2$$

$$\Gamma(K^{*+} \to K^{+}\gamma) = \frac{4}{3}\alpha \frac{k_{K\gamma}^{\star 3}}{m_K^2} |f|^2$$

loss of rigor here
this is not the pole residue

$$|f_{\text{pdg}}| = 0.206(10)$$

summary

stress-tested the **1+J→2** finite-volume formalism in a case with an ‘unwanted’ lower partial wave

consistent **production amplitude** at 128 kinematic points, shows expected **mild energy dependence**

K^* **transition form-factor** extracted from scattering resonance pole,
reasonable ball-park agreement with experiment (considering computation at ‘wrong’ light quark mass)