Ab initio nuclear correction to the Lamb shift

Testing fundamental physics with light muonic atoms

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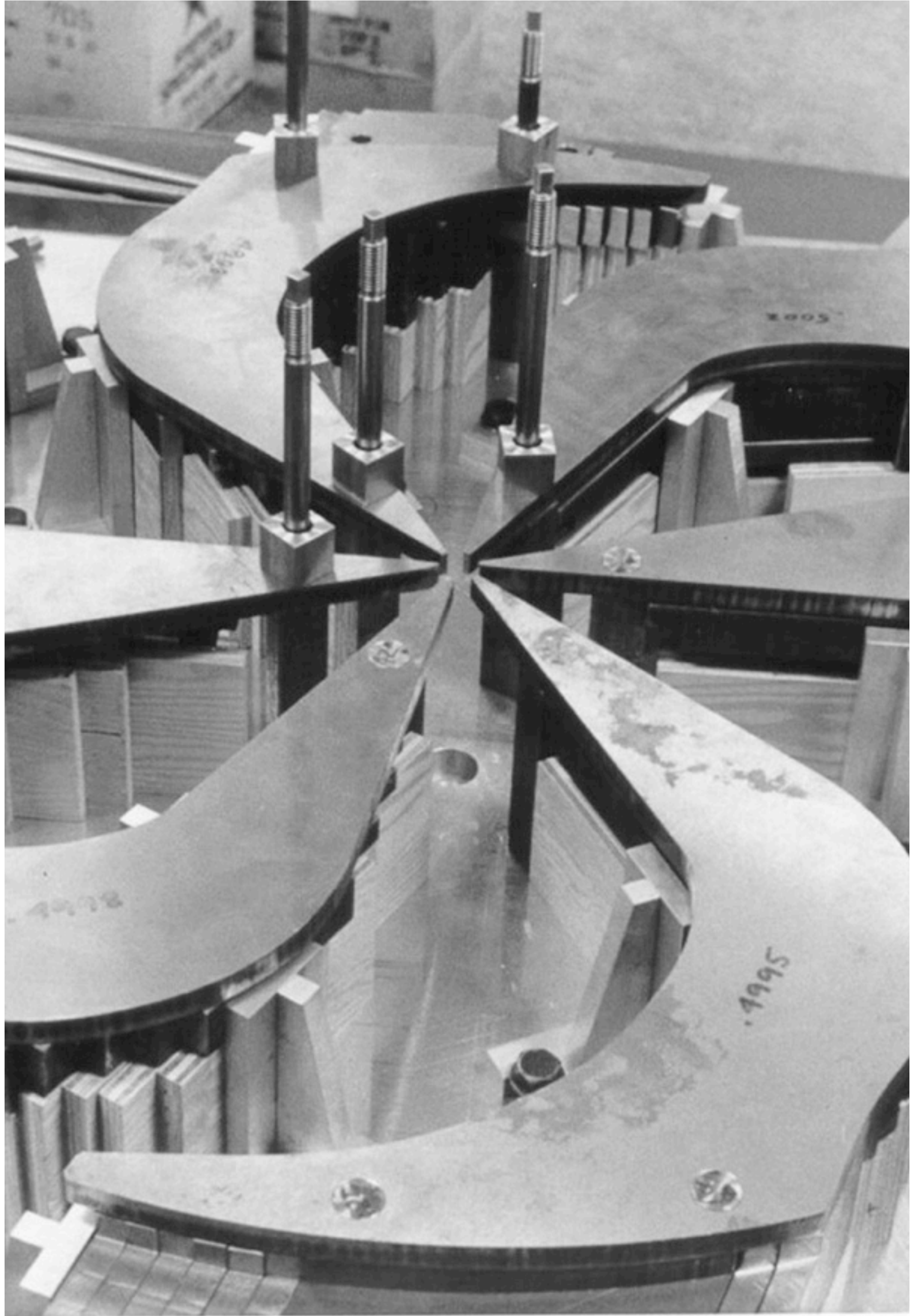
Fundamental Physics with Radioactive Molecules

INT Seattle - 15th of March 2023



Discovery, accelerated

The muonic Lamb Shift as a precision probe

A key probe to develop the Standard Model…

~1900 — **Observation of atomic spectra**

Discrete energy levels
Quantum theory

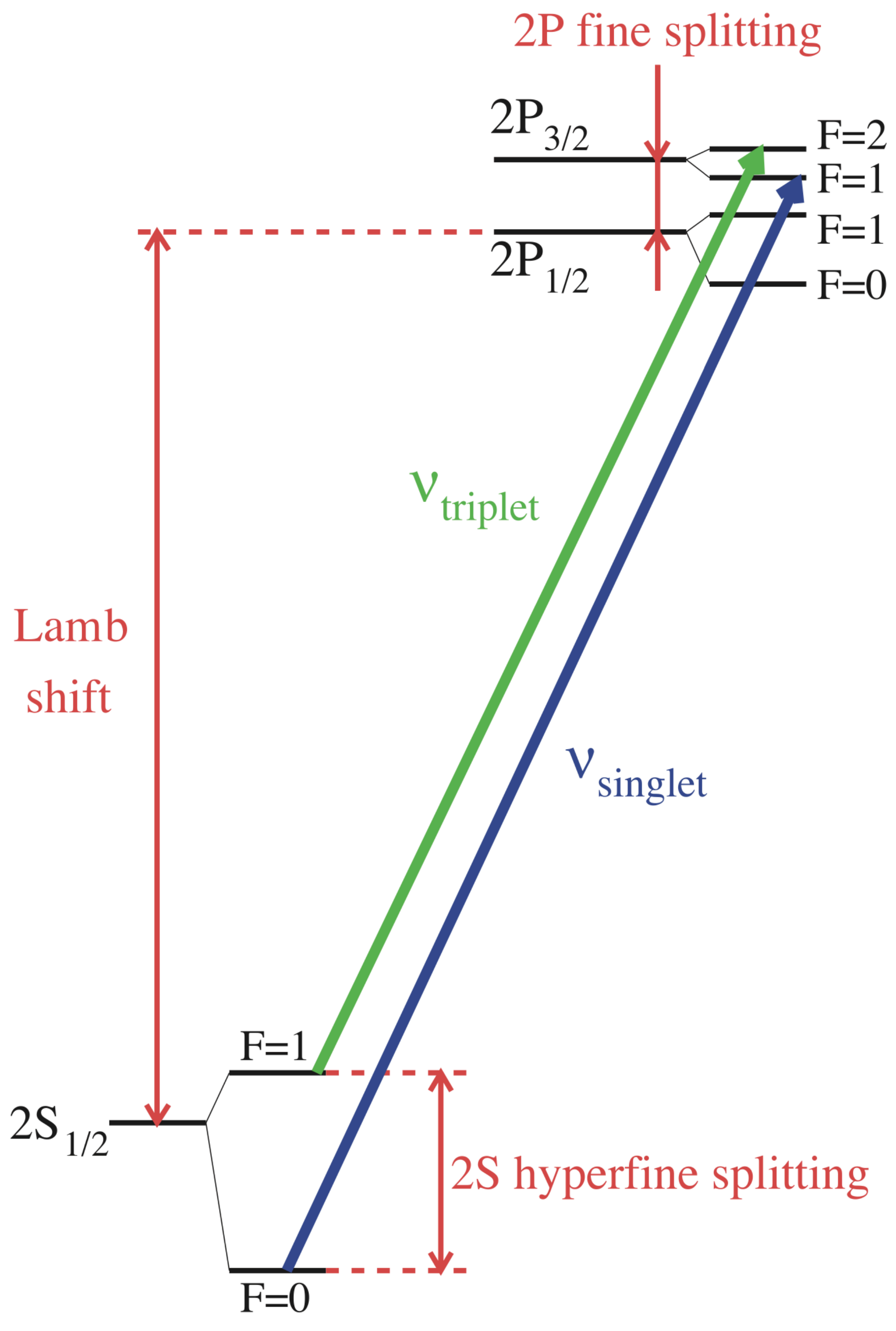
~1947 — **Observation of Lamb shift between $2S_{1/2} - 2P_{1/2}$ states**

Vacuum polarization effect
Renormalization theory

~1965 — **Quantum Electrodynamics**

Non-abelian generalization

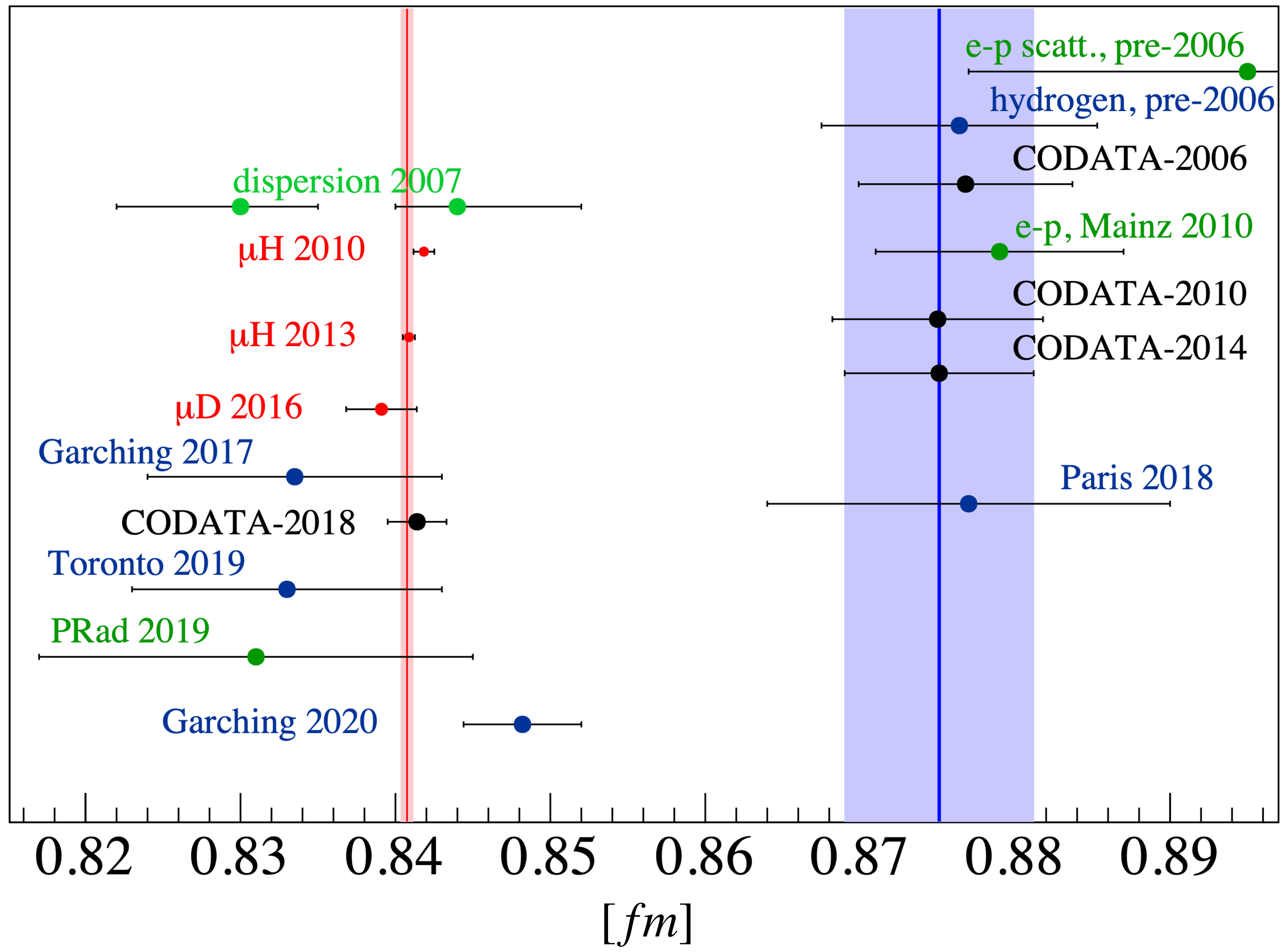
~1970 — **Standard Model of particle physics**



[Antognini et al, Science (2013)]

And much more !!

… and pushing the precision frontier further



- Precise measurement of proton radius:

[CODATA 2018]

$$r_p = 8.414(19) \times 10^{-16}\mathrm{m}$$

- Rydberg constant re-evaluation:

[CODATA 2018]

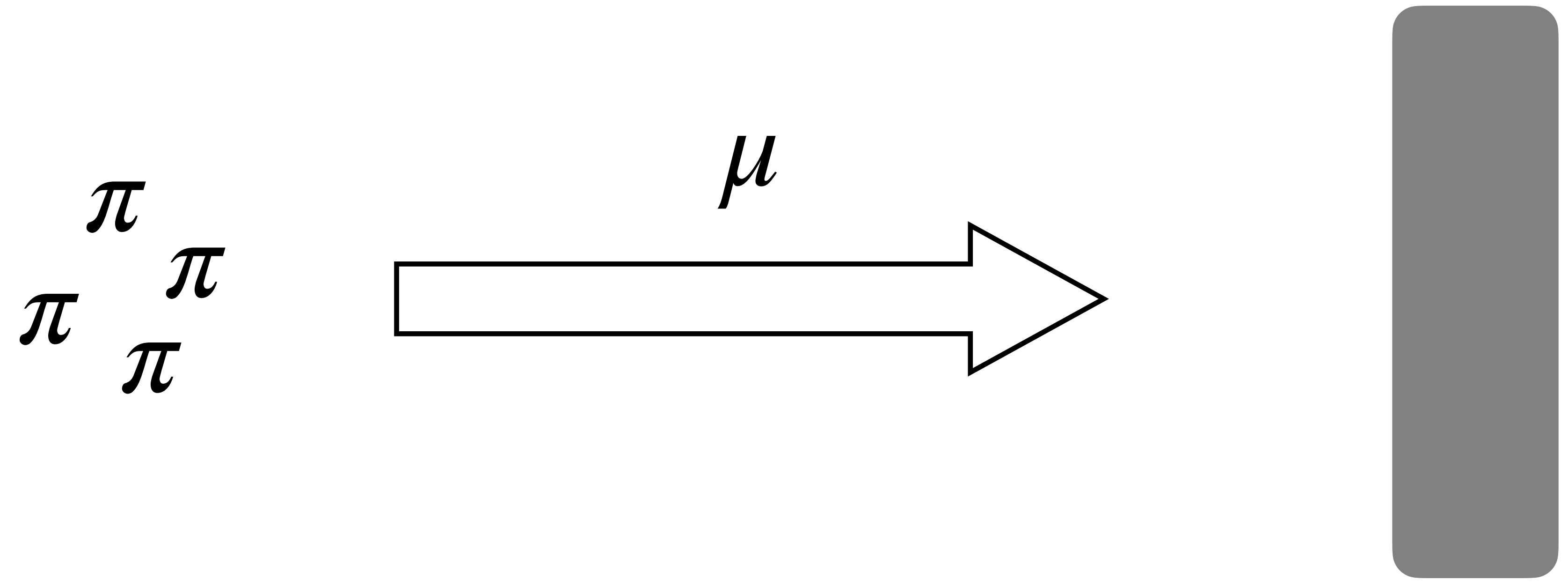
$$R_\infty = 10\ 973\ 731.568160(21)\ \mathrm{m}^{-1}$$

Outline

- **Muonic atom spectroscopy**
  - X-ray spectroscopy
  - On-going experimental efforts
- **Theoretical modeling**
  - Lamb-shift to atomic energy levels
  - Two-photon exchange corrections
- **Numerical implementation**
  - No-Core Shell Model
  - Nuclear polarizability of $^7\mathrm{Li}$

Observing muonic atoms with X-rays

How to make muonic atom



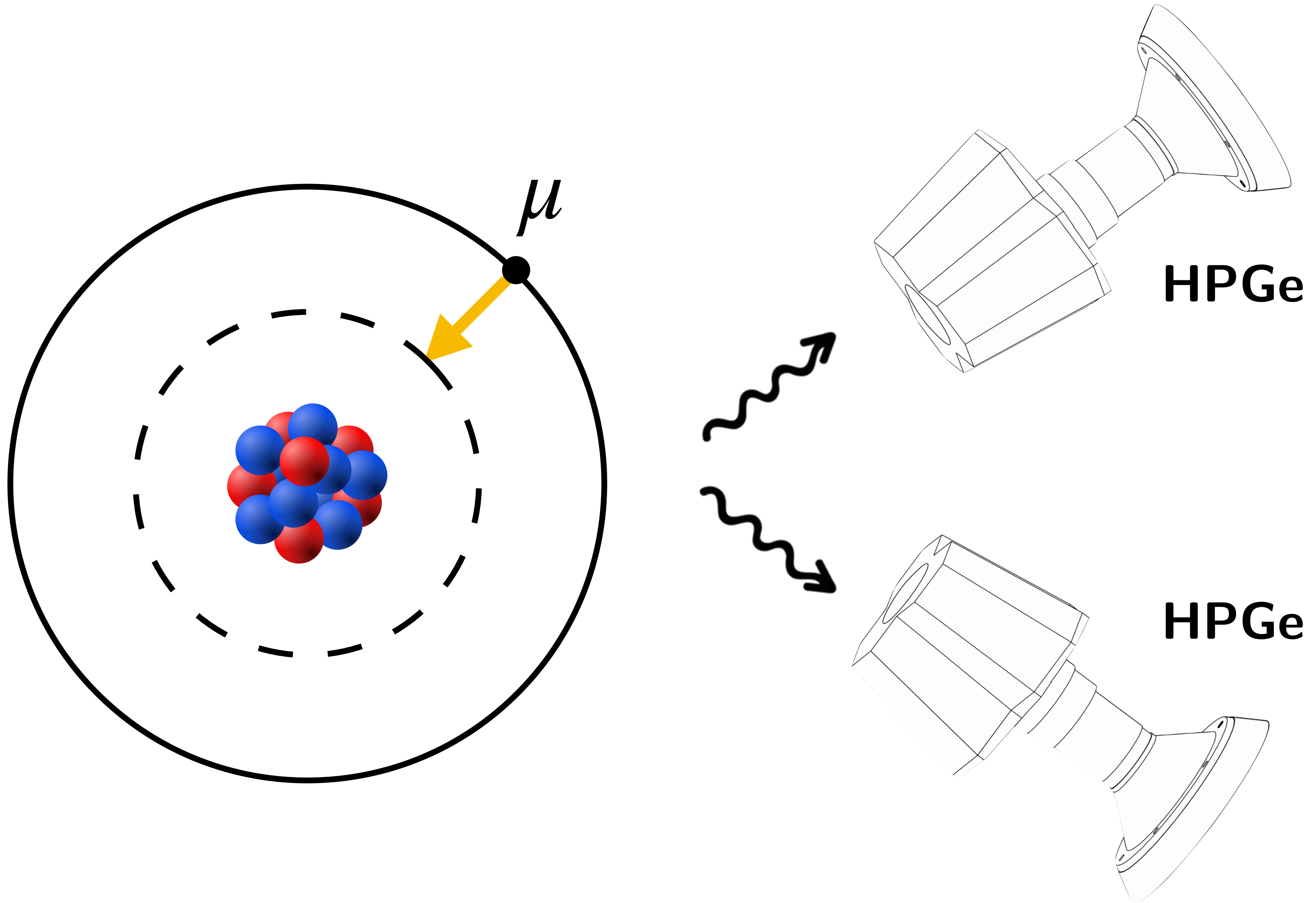
(i) Pion decay: muon source

(ii) High intensity beam: momentum filtering, ...

(ii) Thick target: capture muons

Typically muons captured on orbitals with $n \sim \sqrt{\frac{m_\mu}{m_e}} \sim 14$

Observing characteristic X-rays



Muonic X-ray achievements

- ◉ Precise spectroscopy of almost all stable elements
- ◉ Specific transition targeted with low-latency lasers
- ◉ Absolute charge radii extracted ⇒ **highest accuracy**

➡ **Higher sensitivity due to higher overlap** $\sim \left(\frac{m_\mu}{m_e}\right)^3 \sim 10^7$

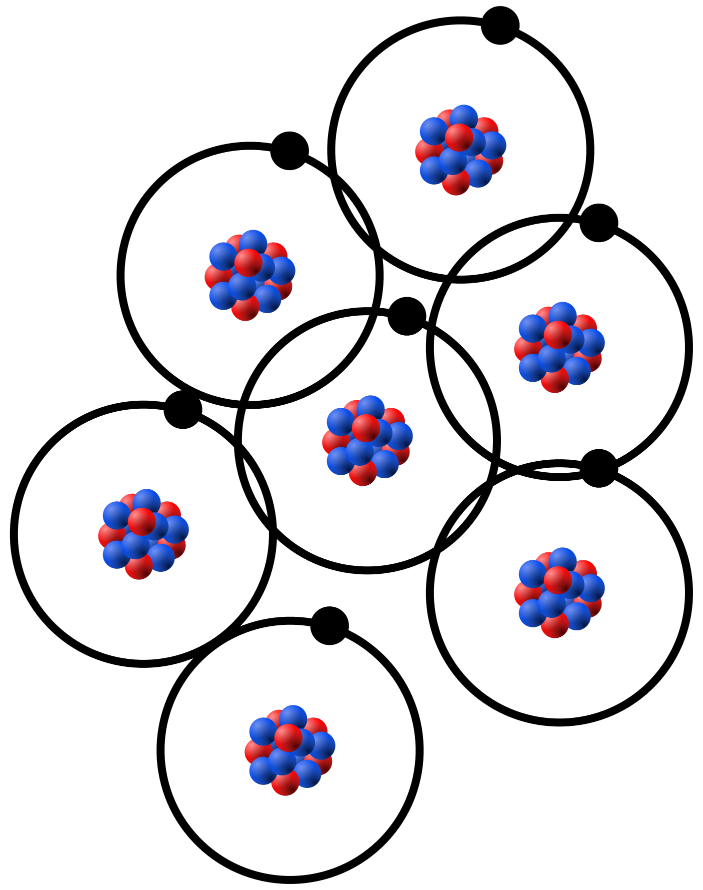
Practical limitations

- × <u>In general</u>: limitations are very experiment dependent
- × Never observe a **unique** muonic atom in the **vacuum**
- × Never with a perfect energy resolution

➡ **Many experimental challenges !**

Real life X-ray measurements

Cascade modelling



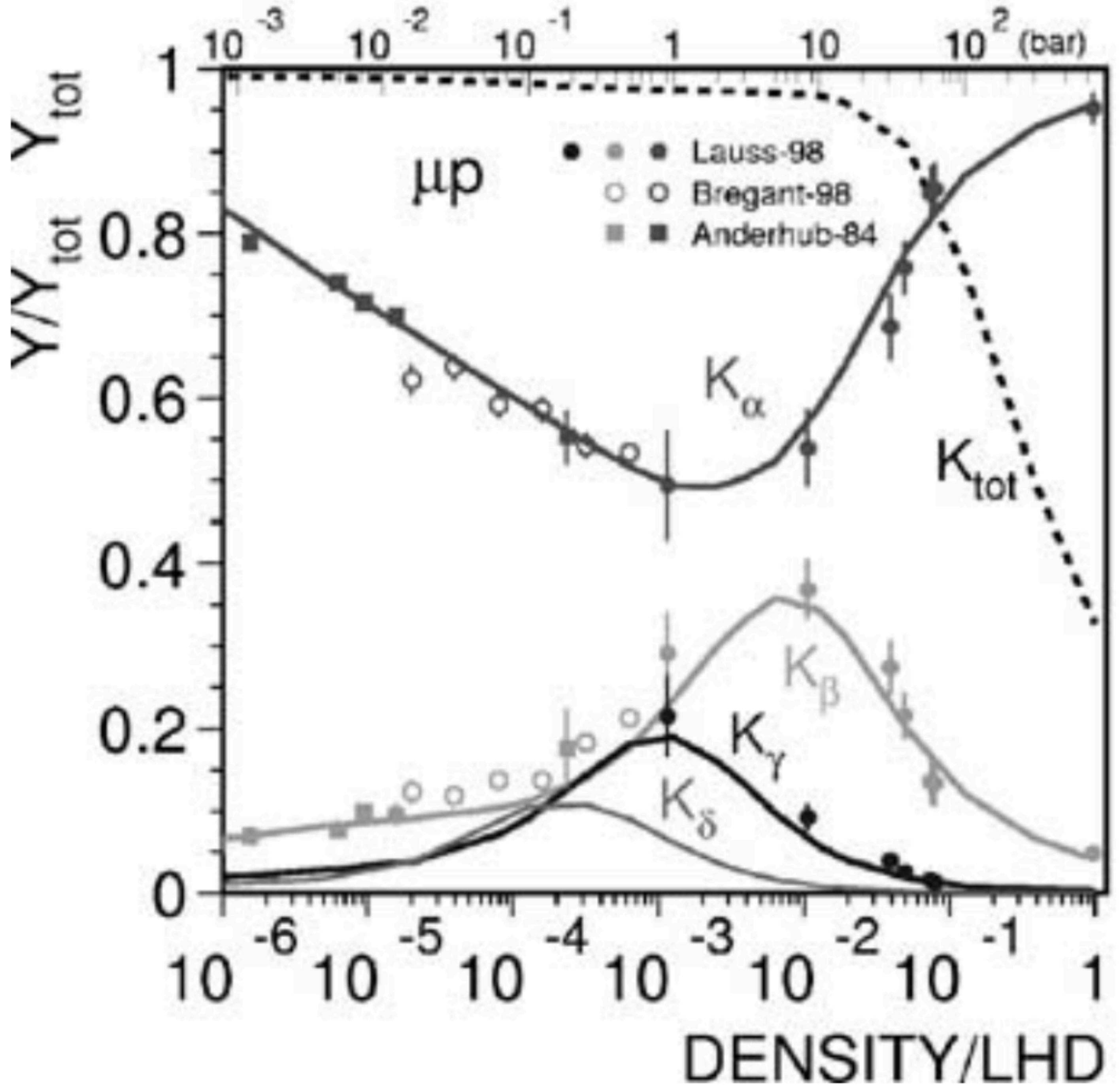
- Many possible processes after a muon reaches the g.s.
  - Radiative transitions
  - External Auger effect
  - Stark mixing
  - ...

Standard Cascade Model

➡ **Competition: radiative vs collisional processes**
➡ **Similar issue for metastable 2S used with lasers**

Mechanism	Example
Radiative	$(\mu p)_i \rightarrow (\mu p)_f + \gamma$
External Auger effect	$(\mu p)_i + \mathrm{H}_2 \rightarrow (\mu p)_f + e^- + \mathrm{H}_2^+$
Stark mixing	$(\mu p)_{nl} + \mathrm{H} \rightarrow (\mu p)_{nl'} + \mathrm{H}$
Elastic scattering	$(\mu p)_n + \mathrm{H} \rightarrow (\mu p)_n + \mathrm{H}$
Coulomb transitions	$(\mu p)_{n_i} + p \rightarrow (\mu p)_{n_f} + p,\, n_f < n_i$
Transfer (isotope exchange)	$(\mu p)_n + d \rightarrow (\mu d)_n + p$
Absorption	$(\pi^- p)_{nS} \rightarrow \pi^0 + n,\ \gamma + n$

[Markushin, Hyperfine Interactions (1999)]



[Markushin, Jensen, Hyperfine Interactions (2001)]

In general many requirement on experiments

- High-precision laser spectroscopy measurements
  ⇒ **requirement on density of gaseous target**
- Radioactive nuclei ⇒ **difficult to build thick target**
- High precision ⇒ **good energy resolution**

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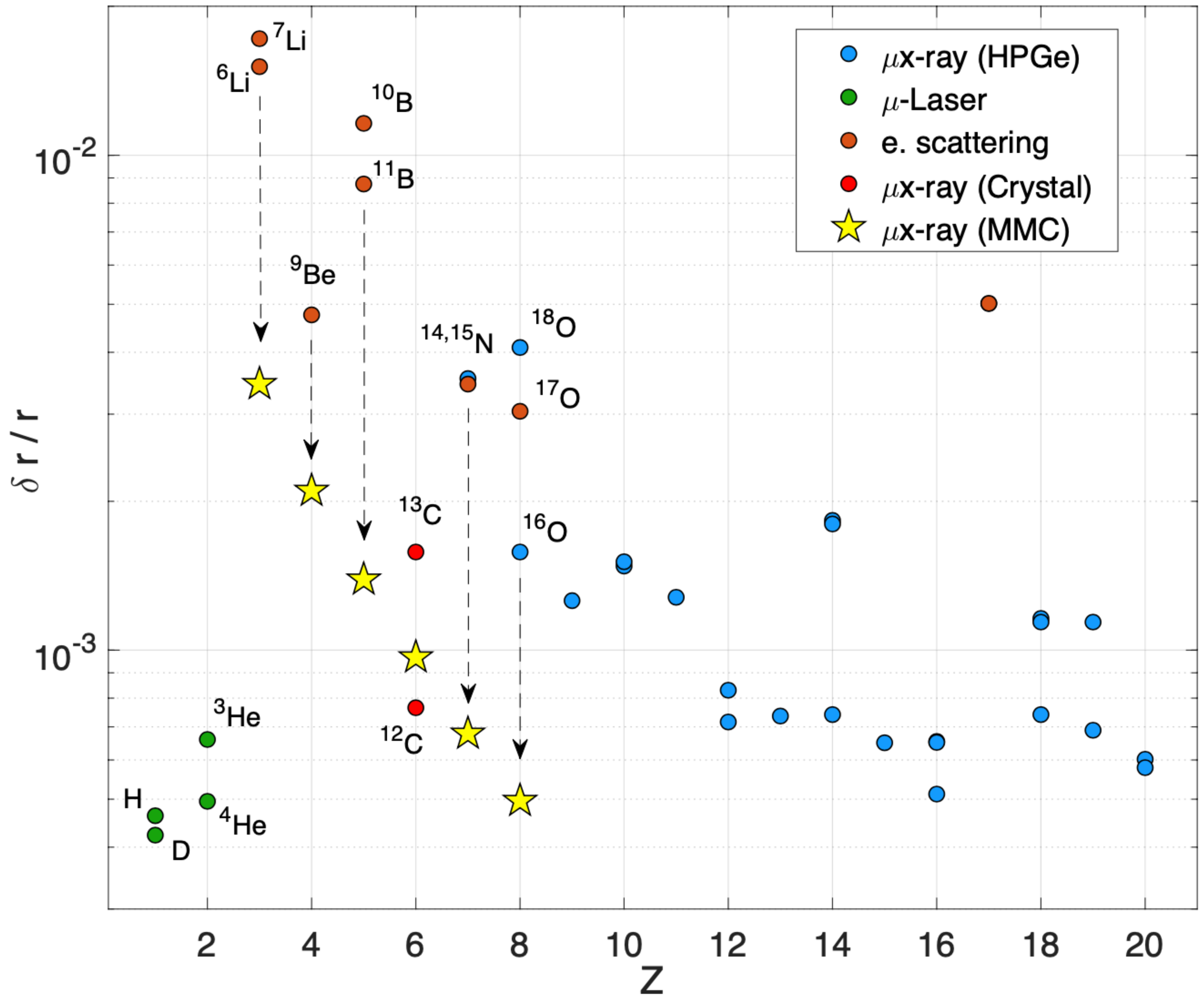
Reaching high resolution for light nuclei



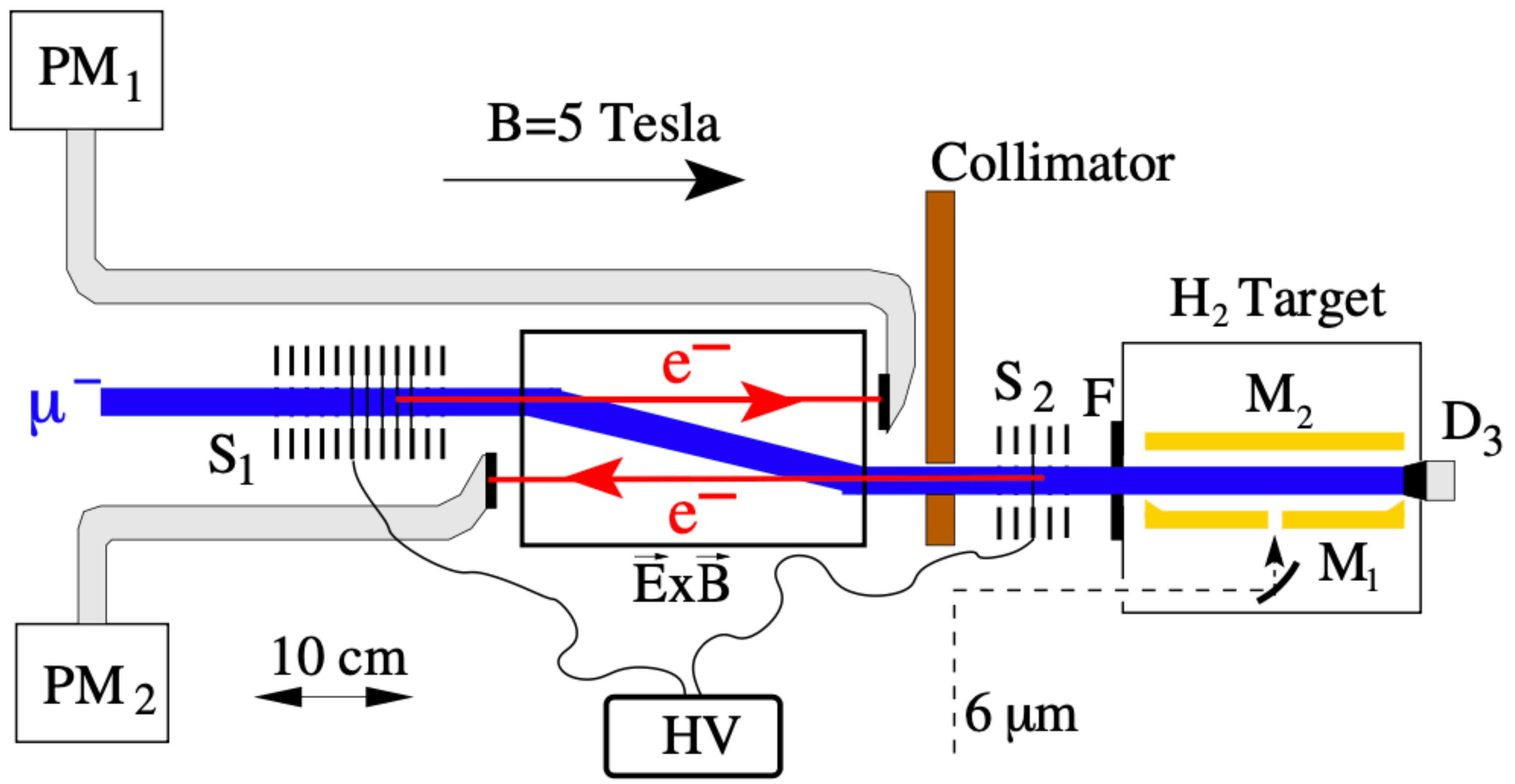
[Antognini et al, arXiv:2210.16929]
NuPECC Long Range Plan 2024

Energy resolution issue

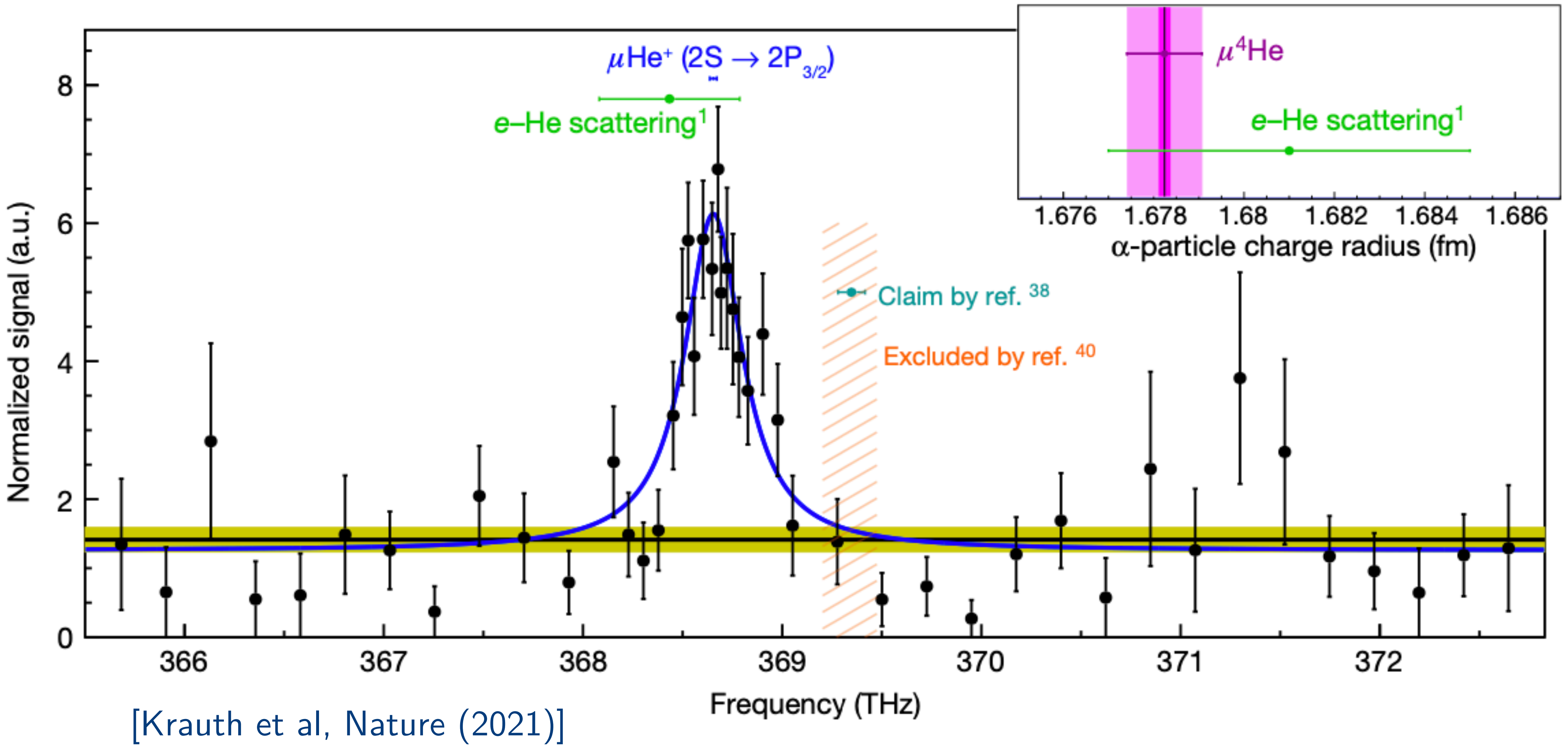
- Intrinsic energy resolution of semi-conductor
  - Not great for $E \leq 200$ keV
  - Limits their usage to $Z \geq 10$
- Light nuclei radii measurements
  - e-scattering data ⇒ low precision
  - Crystal spectrometer ⇒ low efficiency
- New collaboration QUARTET
  - Aim to develop a quantum sensor to reach low-Z nuclei
  - Idea: X-ray ⇒ heat ⇒ magnetization ⇒ SQUID detector
  - **On-going work at PSI**

Even more promising: muonic laser experiments !

Laser spectroscopy for light atoms



[Antognini, Kottman ent Pohl, SciPost (2021)]

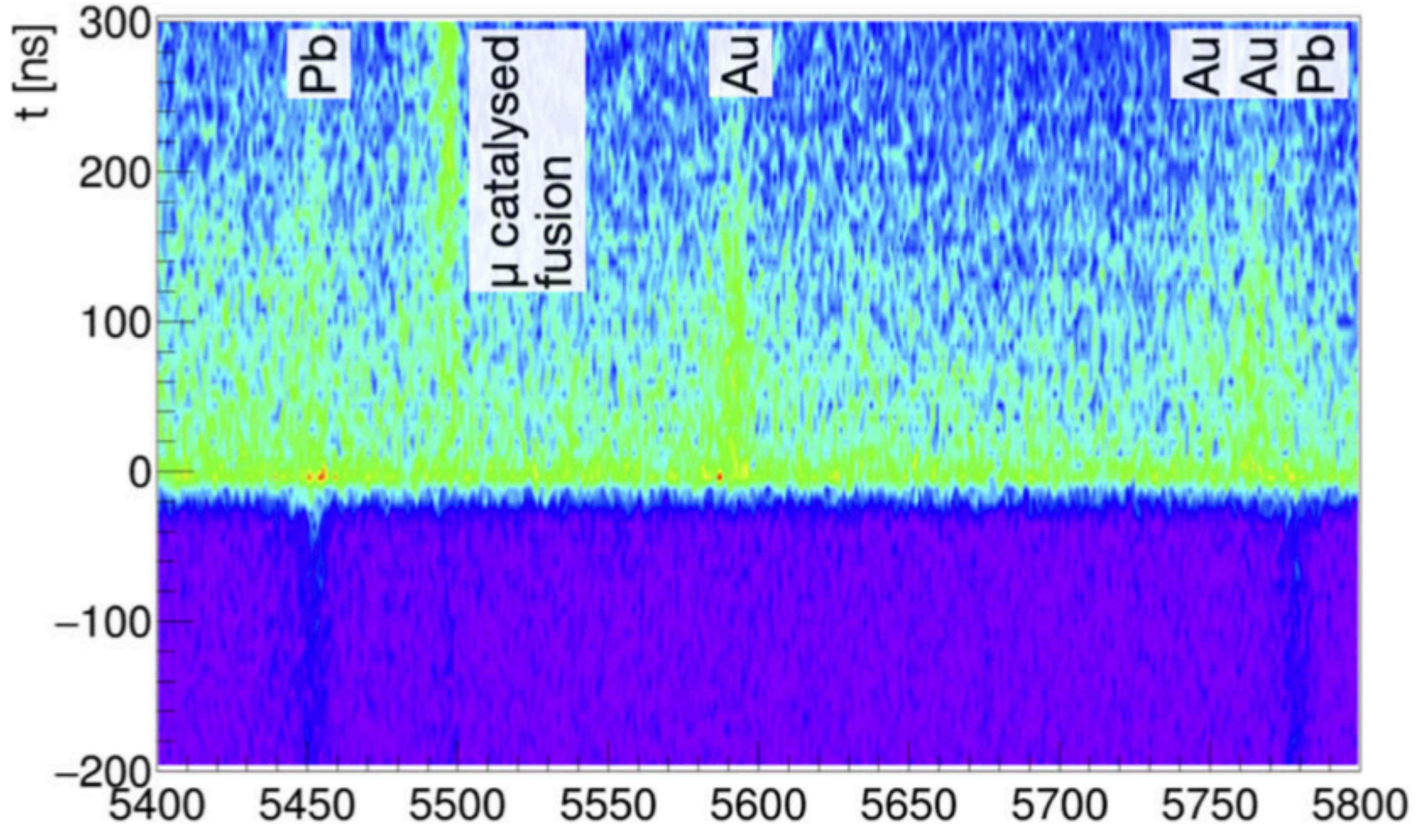


[Krauth et al, Nature (2021)]

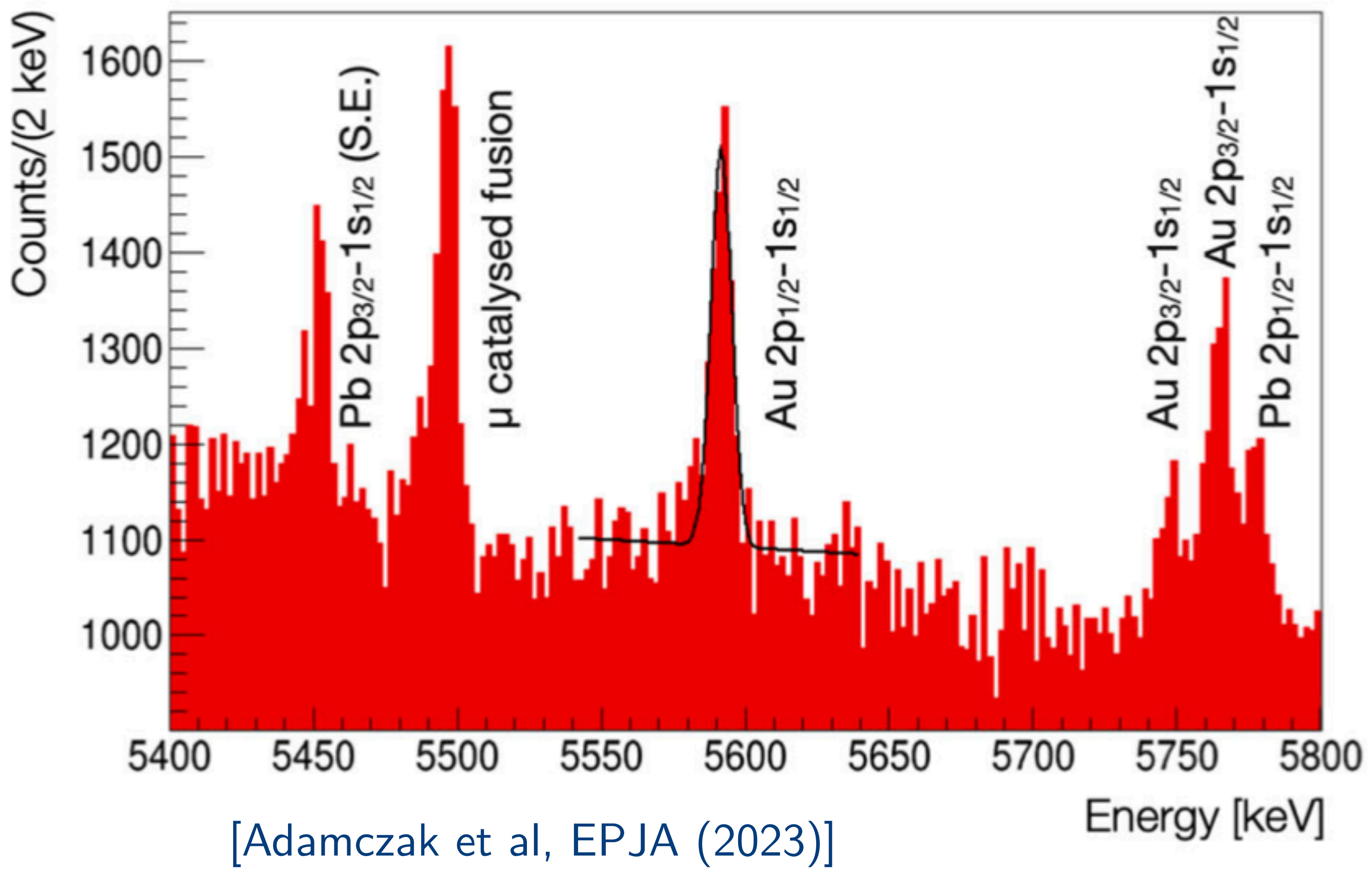
Laser spectroscopy for light muonic atoms

- General idea
  - Muon detected via electron emission in coincidence
  - ➡ triggers laser pumping on $2S \to 2P$
- Laser pumping
  - muon decays to 2S state ⇒ driven by E1 (~ 100 ns)
  - decay to 1S limited (2-photon decay) ⇒ $2S \sim 1\mu s$ lifetime
  - long enough for pulsed laser
- CREMA collaboration
  - Already developed for μH, μD and μHe
  - Main goal: develop a target made of dense cloud of Li
  - **On-going work at PSI**

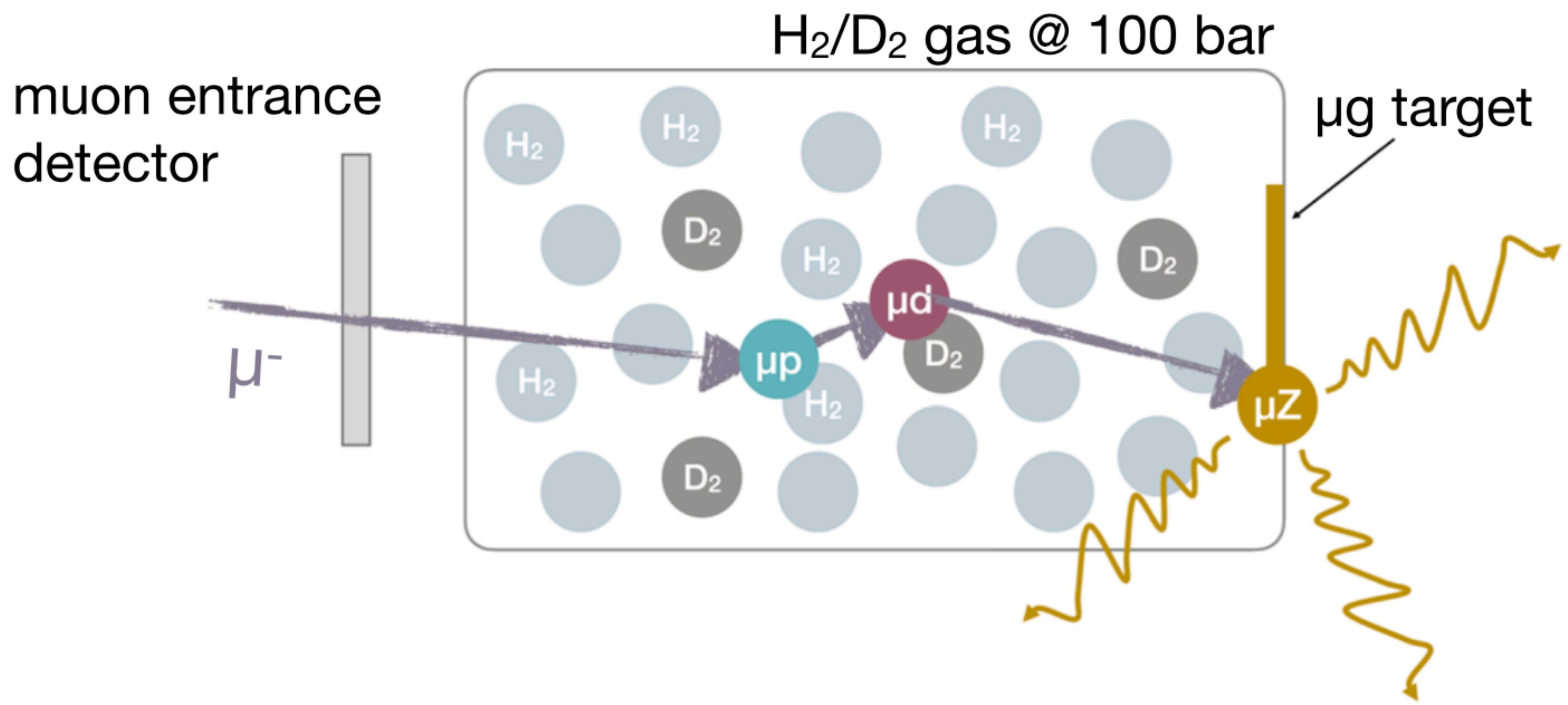
X-ray spectroscopy for radioactive target



(a)



[Adamczak et al, EPJA (2023)]



Reducing the required size of the target

- General idea
  - (i) Muons are captured by protons
  - (ii) In ~ 100 ns, muon transfers to D with 45 keV kinetic gain
  - (iii) μD slows down to 4 eV where cross-section is low $\Rightarrow$ **high mobility**
  - (iv) Muons transfer to high-Z atoms
- muX collaboration
  - Optimized the H/D gas mix to get best performance ($\sim 0.25\%$)
  - Tested on $5\mu g$ Au target instead of standard amount of $\sim 100\ mg$
  - Preliminary results on radioactive ^{248}Cm and ^{226}Ra targets
  - End goal: search **parity violating E1** in $2S - 1S$ transition

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From energy levels to nuclear structure

Converting experimental data

- What to do once precise value of energy levels is known ?
  - Can be used to **test fundamental constants** like R_∞, α, m_e
  - Can be used to extract **nuclear structure information** like r_c
  - Can be used to test validity of **many-body calculations**

- Example in practice: Lamb shift in meV $2S_{1/2} - 2P_{1/2}$ (r_x in fm)

  [Antognini et al, SciPost (2021)]

$$\Delta E(\mu \mathrm{H}) = 206.0336(15) - 5.2275(10) \times r_p^2 + 0.0332(20)$$
$$\Delta E(\mu \mathrm{D}) = 228.7767(10) - 6.1103(3) \times r_\mathrm{D}^2 + 1.7449(200)$$
$$\Delta E(\mu^4\mathrm{He}) = 1668.489(14) - 106.220(8) \times r_\alpha^2 + 9.201(291)$$

Radius extraction master formula

$$\boxed{\delta_{LS} = \delta_{QED} + \mathscr{C}\ r_c^2 + \delta_{NS}}$$

Fixed point-like nucleus

Finite nucleus size effect

Nuclear structure dependent

General many-body problem

- Degrees of freedom
  - Muon $\rightarrow \psi_\mu$; Nucleons $\rightarrow N$; photon $\rightarrow A$

- Hamiltonian
  - For simplicity assume non-relativistic nucleons of equal mass

$$H = H_{Nucl} + e \int \mathrm{d}^3x\ J_\mu(x) A^\mu(x) \quad \text{[Friar, Rosen, Annals of Physics (1974)]}$$
$$+ \frac{e^2}{2m} \int \mathrm{d}^3x \mathrm{d}^3y\ f_{SG}(x,y)\ \vec{A}(x).\vec{A}(y)$$
$$+ H_{QED}$$

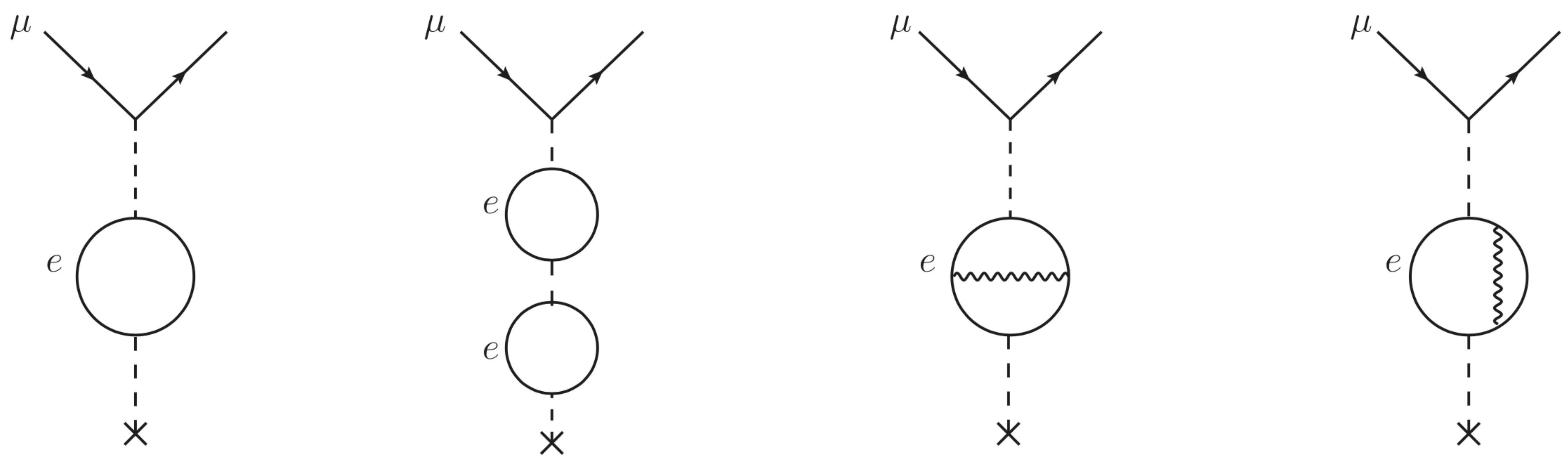
- General approach to compute bound state of H
  - ✗ In principle use Bethe-Salpeter $\Rightarrow$ bound states $\equiv G_2$ poles
  - ✓ In practice use **effective instantaneous potential**
  - DWB correction up to $(Z\alpha)^5$ to match exp accuracy

Bound states QED contributions

Bound muon within potential

- Zero-order: one-body Coulomb interaction
  - Solve exactly for $H_0 = \dfrac{\vec{p}^2}{2m_r} - \dfrac{Z\alpha}{r}$
  - $E_{nl} = -\dfrac{(Z\alpha)^2 m_r}{2n^2} \equiv E^{(0)}$
- Effective potential applied on muon
  - What is relativistic extension to Coulomb ?
  - Define effective potential to reproduce E_{nl} at a given order
  - Power-counting $\Rightarrow$ DWB on H_0
- Main type of contributions
  - Electron vacuum polarization: $a_\mu \sim \lambda_e \Rightarrow$ **main correction!**
  - Finite nuclear mass $\Rightarrow$ recoil and relativistic corrections
  - muon self-energy terms

Example: electron vacuum polarization corrections



$$-\frac{g^{\mu\nu}}{q^2} \rightarrow \frac{g^{\mu\nu}}{q^2(1+\bar{\omega}(\frac{q^2}{m_e}))}$$

where $\bar{\omega} \equiv$ 1PI expanded in powers of α

$$-\frac{g^{\mu\nu}}{q^2} \rightarrow \frac{g^{\mu\nu}}{q^2}(1+\rho^{(1)}+\rho^{(2)}+\dots)$$

$$\Rightarrow \boxed{V^{(i)}(r) = -(Z\alpha)\int \frac{d^3\vec{q}}{(2\pi)^3}\frac{4\pi}{\vec{q}^{\,2}}\;\rho^{(i)}(-\vec{q}^{\,2})\;e^{i\vec{q}.\vec{r}}}$$

$$\Rightarrow E_{nl} = E^{(0)} + \langle V^{(1)}\rangle + \langle V^{(2)}\rangle + \langle V^{(1)}\frac{1}{(E_0-H_0)'}V^{(1)}\rangle + \dots$$

[Pachucki et al. Review of Modern Physics (2024)]

Bound states QED contributions

Section	Order	Correction	μH	μD	μ^3He$^+$	μ^4He$^+$
III.A	$\alpha(Z\alpha)^2$	eVP$^{(1)}$	205.007 38	227.634 70	1641.886 2	1665.773 1
III.A	$\alpha^2(Z\alpha)^2$	eVP$^{(2)}$	1.658 85	1.838 04	13.084 3	13.276 9
III.A	$\alpha^3(Z\alpha)^2$	eVP$^{(3)}$	0.007 52	0.008 42(7)	0.073 0(30)	0.074 0(30)
III.B	$(Z, Z^2, Z^3)\alpha^5$	Light-by-light eVP	−0.000 89(2)	−0.000 96(2)	−0.013 4(6)	−0.013 6(6)
III.C	$(Z\alpha)^4$	Recoil	0.057 47	0.067 22	0.126 5	0.295 2
III.D	$\alpha(Z\alpha)^4$	Relativistic with eVP$^{(1)}$	0.018 76	0.021 78	0.509 3	0.521 1
III.E	$\alpha^2(Z\alpha)^4$	Relativistic with eVP$^{(2)}$	0.000 17	0.000 20	0.005 6	0.005 7
III.F	$\alpha(Z\alpha)^4$	μSE$^{(1)}$ + μVP$^{(1)}$, LO	−0.663 45	−0.769 43	−10.652 5	−10.926 0
III.G	$\alpha(Z\alpha)^5$	μSE$^{(1)}$ + μVP$^{(1)}$, NLO	−0.004 43	−0.005 18	−0.174 9	−0.179 7
III.H	$\alpha^2(Z\alpha)^4$	μVP$^{(1)}$ with eVP$^{(1)}$	0.000 13	0.000 15	0.003 8	0.003 9
III.I	$\alpha^2(Z\alpha)^4$	μSE$^{(1)}$ with eVP$^{(1)}$	−0.002 54	−0.003 06	−0.062 7	−0.064 6
III.J	$(Z\alpha)^5$	Recoil	−0.044 97	−0.026 60	−0.558 1	−0.433 0
III.K	$\alpha(Z\alpha)^5$	Recoil with eVP$^{(1)}$	0.000 14(14)	0.000 09(9)	0.004 9(49)	0.003 9(39)
III.L	$Z^2\alpha(Z\alpha)^4$	nSE$^{(1)}$	−0.009 92	−0.003 10	−0.084 0	−0.050 5
III.M	$\alpha^2(Z\alpha)^4$	$\mu F_1^{(2)}$, $\mu F_2^{(2)}$, μVP$^{(2)}$	−0.001 58	−0.001 84	−0.031 1	−0.031 9
III.N	$(Z\alpha)^6$	Pure recoil	0.000 09	0.000 04	0.001 9	0.001 4
III.O	$\alpha(Z\alpha)^5$	Radiative recoil	0.000 22	0.000 13	0.002 9	0.002 3
III.P	$\alpha(Z\alpha)^4$	hVP	0.011 36(27)	0.013 28(32)	0.224 1(53)	0.230 3(54)
III.Q	$\alpha^2(Z\alpha)^4$	hVP with eVP$^{(1)}$	0.000 09	0.000 10	0.002 6(1)	0.002 7(1)

[Pachucki et al. Review of Modern Physics (2024)]

Finite size nuclear contributions

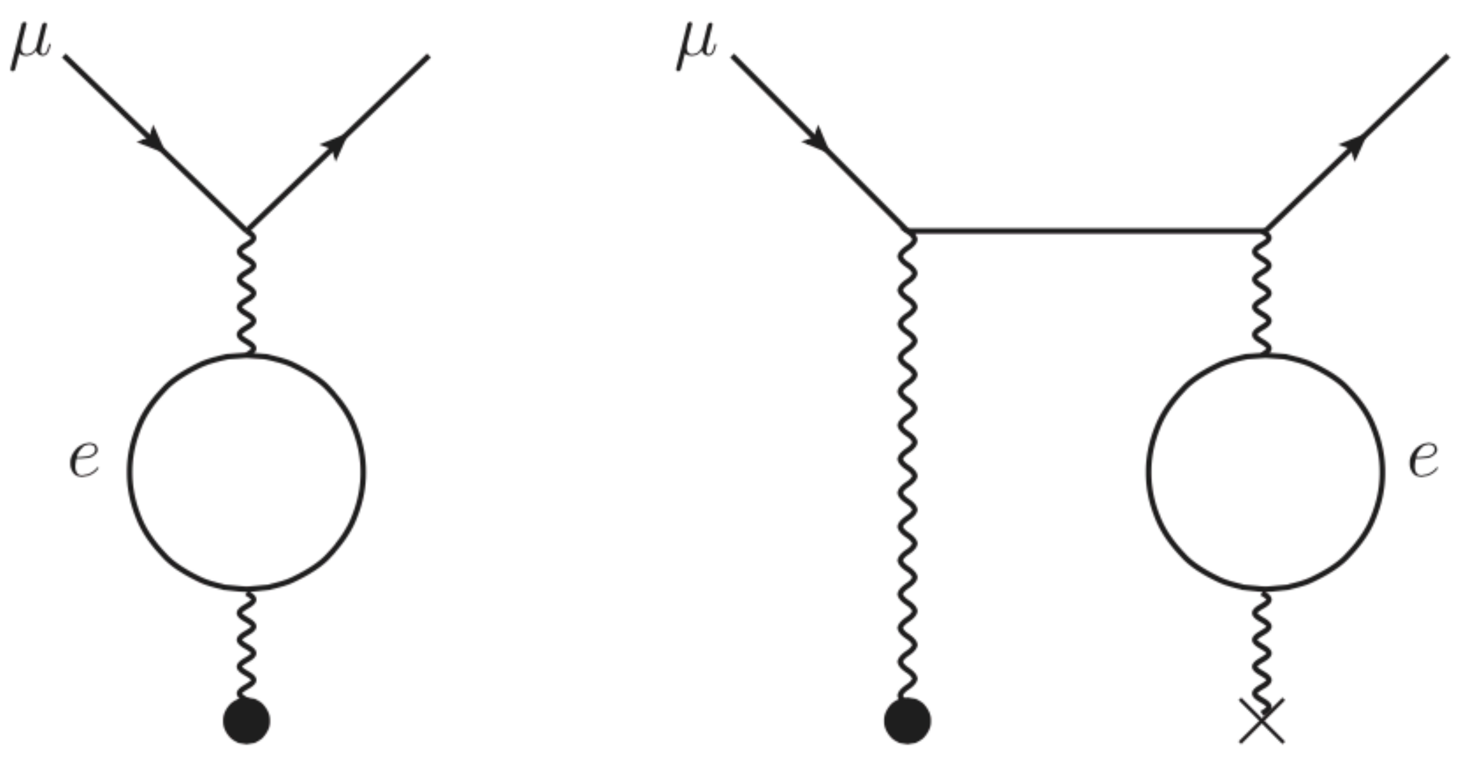
Finite nuclear size contribution

- Correction to account for non-point like nucleus
  - Similar approach as pure QED contributions
  - Multipole expansion of charge distribution

  ➡ Main contributions $\propto r_c^2$

- Beyond charge radius contributions
  - In principle higher order terms leads to multipoles of ρ
  - Experiments not precise enough for now
  - CREMA = on-going attempt to measure **HFS for proton!**

Examples taking into account electron vacuum polarization



$\Rightarrow$ $\mathscr{C} r_c^2$ term in δ_{LS}

[Pachucki et al. Review of Modern Physics (2024)]

Section	Order	Correction	μH	μD	μ^3He$^+$	μ^4He$^+$
IV.A	$(Z\alpha)^4$	r_C^2	$-5.1975r_p^2$	$-6.0732r_d^2$	$-102.523r_h^2$	$-105.322r_\alpha^2$
IV.B	$\alpha(Z\alpha)^4$	eVP$^{(1)}$ with r_C^2	$-0.0282r_p^2$	$-0.0340r_d^2$	$-0.851r_h^2$	$-0.878r_\alpha^2$
IV.C	$\alpha^2(Z\alpha)^4$	eVP$^{(2)}$ with r_C^2	$-0.0002r_p^2$	$-0.0002r_d^2$	$-0.009(1)r_h^2$	$-0.009(1)r_\alpha^2$

Nuclear structure dependent corrections

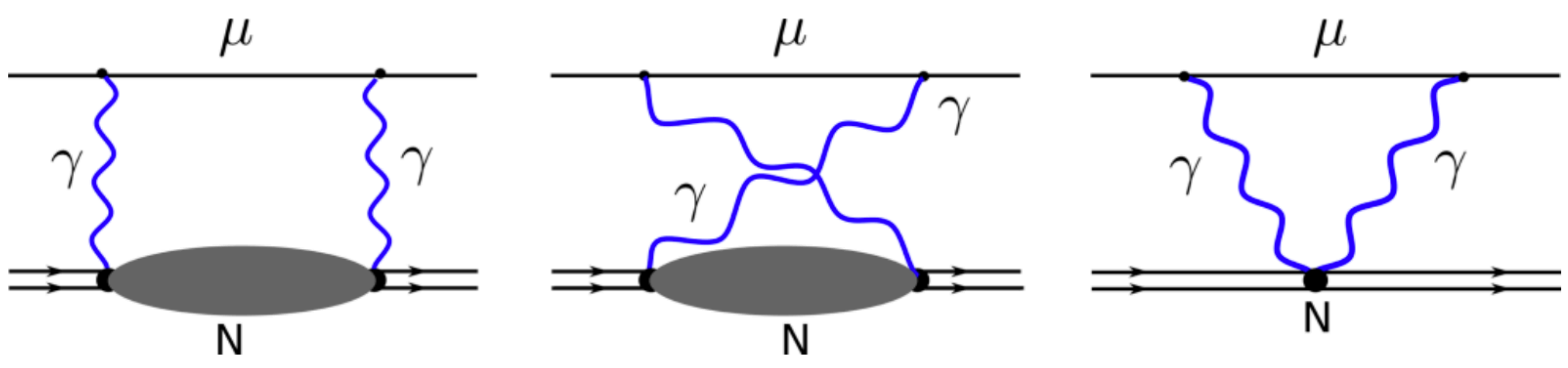
Nuclear structure effects

- Corrections accounting for non static effects
  - The nucleus is no longer treated as an external potential
  - Main contribution comes from **two-photon exchange** δ_{TPE}
  - **Nuclear excited states** become necessary to be accounted for

➡ δ_{TPE} contributes at $(Z\alpha)^5$

- Beyond TPE
  - Further corrections three-, four-, … photon exchange
  - can also be combined with vacuum polarization, …

Two photon exchanges contributions



$$\boxed{\Delta E_{nl} = -\frac{(4\pi Z\alpha)}{m_r}|\phi_{nl}(0)|^2 \mathrm{Im}\int \frac{d^4q}{(2\pi)^4} D^{\mu\rho}(q) D^{\nu\tau}(-q)\ t_{\mu\nu}(q,k)\ T_{\rho\tau}(q,-q)}$$

with:

- $D^{\mu\nu}(q) \equiv$ the photon propagator
- $t_{\mu\nu} \equiv$ the lepton tensor
- $T_{\mu\nu} \equiv$ the hadronic tensor
- $k \equiv (m_r, 0)$

[Bernabeu et al, Nuclear Physics A (1974)]
[Rosenfelder Nuclear Physics A (1983)]
[Hernandez et al. Physical Review C (2019)]

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Nuclear Compton Tensor

Pure electromagnetic part

- Leptonic tensor:
  - Wave-function approximation
  - ➡ Free muon propagator + $\phi_{1s}(0)$ decoupled from nuclear currents

$$t_{\mu\nu}(q,k) = \frac{\frac{1}{4}\mathrm{Tr}\left[\gamma_\mu(\not{k}-\not{q}+m_r)\gamma_\nu(\not{k}+m_r)\right]}{(k-q)^2-m_r^2+i\epsilon}$$

- Photon propagator:
  - Use Coulomb gauge
  - ➡ Convenient to split charge and transverse contributions

$$D^{\mu\nu}(q) = \begin{pmatrix} \frac{1}{\vec{q}^{\,2}} & 0 \\ 0 & \frac{1}{q^2}\left(\delta_{ij} - \frac{q_i q_j}{\vec{q}^{\,2}}\right) \end{pmatrix}$$

Overall relatively well under-controlled

Hadronic part

- Hadronic tensor:
  - Approximations: no recoil + $p_\mu \ll m_\mu$
  - ➡ Only **forward** Compton tensor

Seagull term

$$\begin{aligned} T_{\mu\nu}(q) =& \delta_{\mu\nu}\left\langle \Psi \left| \int \mathrm{d}^3x e^{iq.x} f_{SG}(x,0) \right| \Psi \right\rangle \\ &+ \sum_{N\neq 0}\left[\frac{\langle \Psi \,|\, J_\mu(0) \,|\, N\vec{q}\rangle \langle N\vec{q} \,|\, J_\nu(0) \,|\, \Psi\rangle}{E_0 - E_N + q_0 + i\epsilon} \right. \\ &\left. + \frac{\langle \Psi \,|\, J_\nu(0) \,|\, N-\vec{q}\rangle \langle N-\vec{q} \,|\, J_\mu(0) \,|\, \Psi\rangle}{E_0 - E_N - q_0 + i\epsilon} \right] \end{aligned}$$

[Bernabeu et al, Nuclear Physics A (1974)]

[Friar, Annals of Physics (1976)]

- Seagull: necessary to cancel divergence + use dispersion relation

Decomposition of two-photon exchange

- Nucleon/Nucleus decomposition: (in the end use DR to model $T_{\mu\nu}$)

$$\boxed{\delta_{TPE} = (\delta^N_{el} + \delta^N_{pol}) + (\delta^A_{el} + \delta^A_{pol})}$$

Nuclear modeling

Model used of nuclear currents

- **Multipole decomposition of nuclear currents**
  [Donnelly, Haxton, Atomic and Nuclear Data Tables (1979)]
  - $M_{JM_J;TM_T}(q) \equiv \int d^3x \, \mathbf{M}_J^{M_J}(qx) J_0(x)_{TM_T}$
  - $T^E_{JM_J;TM_T}(q) \equiv \int d^3x \left[\frac{1}{q}\nabla \times \vec{\mathbf{M}}^{M_J}_{JJ}(qx)\right] . \vec{J}(x)_{TM_T}$
  - $T^M_{JM_J;TM_T}(q) \equiv \int d^3x \vec{\mathbf{M}}^{M_J}_{JJ}(qx) \, . \vec{J}(x)_{TM_T}$

  ➡ **Truncation at $J = 1$**

- **Electromagnetic current modeling**
  - Decomposed within the seven operator basis
  - Form factors given by the isovector dipole model
  - $f_{SN}(q) = \left(1 + \frac{q^2}{M_V^2}\right)^{-2} \quad , \quad F^{(T)}_{1,2}(q) = F^{(T)}_{1,2}(0)\, f_{SN}(q)$

  where $F^{(T)}_{1,2}(0)$ based on $\mu^{S,V}$ (nucleon magnetic moments)

Model used of nuclear many-body state

- **Ab initio nuclear interaction** [Entem et al. (2017)] [Somà et al. (2020)]
  - Chiral interaction: 2N-N4LO(500) + 3N(lnl)-N2LO
  - Eventually will have to be varied
- **Model space**
  - Harmonic oscillator Slater determinant

  ➡ **Sub-space truncation: $\hbar\Omega = 18$ MeV, $N_{max} = 7$**
- **Many-body approximation**
  - No-Core Shell Model
  - More details in next section

Need expression of δ^A_{pol} in terms of multipole currents !

Master formula

Inputs to evaluate nuclear polarizability

- Charge spectral function

$$S_{C,J}(\omega,q) \equiv \sum_{N\neq 0} |\langle N \,|\, M_{J0}(q) \,|\, \Psi\rangle|^2 \delta(E_N - E_0 - \omega)$$

- Transverse electric spectral function

$$S^E_{T,J}(\omega,q) \equiv \sum_{N\neq 0} |\langle N \,|\, T^E_{J0}(q) \,|\, \Psi\rangle|^2 \delta(E_N - E_0 - \omega)$$

- Transverse magnetic spectral function

$$S^M_{T,J}(\omega,q) \equiv \sum_{N\neq 0} |\langle N \,|\, T^M_{J0}(q) \,|\, \Psi\rangle|^2 \delta(E_N - E_0 - \omega)$$

Non-relativistic reduction

- Limit: $q \ll m_r$
  - ➡ Only **charge** kernel remains ⇒ simpler + consistency check

$$K_C(\omega,q) \to K_{NR}(\omega,q) = \frac{1}{q^2\left(\frac{q^2}{2m_r}+\omega\right)}$$
$$K_L(\omega,q) \to 0$$
$$K_S(\omega,q) \to 0$$

Relativistic formulation

[Rosenfelder Nuclear Physics A (1983)]

- Decomposition of nuclear polarizability: [Hernandez et al. Physical Review C (2019)]
  - Contribution from **charge**, **transverse electric** and **magnetic**

➡ $\delta^A_{pol} = \Delta_C + \Delta_{T,E} + \Delta_{T,M}$

$$\Delta_C = -8(Z\alpha)^2|\phi_{2S}(0)|^2 \int \mathrm{d}q \int \mathrm{d}\omega\ K_C(\omega,q) S_C(\omega,q)\ ,$$
$$\Delta_{T,E} = -8(Z\alpha)^2|\phi_{2S}(0)|^2 \int \mathrm{d}q \int \mathrm{d}\omega\ K_T(\omega,q) S^E_T(\omega,q) + K_S(\omega,q) S^E_T(\omega,0)\ ,$$
$$\Delta_{T,M} = -8(Z\alpha)^2|\phi_{2S}(0)|^2 \int \mathrm{d}q \int \mathrm{d}\omega\ K_T(\omega,q) S^M_T(\omega,q)$$

- Kernels in the integrals:

$$K_C(\omega,q) = \frac{1}{E_q}\left[\frac{1}{(E_q-m_r)(\omega+E_q-m_r)} - \frac{1}{(E_q+m_r)(\omega+E_q+m_r)}\right]$$
$$K_L(\omega,q) = \frac{q^2}{4m_r^2}K_C(\omega,q) - \frac{1}{4m_r q}\frac{\omega+2q}{(\omega+q)^2}$$
$$K_S(\omega,q) = \frac{1}{4m_r\omega}\left[\frac{1}{q} - \frac{1}{E_q}\right]$$

Outline

- **Muonic atom spectroscopy**
  - X-ray spectroscopy
  - On-going experimental efforts
- **Theoretical modeling**
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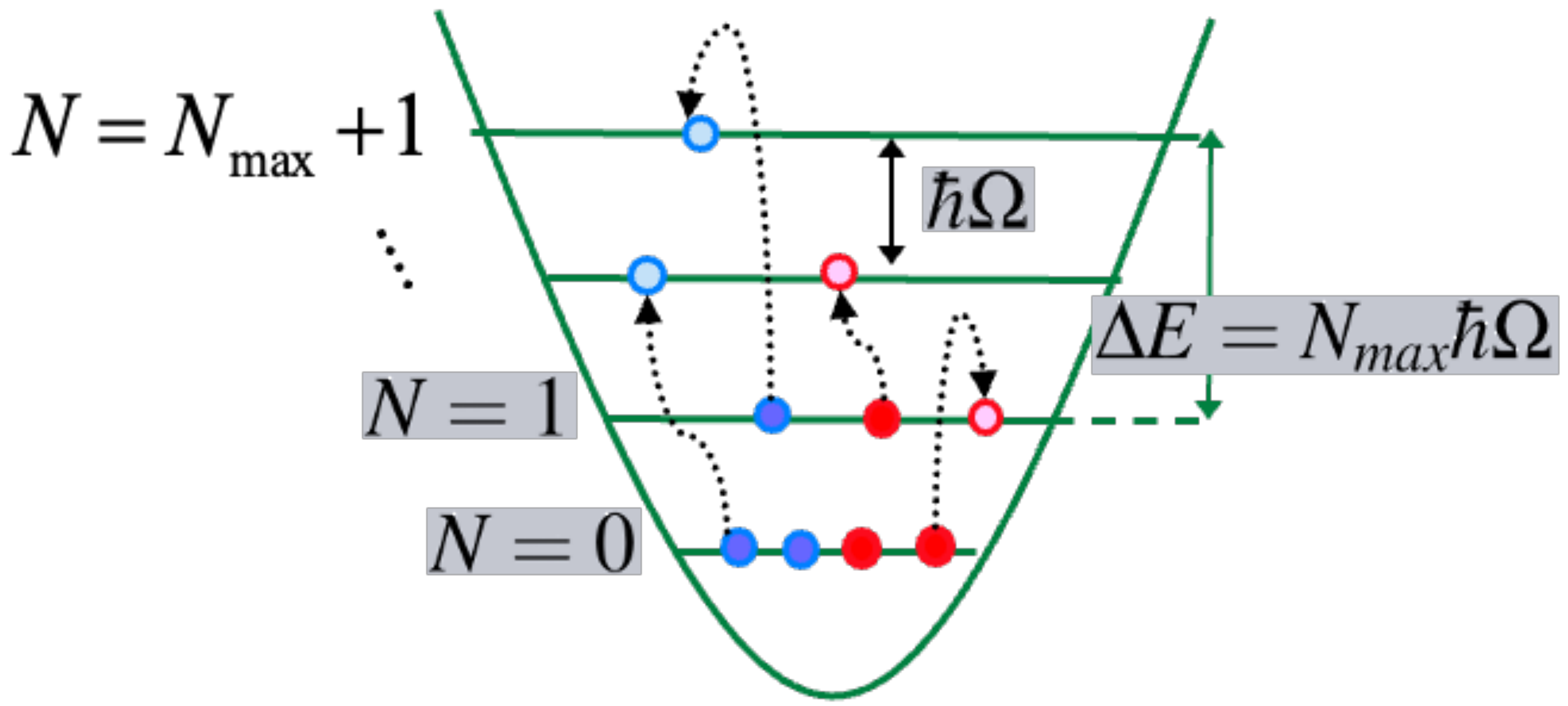
  - No-Core Shell Model
  - Nuclear polarizability of $^{7}\mathrm{Li}$

The No-Core Shell Model

Lanczos tridiagonalization algorithm [Lanczos (1950)]

- Initialization: normalized pivot $|\phi_1\rangle$
- Recursion: α_i, β_i and $|\phi_i\rangle$
  - $\beta_{i+1}|\phi_{i+1}\rangle = H|\phi_i\rangle - \alpha_i|\phi_i\rangle - \beta_i|\phi_{i-1}\rangle$
  - $\alpha_i = \langle\phi_i|H|\phi_i\rangle$ and β_{i+1} st $\langle\phi_{i+1}|\phi_{i+1}\rangle = 1$
- Output:
  - Lanczos basis and coefficients $\{|\phi_i\rangle, \alpha_i, \beta_i\}$ ➡ $\underbrace{\begin{pmatrix} \alpha_1 & \beta_2 & & & & \\ \beta_2 & \alpha_2 & \beta_3 & & & \\ & \beta_3 & \alpha_3 & \ddots & & \\ & & \ddots & \ddots & \beta_{k-1} & \\ & & & \beta_{k-1} & \alpha_{k-1} & \beta_k \\ & & & & \beta_k & \alpha_k \end{pmatrix}}_{H \text{ in Lanczos basis}}$
  - Lanczos basis ≡ orthonormal basis in Krylov space $\left\{|\phi_1\rangle, H|\phi_1\rangle, \dots, H^{N_L}|\phi_1\rangle\right\}$

Anti-symmetrized products of many-body HO states



Application to nuclear structure

- Extraction of spectra
  - Selection rules sparsity ⇒ **Fast matrix-vector multiplication**
  - In practice: $N_L \sim 100 - 200$ is sufficient to converge low-lying states
  - Cost of diagonalization of the tridiagonal matrix is negligible

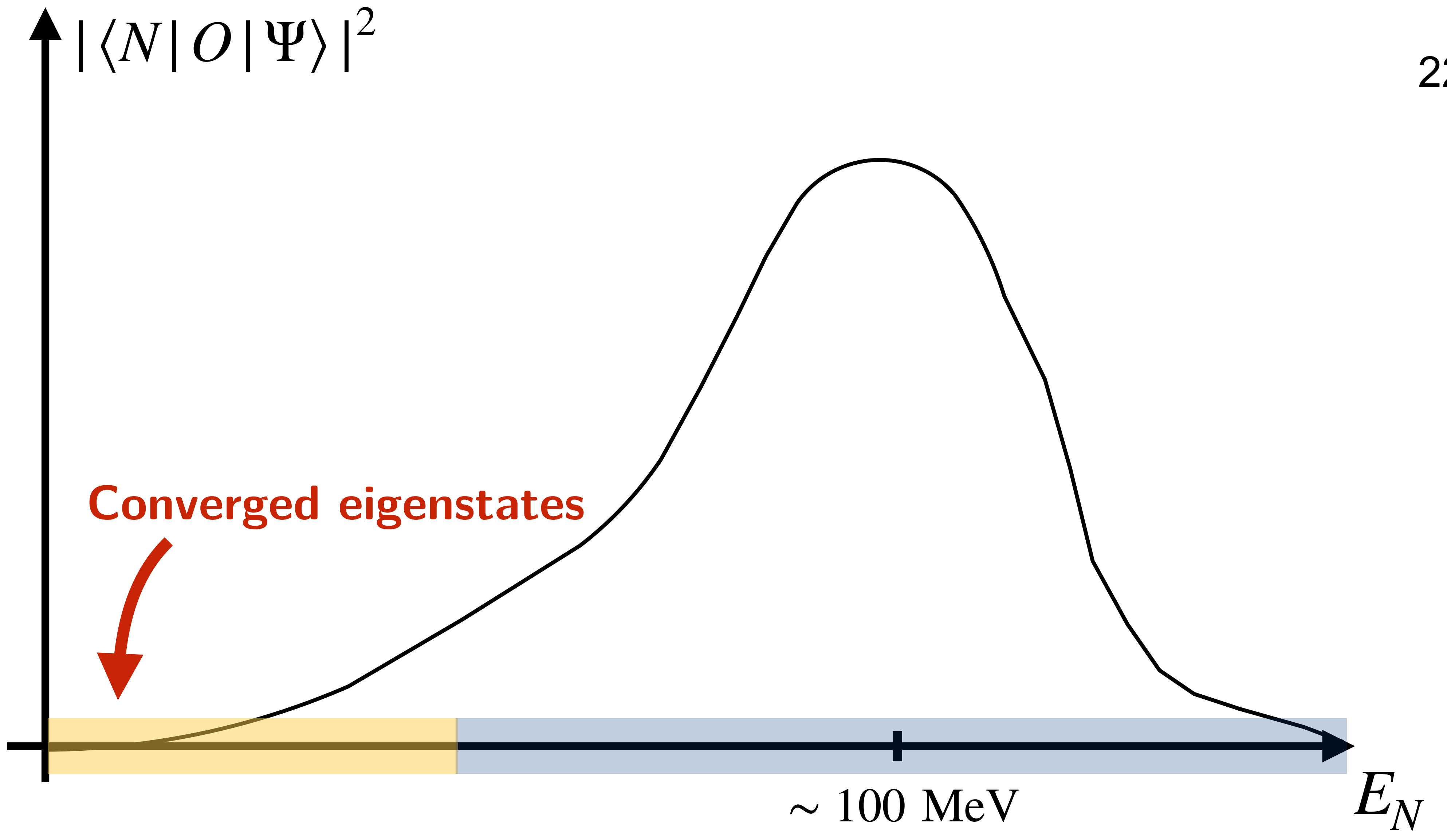
Application to $^7\mathrm{Li}$

- Parameters of many-body calculation
  - $N_L = 250$ for $N_{max} = 7$ and $\hbar\Omega = 18$ MeV
- Results
  - Ground-state of $^7\mathrm{Li}$ $|\Psi\rangle$ ⇒ **Used as a test for δ_{LS}**

The Lanczos strength algorithm

Strength functions

- We need to compute
  - Eigenvalues: $E_N \Rightarrow$ obtained already with Lanczos
  - Overlaps: $|\langle N|O|\Psi\rangle|^2$ for each eigenstate and operator $\Rightarrow$ **expansive**
- Lanczos strength algorithm
  - Efficient variant of Lanczos: extract **relevant information** strength functions



Idea of the algorithm

- For each operator O
  - Compute $\dfrac{O|\Psi\rangle}{\sqrt{\langle\Psi|O^\dagger O|\Psi\rangle}}$ $\Rightarrow$ **Pivot $|\phi_1\rangle$ for 2^{nd} Lanczos**
- Extract strength from orthonormality of Lanczos basis
  - $\langle\Psi|O|N\rangle = \sqrt{\langle\Psi|O^\dagger O|\Psi\rangle} \times \langle\phi_0|N\rangle$ ➡ **Obtained for free during diagonalization**

Sum rules convergence

- Convergence problem
  - Often the **strength is fragmented**
  - Only low-lying states converged in general
- Lanczos strength algorithm
  - Recover exactly $\int \mathrm{d}\omega\ S_O(\omega)\ \omega^n$ for any $n \leq 2N_L$

➡ **Fast convergence of** $\int \mathrm{d}\omega\ f(\omega) S_O(\omega)$ (if f $\sim P_{100}(\omega)$)

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Testing convergence of sum rules for δ^A_{pol}

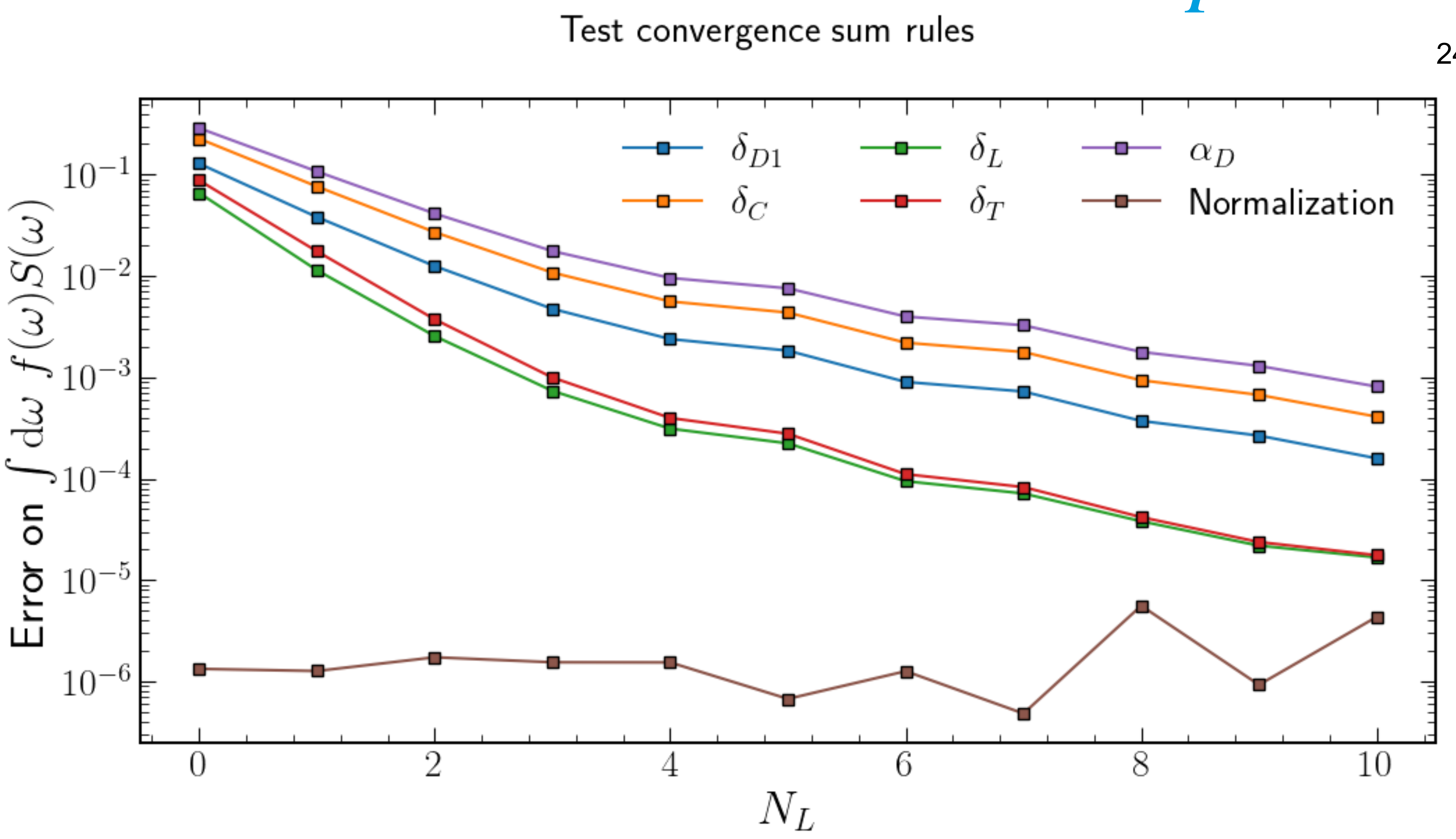
First tests of sum rule convergence

- Before running expansive $q-$dependent
  - Test convergence of strength integrals
  - Cases tested based on **electric dipole operator**
- Sum rules tested: $\int d\omega\, f(\omega) S_D(\omega)$
  - $f_{norm}(\omega) = 1$
  - $f_{D1}(\omega) = \sqrt{\frac{2m_r}{\omega}}$
  - $f_C(\omega) = \frac{m_r}{\omega} \ln \frac{2(Z\alpha)^2 m_r}{\omega}$
  - (+ more complicated one)

Leading order η-expansion of δ^A_{pol}

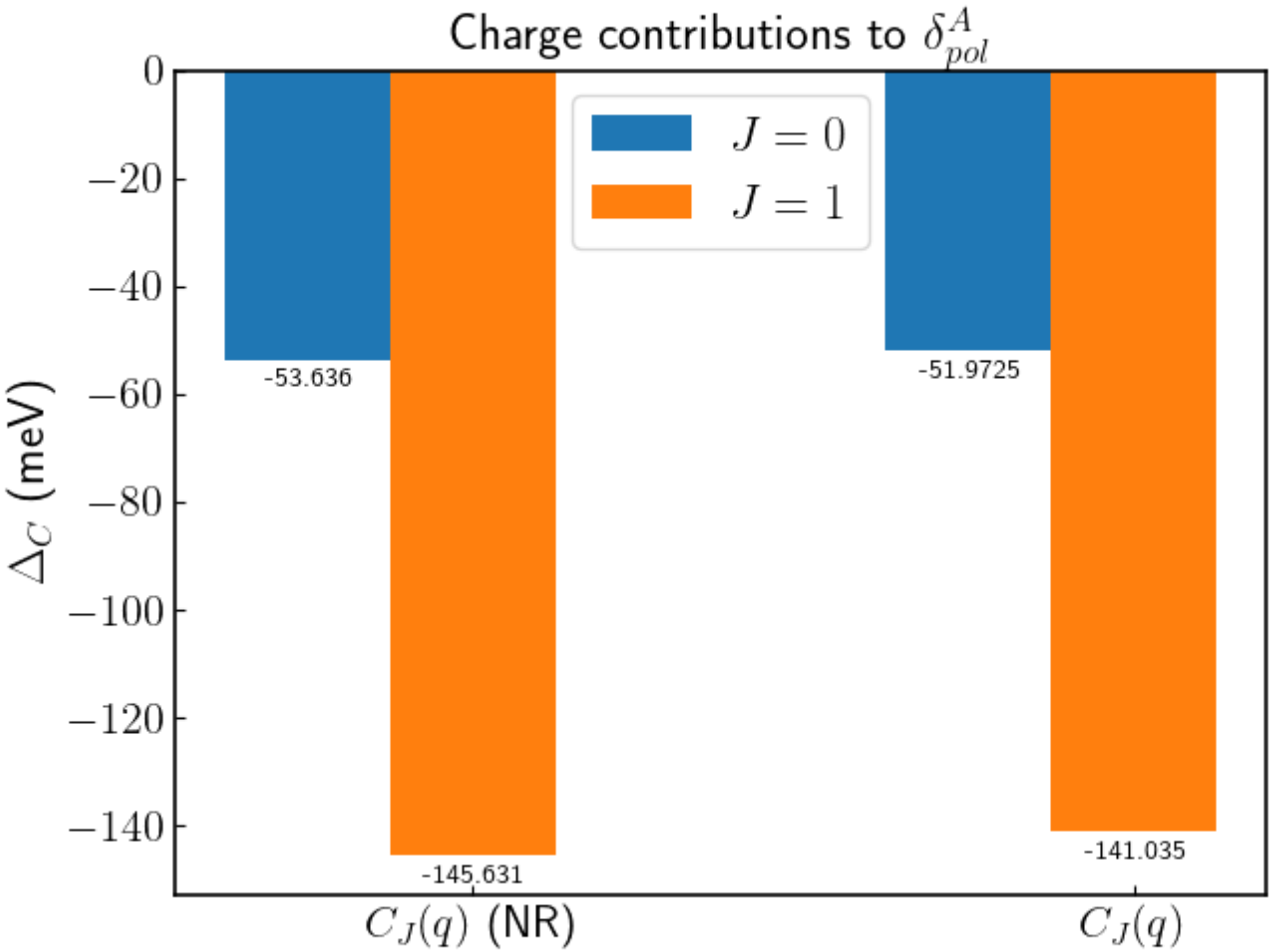
[Hernandez et al. PRC (2019)]

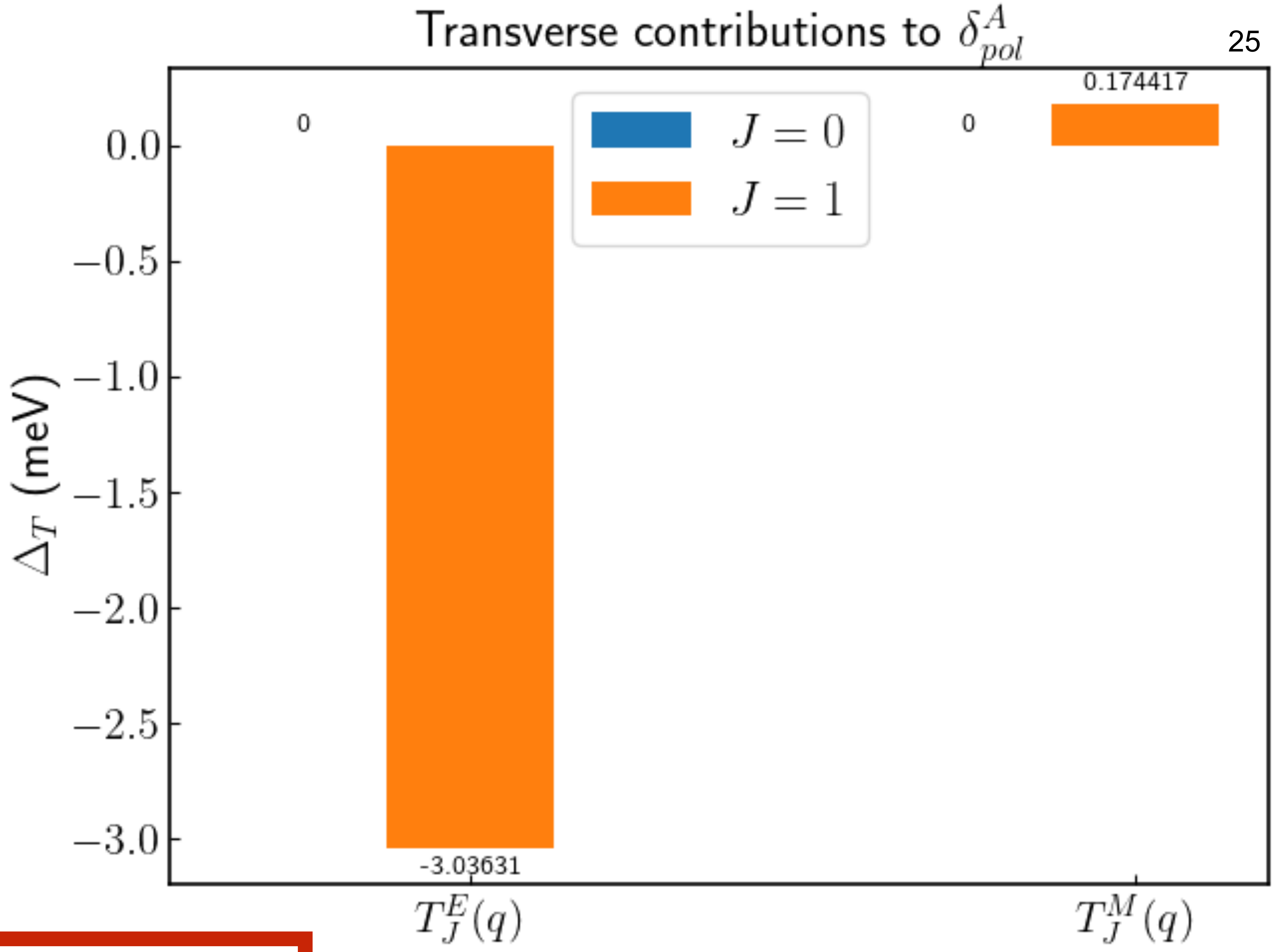
- Observations
  - Sum rules converge quickly $\Rightarrow N_L = 50$ is sufficient
  - Reaches plateau around $\sim 10^{-5}$ relative error



Next step: q-dependent calculations of δ^A_{pol} !

(Very) preliminary results





Many improvements on the way
Simply a proof-of-principle so far

$$\boxed{\begin{aligned}\delta^{A,NR}_{pol} &= -199.26 \text{ meV}\\ \delta^{A}_{pol} &= -195.86 \text{ meV}\end{aligned}}$$

Coherent with previous estimation based on η-expansion

Conclusion

Summary

- <u>Promising muonic spectroscopy are on the way!</u>
  - Laser spectroscopy of muonic atoms

    $\Rightarrow$ an order of magnitude more precise than before
  - Other experimental projects enlarge the nuclei reachable

    $\Rightarrow$ Low-Z with quantum sensor

    $\Rightarrow$ Radioactive nuclei with new target technology
- <u>Involved theoretical calculations</u>
  - Pure particle physics $\Rightarrow$ bound state QED
  - Nuclear structure corrections $\Rightarrow$ **dominant uncertainty**
- <u>Ab initio nuclear calculations of two-photon exchange</u>
  - Critical to reduce uncertainty in radii extraction
  - Proof-of-principle for $^7\mathrm{Li}$ within NCSM

Outlook

- <u>Completing ab initio calculation</u>
  - Convergence studies (in $\int \mathrm{d}q$, in J_{max}, in N_{max})

  ➡ Essential for **uncertainty estimation**
- <u>Including nucleon contribution</u>
  - Similar to super allowed beta decay
  - Combine nucleon/nucleus models using dispersion relation

  ➡ Expect **reduction of uncertainty**
- <u>Controlling theoretical uncertainty</u>
  - Necessary to combine multiple scales
  - Critical to have reliable uncertainty estimations

  ➡ Developing a **complete network of EFTs**

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Thank you
Merci

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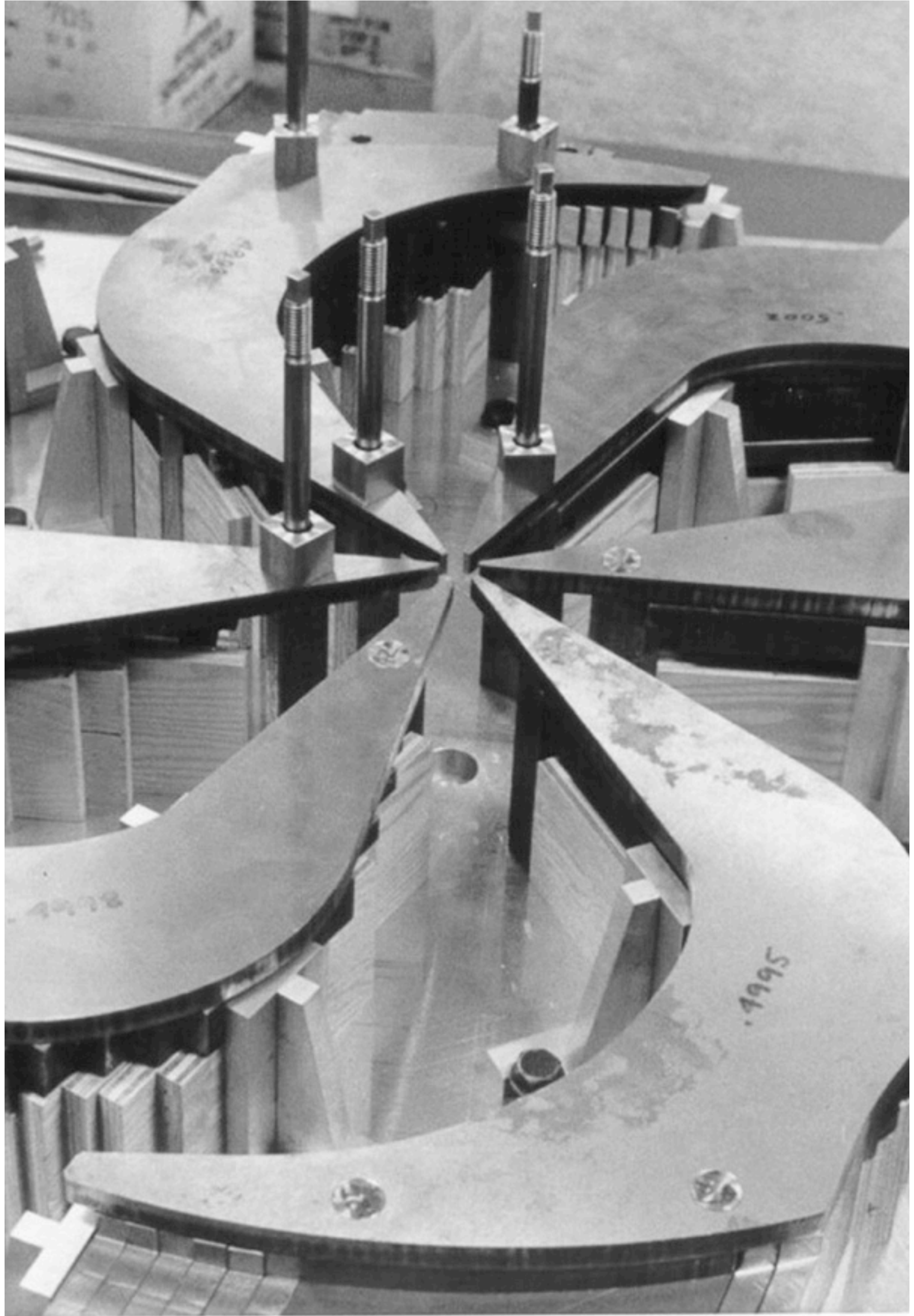

Discovery, accelerated