Three-body systems from a finite volume

with a unitary amplitude

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INT WORKSHOP INT-20R-2C

Accessing and Understanding the QCD Spectra

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Review 2B-lattice: [Briceno]
Reviews 3B-lattice: [Hansen] [Mai]
Review hadron resonances: [Mai]

Key publications Finite-Volume Unitary (FVU) approach:
- Three-body unitarity [Mai]
- Three-body unitarity finite volume [Mai]
- a_1 in finite volume & results from lQCD [Mai]

Talk outline:
- 3-body unitarity
- a_1 in infinite volume
- a_1 and other systems in finite volume

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Progress in last three years alone (narrowly defined for 3B)

- **Whitepapers**: Snowmass whitepaper amplitude analysis: [1], Snowmass whitepaper lattice: [2]
- **FVU papers**: a_1 pole phenomenological: [3], $a_1 \to \pi\sigma$ inf. volume: [4], a_1 lQCD/PRL: [5], Review 3B lattice: [6], 3B force: [7], $3K^+$: [8], a_1 Dalitz: [9], $3\pi^+$ GWQCD data: [10] $3\pi^+$ interpretation Hanlon Data: [11], cross channel $\pi\pi$: [12], Resonance review (preprint): [13], (ρ with ETMC [14], φ^4 equivalence FVU/RFT [15])
- **RFT papers**: $3\pi^+$ HadSpec "Dalitz"/inf. vol. amplitude: [16], Decay amplitude to 3 hadrons: [17], 3 pions all isospins: [18], Review 3B fin vol Hansen: [19], QC $\pi^+\pi^+K^+$: [20], Higher-spin isobars: [21], Non-degenerate scalars 3B: [22] Alternative derivation 3B QC [23], ETMC/Bonn $3\pi^+$: [24]. $3\pi^+$ PRL analysis [25] of Hanlon/Hoerz data: [26]
- **(N)REFT**: Resonance form factor from corr functions [27], Spurious poles [28], EFT Book [29], Rel.-inv. formulation [30], ϕ^4 test scattering [31], Lüscher-Lellouch analog 3-body [32], Analytic energy shift 3B ground state [33], N-particle energy shift [34], Rusetsky Mini-review 3-body [35] Latest (schematic) effort for Roper fin vol [36].
- **Peng/Pang/Koenig, others:** Fin-vol extrapolation eigenvector continuation [37]. 3B resonances pionless EFT [38], Few-body bound states Fin Vol [39], Few-body resonances fin-vol [40], DDK system finite volume [41], Finite volume magnetic field [42, 43], Different fin vol geometries [44], Few-body resonances finite volume [45], Visualization three-body resonances (analytic cont. of L-dependence) [46], Multi-π^+ and analysis of lattice data [47], Threshold expansion N-particle Fin Vol [48], Propagation particle torus [49]
- **inf. vol./Equivalence 3B formalisms:** Equivalence different 3B QC [50], Jackura 3B unitarity PW [51], JPAC hadron physics review [52], 3B unitarity in RFT: [53].

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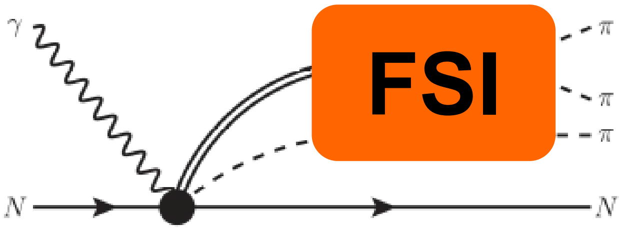
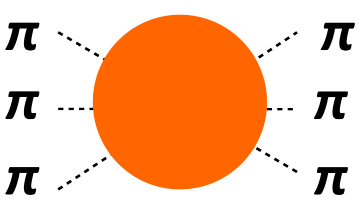
- ….

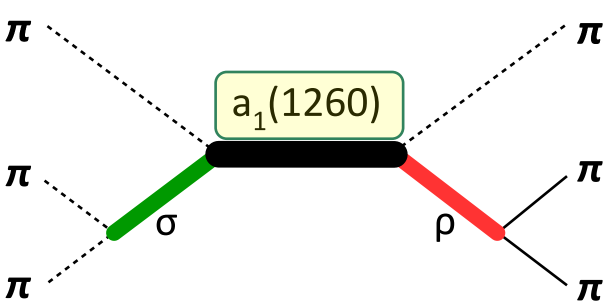
Three-body aspects: $\pi\pi N$ vs. $\pi\pi\pi$

Light mesons

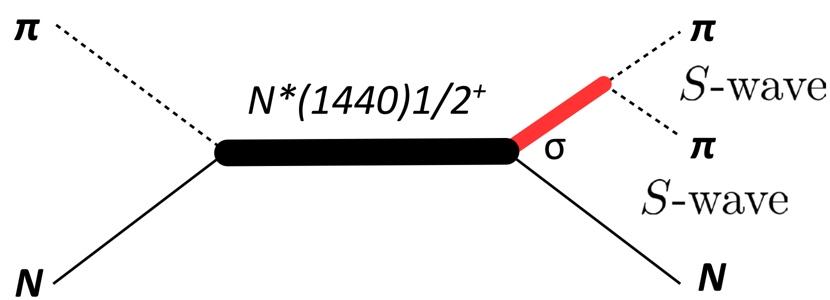
- COMPASS @ CERN: $\pi_1(1600)$ discovery
- GlueX @ Jlab in search of hybrids and exotics,
- Finite volume spectrum from lattice QCD:
  Lang (2014), Woss [HadronSpectrum] (2018)
  Hörz (2019), Culver (2020, 21,…), Fischer (2020),
  Hansen/HadSpec (2020)







Light baryons



- Roper resonance is debated for ~50 years in experiment.
- 1st calculation w. meson-baryon operators on the lattice: **Lang et al. (2017)**

Three-body unitarity with isobars * [Mai 2017]

$$\langle q_1,q_2,q_3|(\hat{T}-\hat{T}^\dagger)|p_1,p_2,p_3\rangle = i\int_P \langle q_1,q_2,q_3|\hat{T}^\dagger|k_1,k_2,k_3\rangle\langle k_1,k_2,k_3|\hat{T}|p_1,p_2,p_3\rangle$$

$$\times \prod_{\ell=1}^{3}\left[\frac{\mathrm{d}^4k_\ell}{(2\pi)^4}(2\pi)\boxed{\delta^+(k_\ell^2-m^2)}\right](2\pi)^4\delta^4\left(P-\sum_{\ell=1}^{3}k_\ell\right)$$

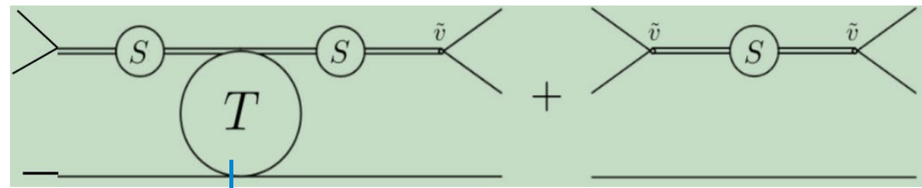
delta function sets all intermediate particles on-shell

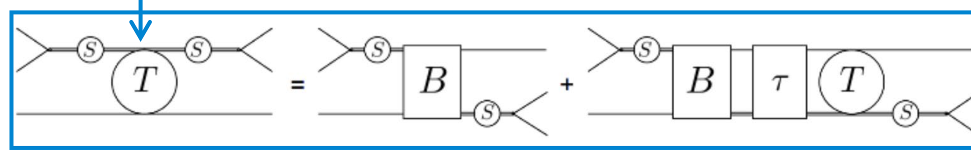
Idea: To construct a 3B amplitude, start directly from unitarity (based on ideas of 60's); match a general amplitude to it

* "Isobar" stands for two-body sub-amplitude which can be resonant or not; can be matched to CHPT expansion to one loop if desired. Isobars are re-parametrization of full 2-body amplitude [Bedaque] [Hammer]

Three-body unitarity

$$\langle q_1, q_2, q_3|(\hat{T}-\hat{T}^\dagger)|p_1, p_2, p_3\rangle \;=\; i\int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$

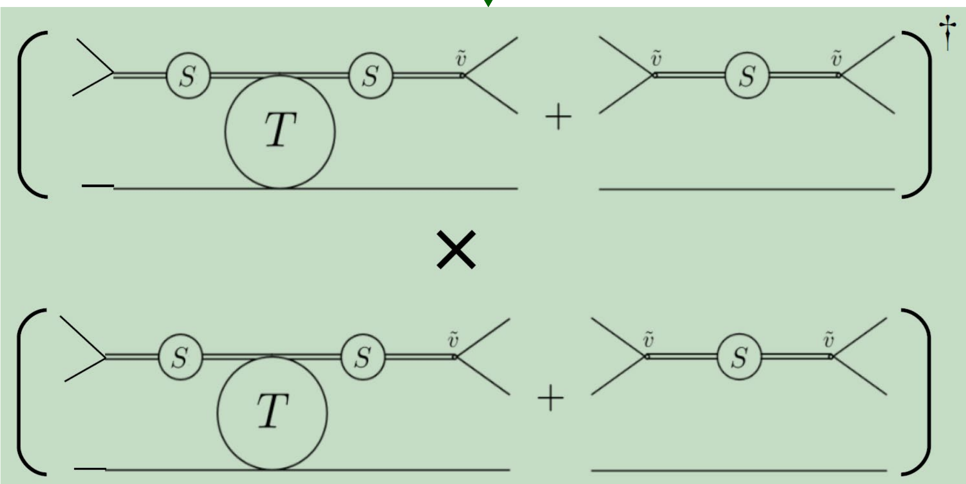




General Ansatz for the isobar-spectator interaction
→ **B & τ** are **new** unknown functions

Three-body unitarity

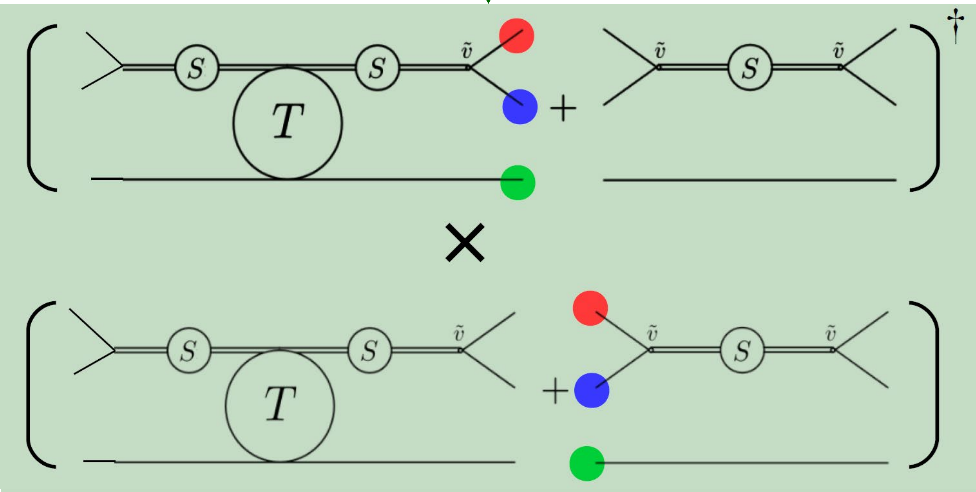
$$\langle q_1, q_2, q_3|(\hat{T}-\hat{T}^\dagger)|p_1, p_2, p_3\rangle \quad = \quad i\int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$

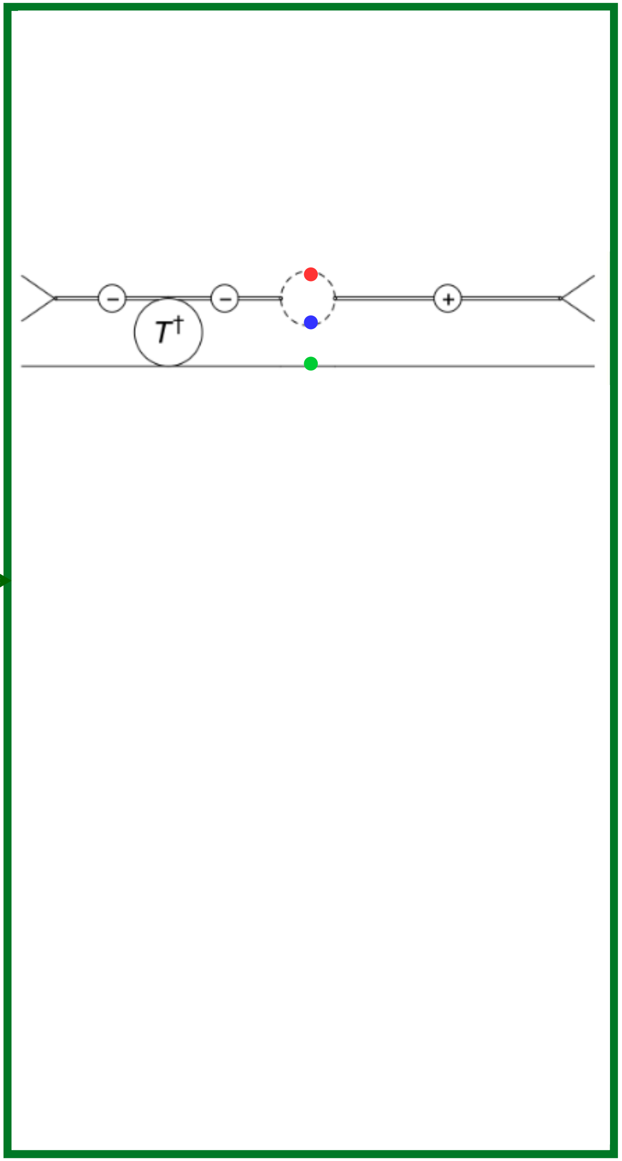


General connected-disconnected structure

Three-body unitarity

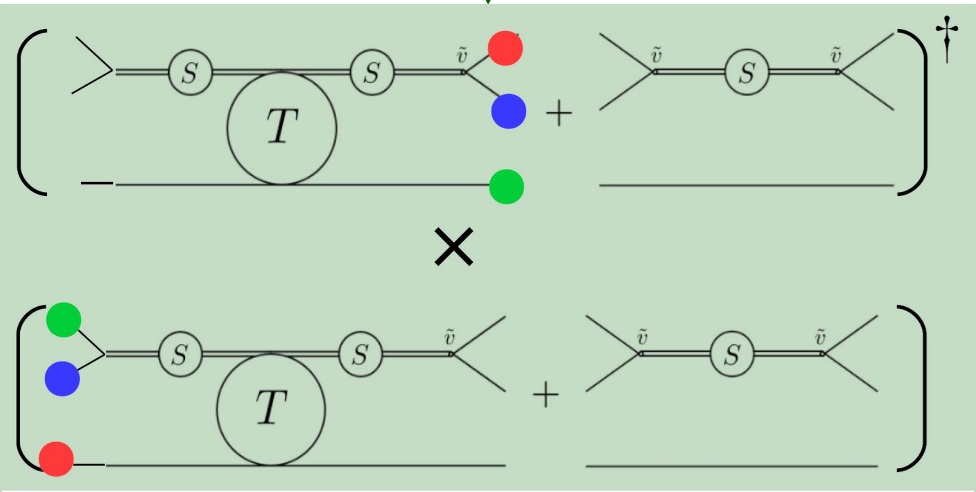
$$\langle q_1,q_2,q_3|(\hat{T}-\hat{T}^\dagger)|p_1,p_2,p_3\rangle = i\int_P \langle q_1,q_2,q_3|\hat{T}^\dagger|k_1,k_2,k_3\rangle\langle k_1,k_2,k_3|\hat{T}|p_1,p_2,p_3\rangle$$

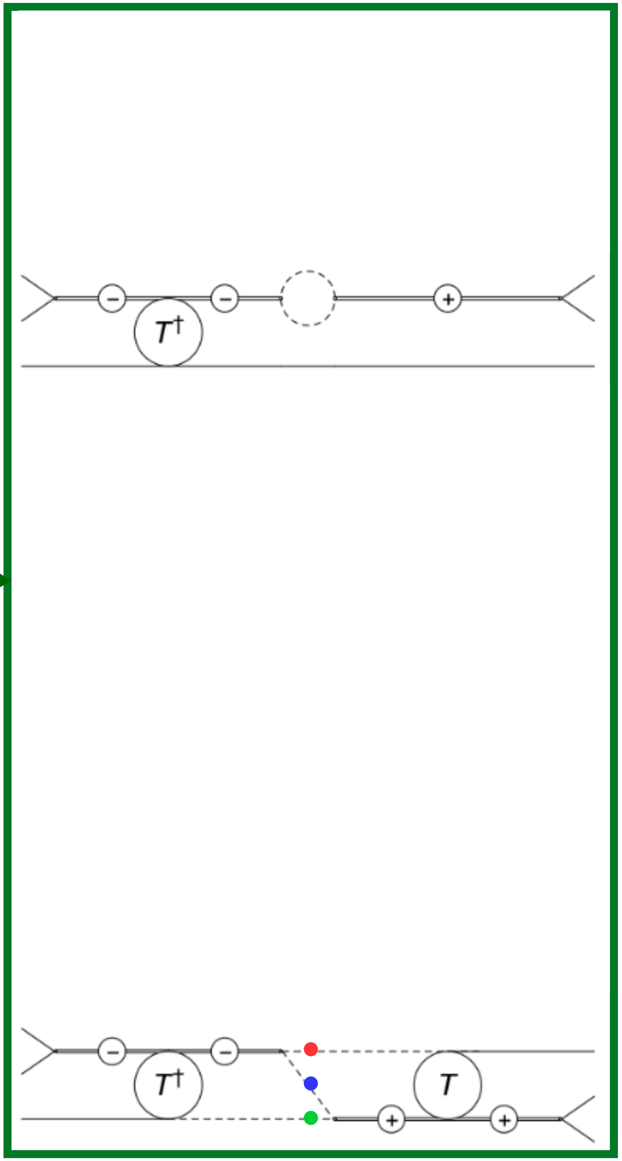




Three-body unitarity

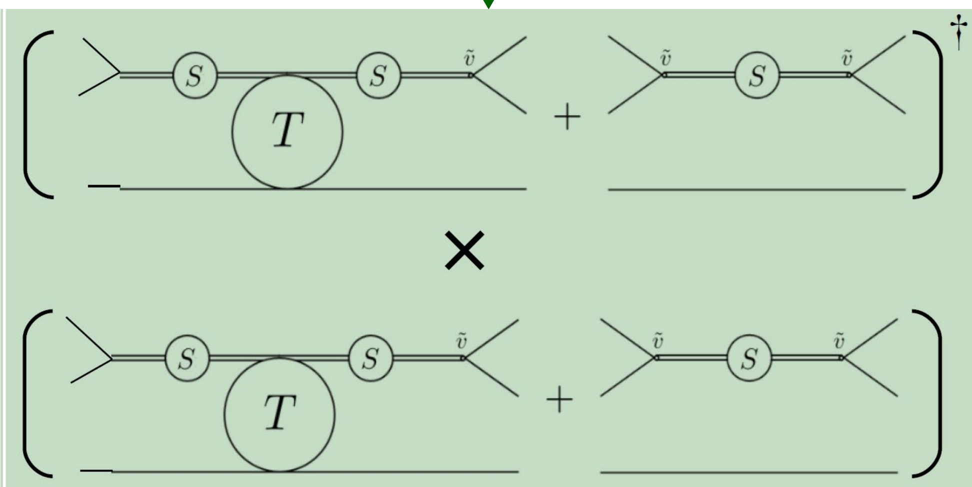
$$\langle q_1,q_2,q_3|(\hat{T}-\hat{T}^\dagger)|p_1,p_2,p_3\rangle = i\int_P \langle q_1,q_2,q_3|\hat{T}^\dagger|k_1,k_2,k_3\rangle\langle k_1,k_2,k_3|\hat{T}|p_1,p_2,p_3\rangle$$



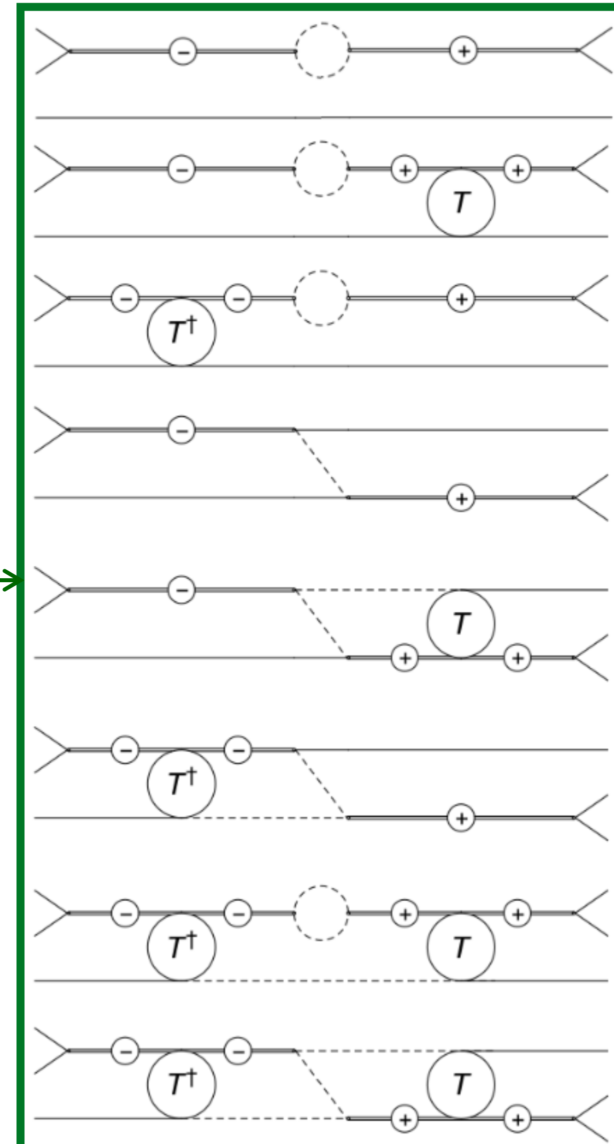


Three-body unitarity

$$\langle q_1, q_2, q_3|(\hat{T}-\hat{T}^\dagger)|p_1, p_2, p_3\rangle \quad = \quad i\int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$

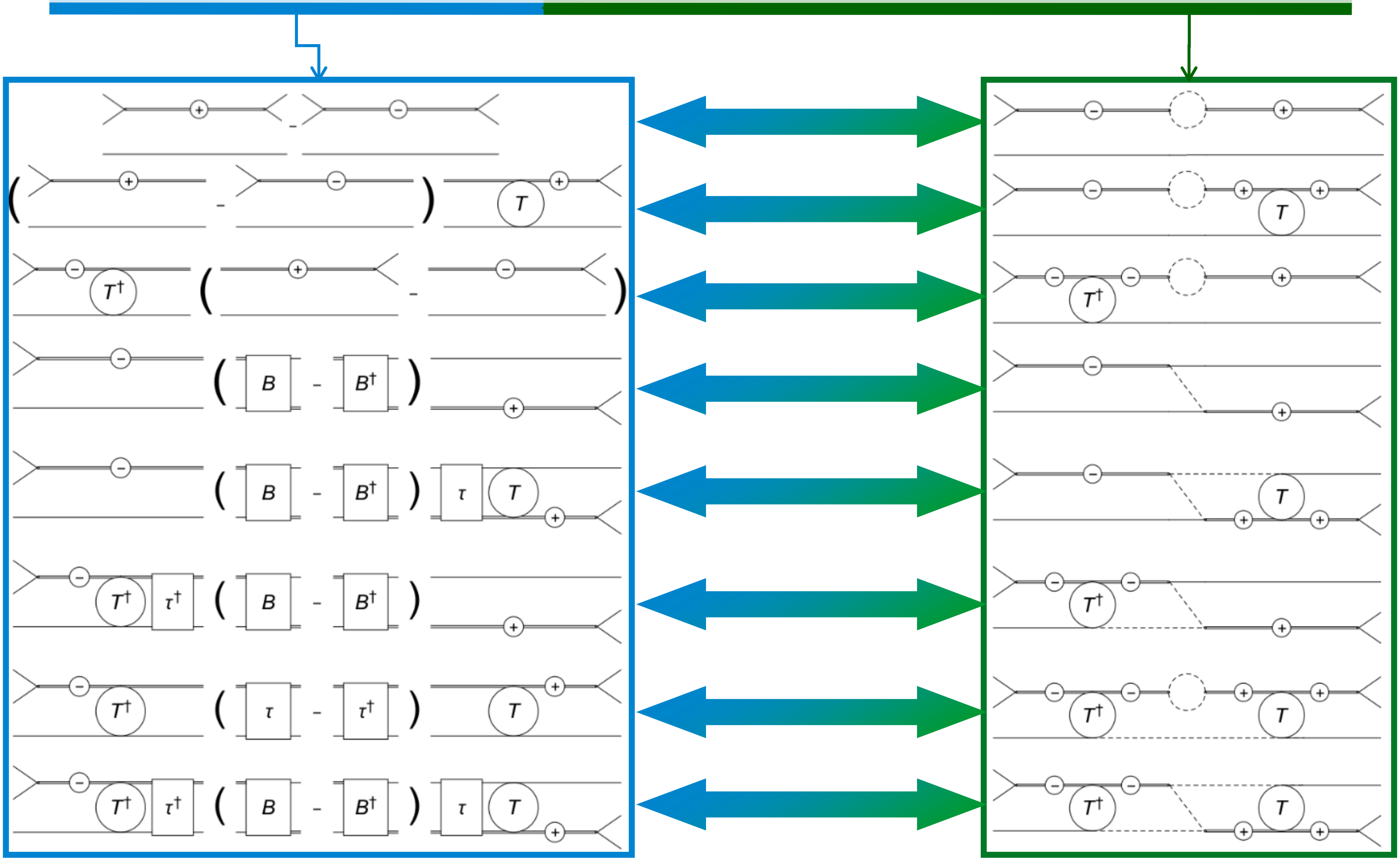


8 topologies



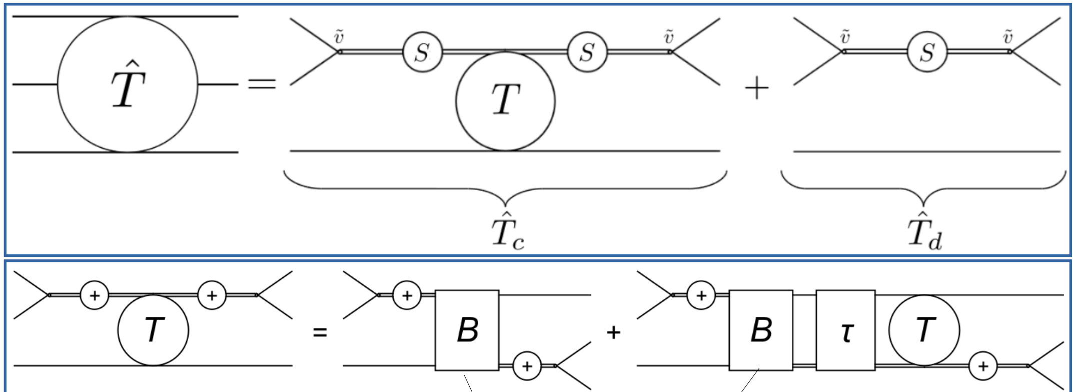
Three-body unitarity

$$\langle q_1, q_2, q_3|(\hat{T} - \hat{T}^\dagger)|p_1, p_2, p_3\rangle \quad = \quad i\int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$



Scattering amplitude

$3 \to 3$ scattering amplitude is a 3-dimensional integral equation



LS-type

- Imaginary parts of ***B, S*** are fixed by **unitarity/matching**
- B, S are determined **consistently** through 8 different relations

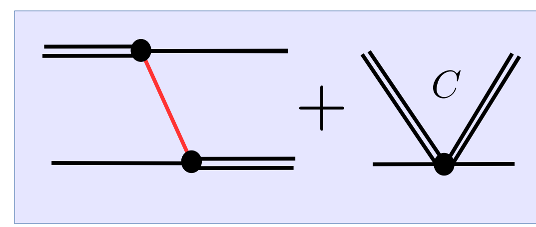
Matching → $$\mathrm{Disc}\, B(u) = 2\pi i \lambda^2 \frac{\delta\left(E_Q - \sqrt{m^2 + \mathbf{Q}^2}\right)}{2\sqrt{m^2 + \mathbf{Q}^2}}$$

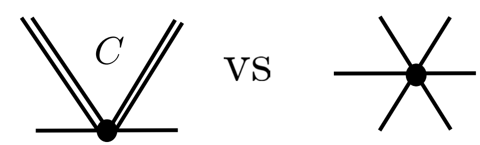
- un-subtracted dispersion relation

$$\langle q|B(s)|p\rangle = -\frac{\lambda^2}{2\sqrt{m^2+\mathbf{Q}^2}\left(E_Q - \sqrt{m^2+\mathbf{Q}^2} + i\epsilon\right)} + C$$

- one-**π** exchange in TOPT → *RESULT, NOT INPUT !*

- One <u>can</u> map to field theory but does not have to. Result is a-priori dispersive.

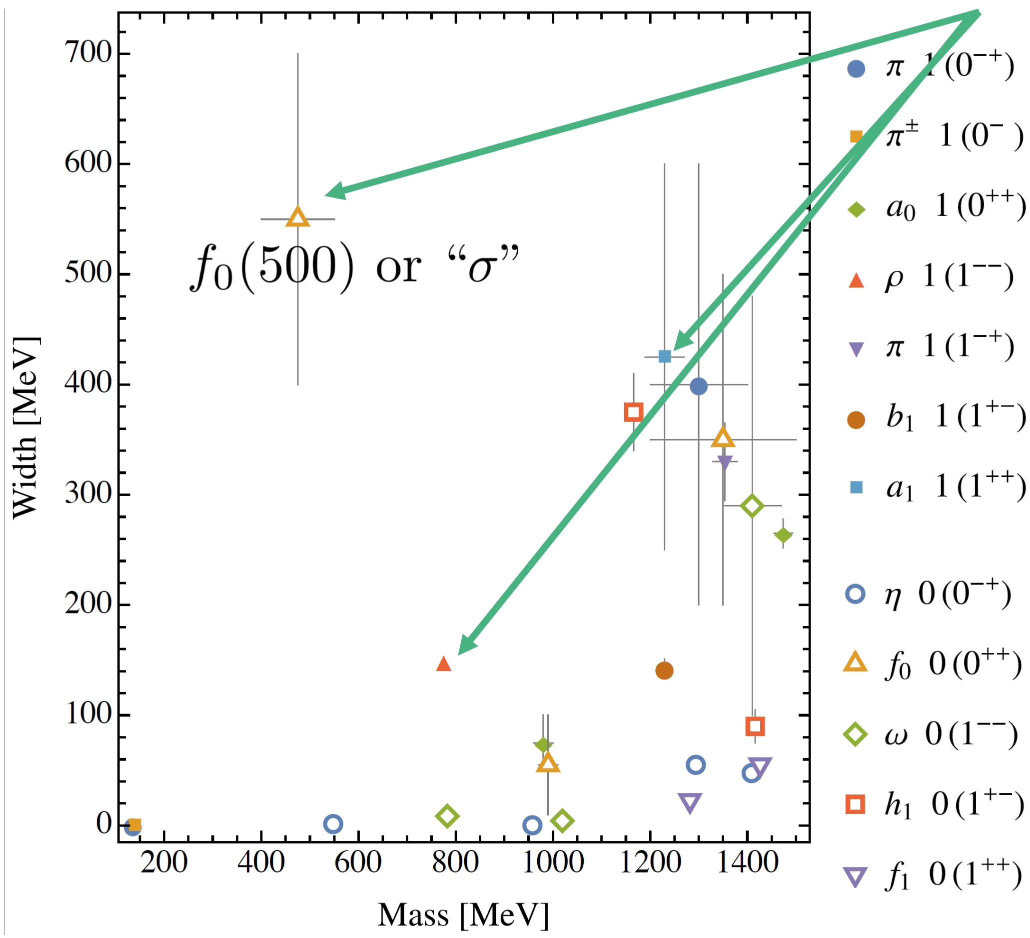




Add. Steps to map to theory might be needed [Brett (2021)]

Study the "intermediate energy region"

Transition region where hadrons are almost confined: "**Resonances**"



We concentrate on these resonances!

Resonances decay, to 2, 3, … particles:

$\rho \rightarrow \pi\pi,\ \gamma\gamma$

$a_1 \rightarrow \pi\pi\pi, K\bar{K}^*, \pi\gamma, 5\pi, \ldots$

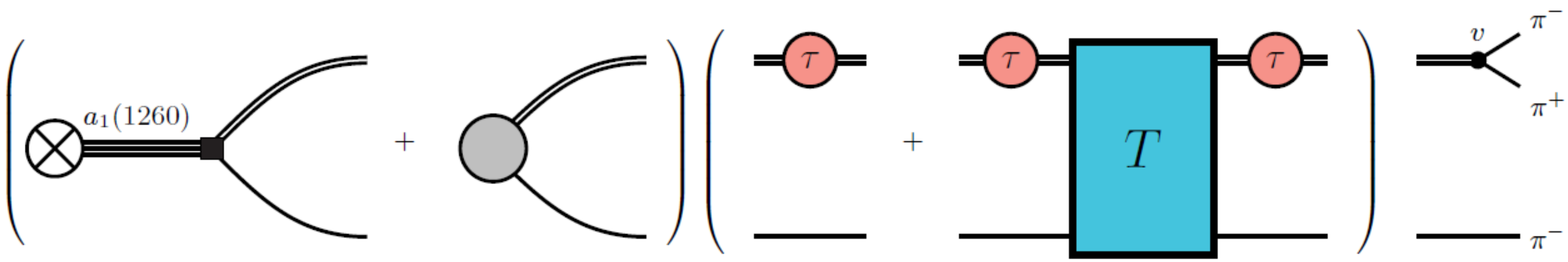
$f_0 \rightarrow \pi\pi,\ \gamma\gamma$

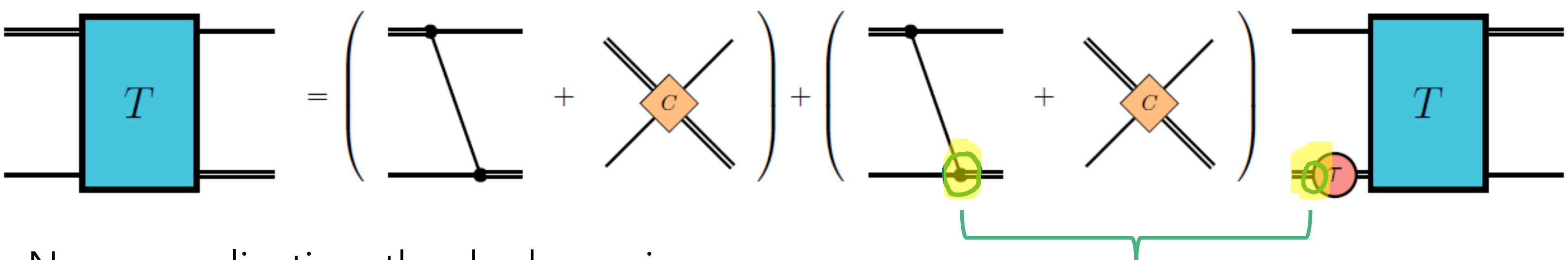
Picture: M. Mai

The $a_1(1260)$ and its Dalitz plots

[Sadasivan 2020]

- Disconnected and connected decays for three-body untarity





- New complication: the rho has spin:

$$T_{\lambda'\lambda}(\boldsymbol{p},\boldsymbol{q}_1) = (B_{\lambda'\lambda}(\boldsymbol{p},\boldsymbol{q}_1) + C) +$$
$$\sum_{\lambda''} \int \frac{d^3\boldsymbol{l}}{(2\pi)^3 2E_l} (B_{\lambda'\lambda''}(\boldsymbol{p},\boldsymbol{l}) + C)\,\tau(\sigma(l))T_{\lambda''\lambda}(\boldsymbol{l},\mathbf{q}_1)$$

3B unitarity allows for form factors for UV regularization IF covariant & consistent for on-shell. E.g.:

$$F(\sigma, Q^2) = \frac{\Lambda^4}{\Lambda^4 + e^{1+(Q^2/4-(\sigma-4m_\pi^2))/(1\mathrm{GeV}^2)}}$$

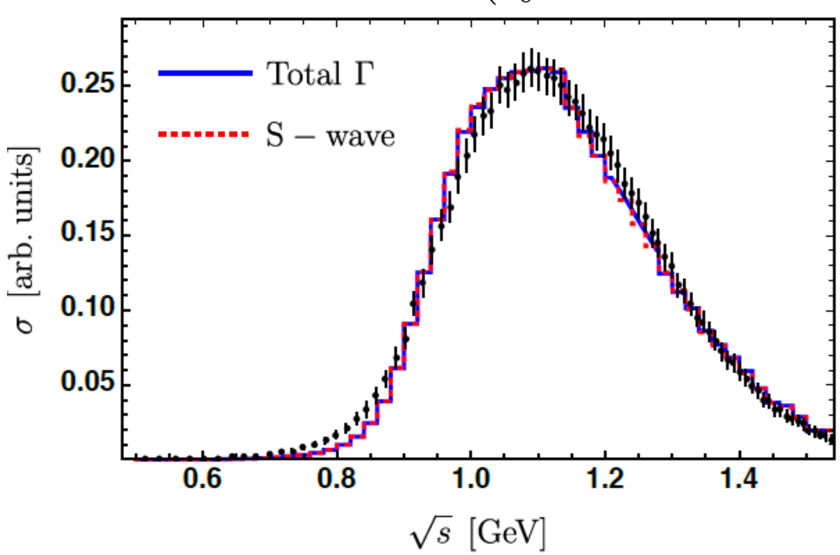
Fitting the lineshape & predicting Dalitz plots

[Sadasivan 2020]

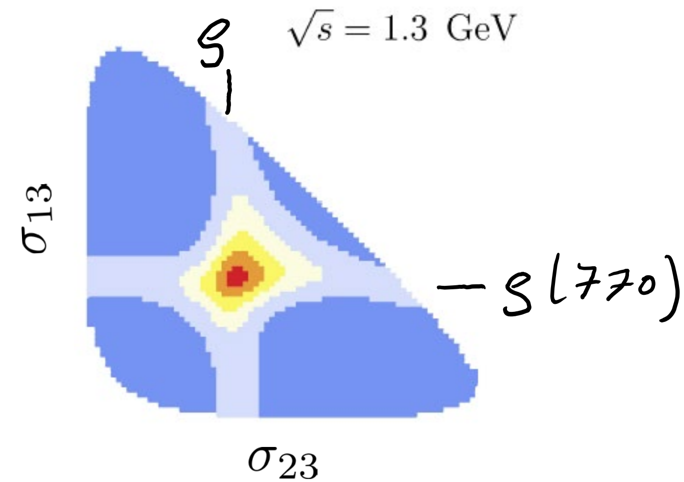
- One can have $\pi\rho$ in S- and D-wave coupled channels
- "Line shape": integrate all three final-state momenta,

$$\mathcal{L}(\sqrt{s}) = \frac{1}{\sqrt{s}} \int \frac{d^3\boldsymbol{q}_1}{(2\pi)^3} \frac{d^3\boldsymbol{q}_2}{(2\pi)^3} \frac{d^3\boldsymbol{q}_3}{(2\pi)^3} \frac{1}{2E_{q_1} 2E_{q_2} 2E_{q_3}} \quad (18)$$
$$\times (2\pi)^4 \delta^4(P_3 - q_1 - q_2 - q_3) \overline{|\Gamma(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3)|}^2 .$$

$a_1 \to \pi^-\pi^-\pi^+$ (symmetrize π^-'s!)



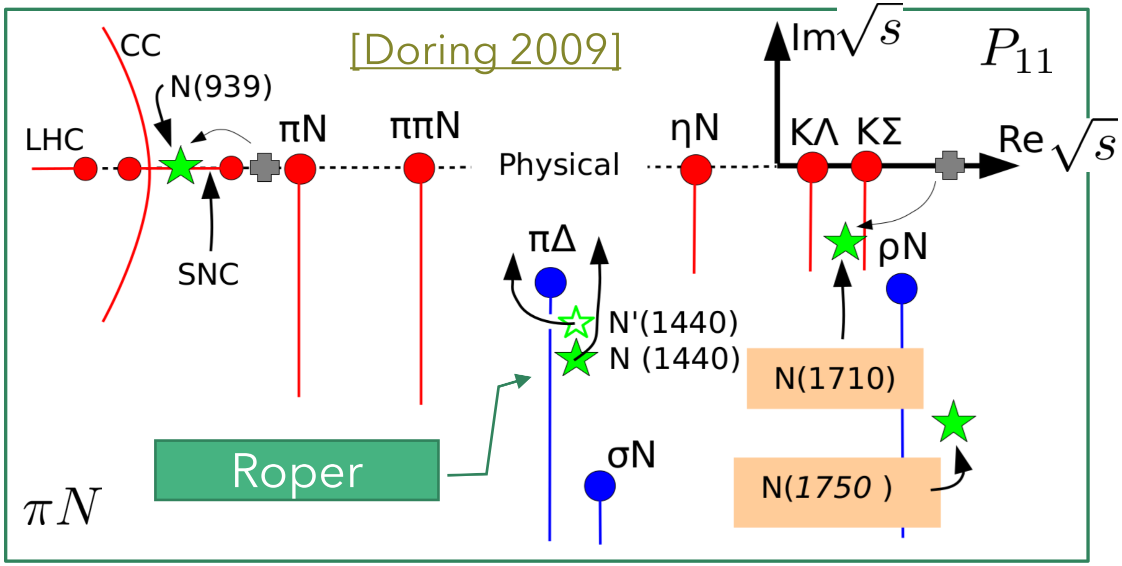
predict →



Where is the resonance pole in s?

Hadronic resonances as poles

- Defining resonances as poles in amplitudes at complex energies resolves all mentioned problems
  - Real part of pole position ⟺ Mass
  - 2x Imaginary part of pole position ⟺ Width
  - Pole residue ⟺ Branching ratio into different channels because amplitudes factorize at poles
  - Next goal: What is this?
    - Red: Real thresholds
    - Blue: resonanat sub-channel thres.
    - Double pole Roper
    - Note partial-wave cuts (CC, SNC) that disappear in plane-wave amplitude



Details on sub-threshold structure

- For πN-system

[Doering (2009)]

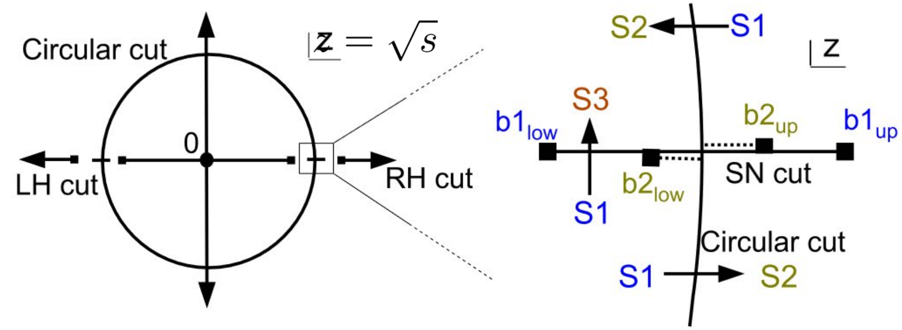
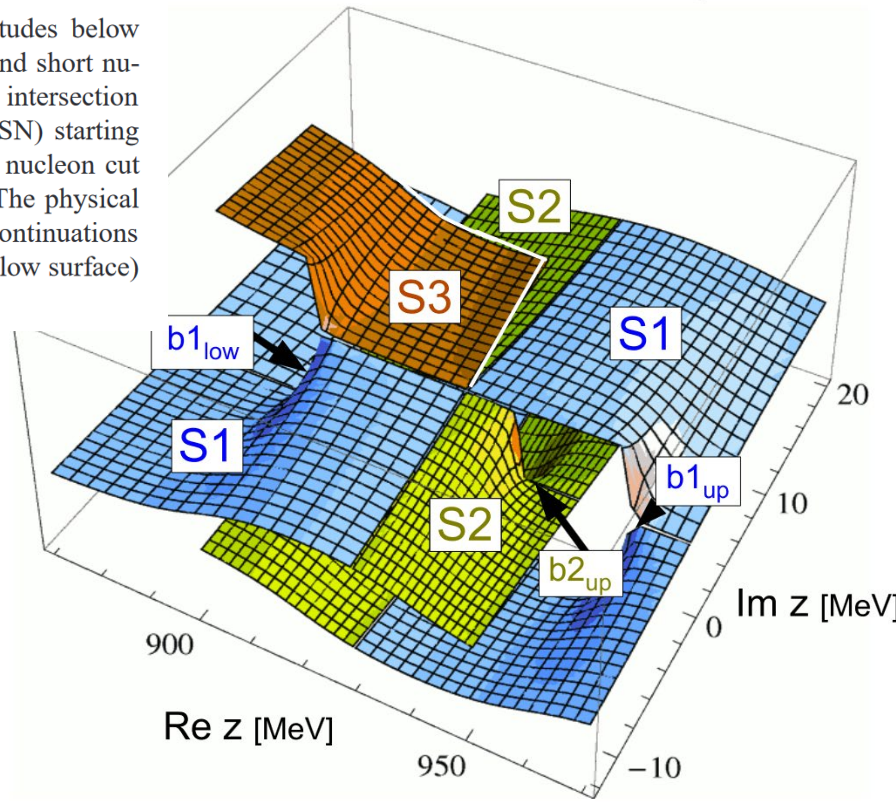
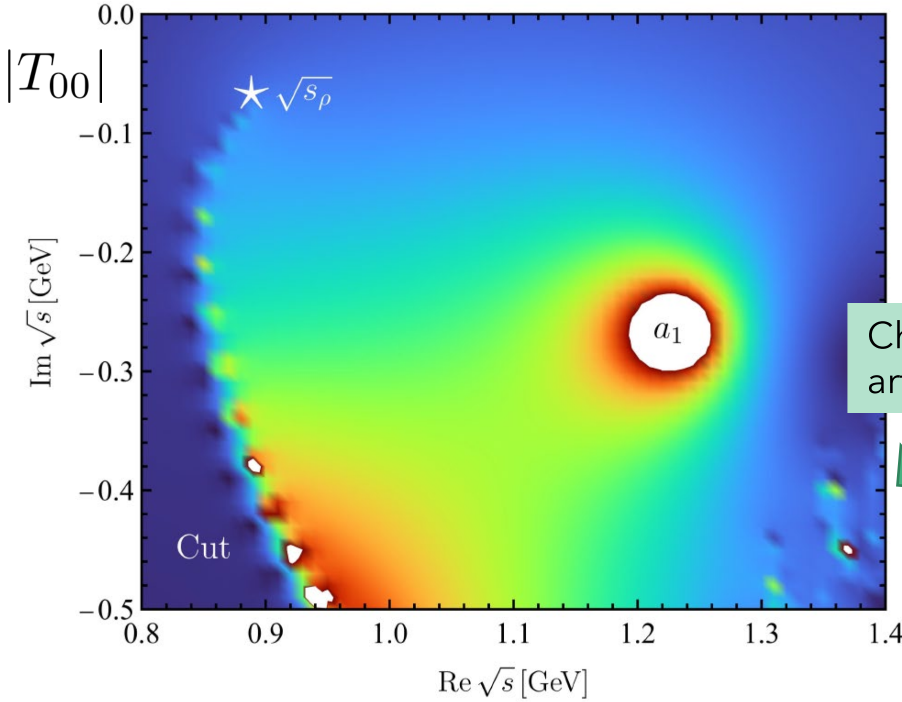


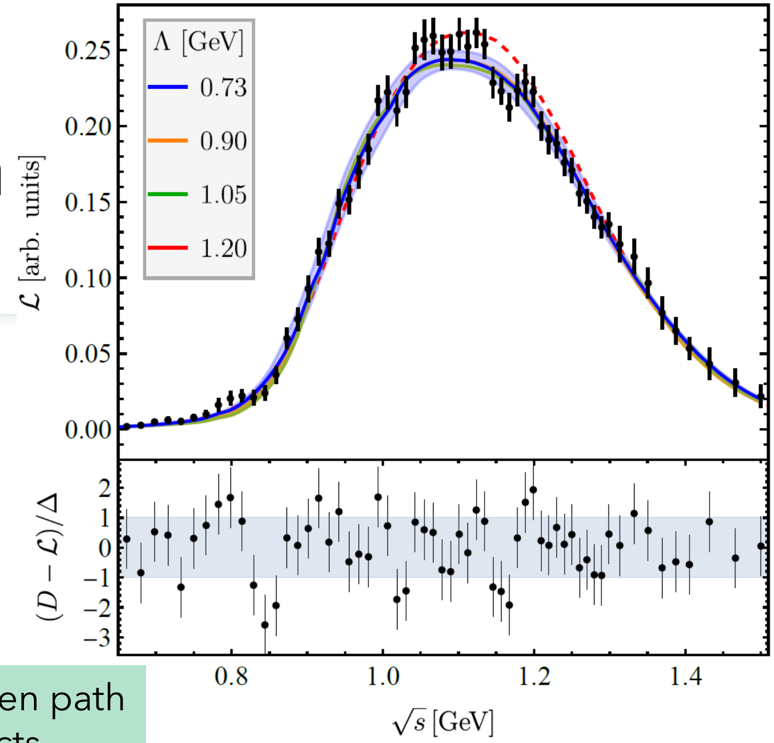
Figure 22: The analytic structure of the πN partial wave amplitudes below threshold. Upper left: right-hand (RH), left-hand (LH), circular and short nucleon cuts in the z-plane. Upper right: Analytic structure at the intersection of circular and short nucleon (SN) cut. The short nucleon cut (SN) starting at $b1_{\mathrm{low}}$ on sheet $S1$ ends at $b2_{\mathrm{up}}$ on sheet $S2$, whereas the short nucleon cut starting at $b2_{\mathrm{low}}$ on sheet S2 ends at $b1_{\mathrm{up}}$ on sheet $S1$. Lower: The physical Riemann sheet is indicated with $S1$ (blue surface), the analytic continuations along the circular and short nucleon cuts are indicated with $S2$ (yellow surface) and $S3$ (orange surface), respectively.



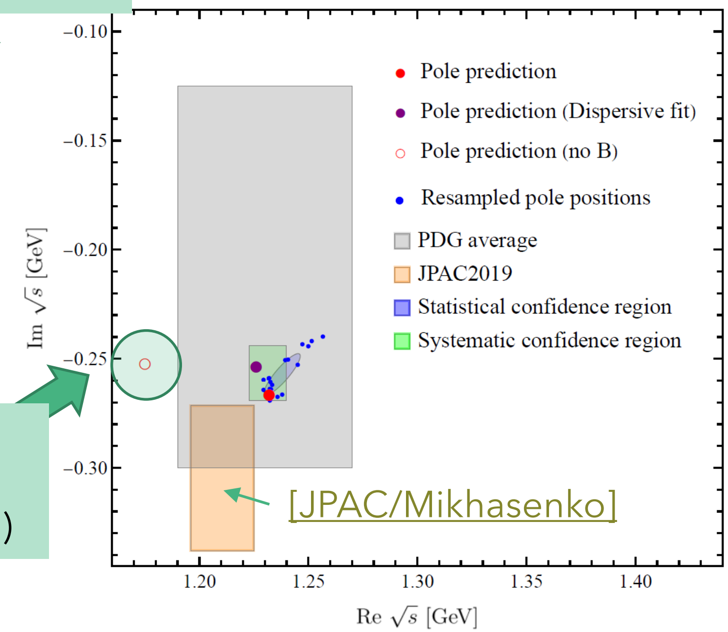
Result: Pole position

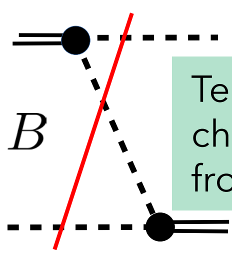
Technicalities analytic cont.: contour deformation } [Sadasivan (2021)] [Doering (2009)]





Chosen path artifacts





Technical challenges from 3B cuts

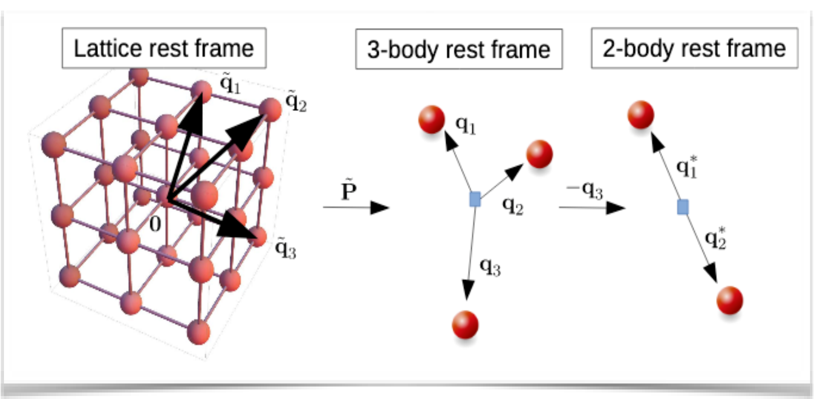
If the B-term is neglected + refit (unitarity violated)

[JPAC/Mikhasenko]

See also: S. M. Dawid [2303.04394 [nucl-th]]

Quantization condition (FVU)

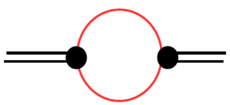
- General procedure:
  - Formulate an amplitude in infinite volume identifying each possible on-shell configuration
  - Discretize all momenta
  - Solve in plane-wave basis,
    project to to irreps then.

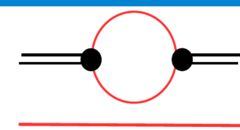


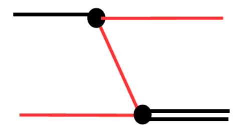
Lattice momenta with boosts from (0,0,0) to (0,
defined in moving frame

Three-body quantization condition/FVU

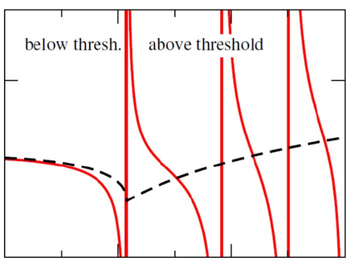
Two-body unitarity	Three-body unitarity
↓	↓
On-shell condition	On-shell condition



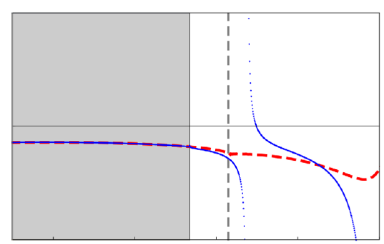


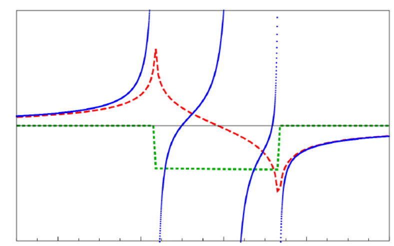


Imaginary parts	Imaginary parts



Infinite → Fin. Vol





Power-law fin-vol. effects	Power-law fin-vol. effects
↓	↓

Lüscher

$$p\cot\delta(p) = -8\pi\sqrt{s}\left(\tilde{G}(E) - \mathrm{Re}\,G(E)\right)$$

.

Quantization Condition

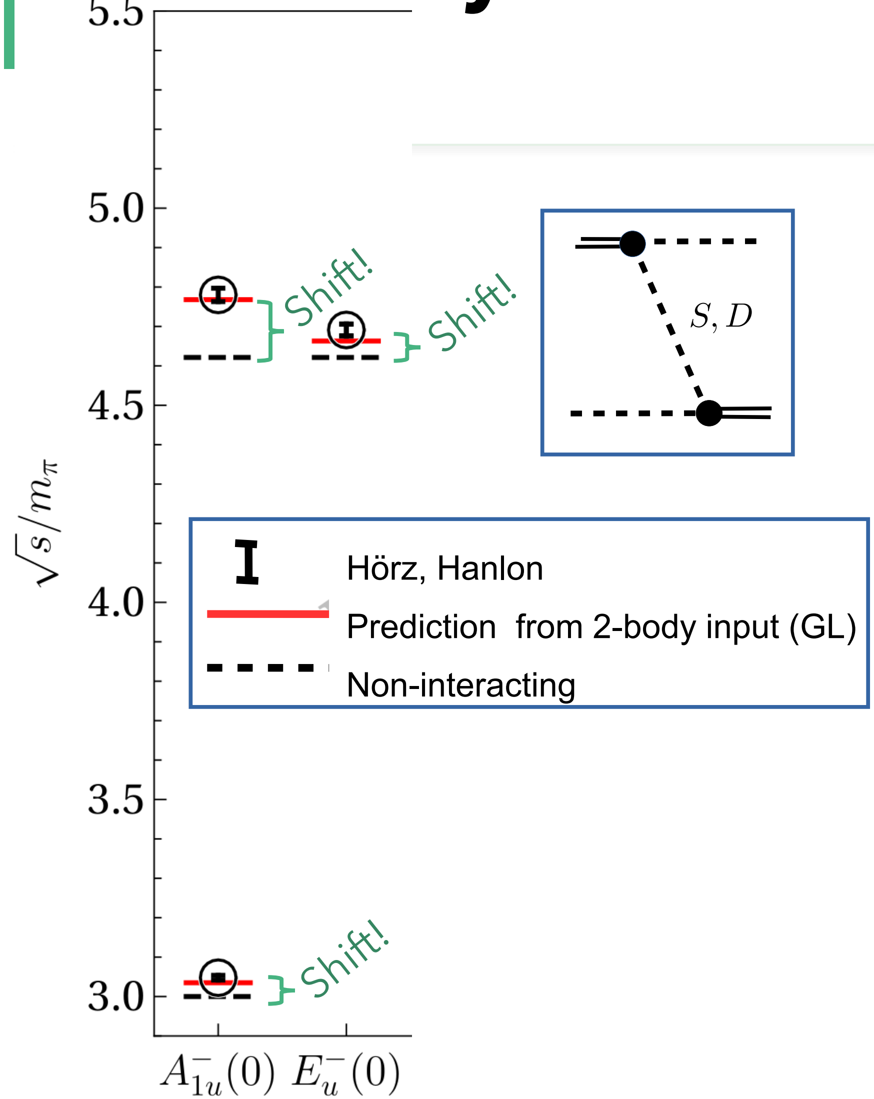
$$\mathrm{Det}\left(\mathbf{B}^{\Gamma \mathbf{ss'}}_{\mathbf{uu'}}(\mathbf{W^2}) + \frac{\mathbf{2E_s\,L^3}}{\vartheta(\mathbf{s})}\tau_{\mathbf{s}}(\mathbf{W^2})^{-1}\delta_{\mathbf{ss'}}\delta_{\mathbf{uu'}}\right) = \mathbf{0}$$

.

[Review Mai et al.]

Unitarity shows in FV spectrum

[GWUQCD/Culver 2019]



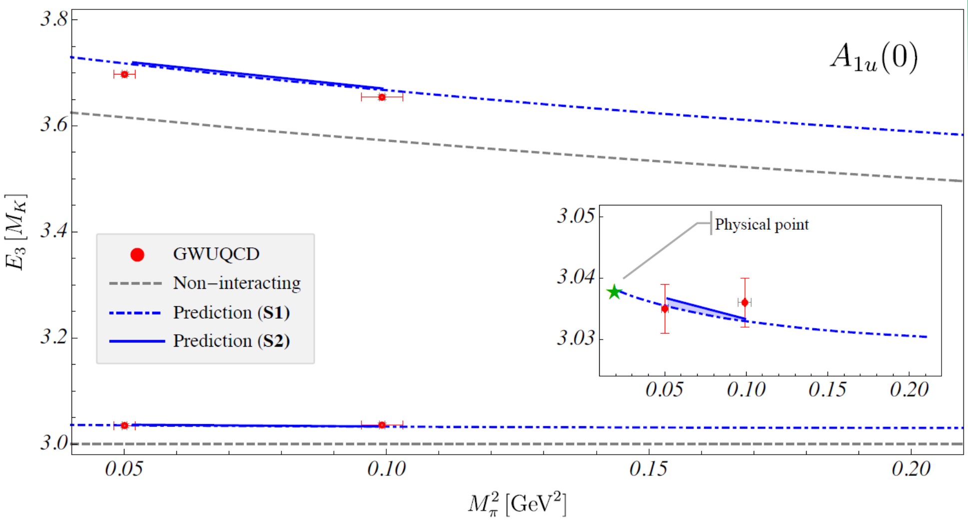
S **D** (lowest participating wave)

- 3-body "force" set to zero
- D-wave prediction qualitatively good
  - → Relative/absolute strength between S- and D-wave matched
  - → Consequence that 3-body interaction dominated by exchange
  - → Consequence of 3-body Unitarity
- Three-body unitarity directly visible in the eigenvalue spectrum of lattice QCD
- Many additional levels, including boosts (not shown)

More examples FVU

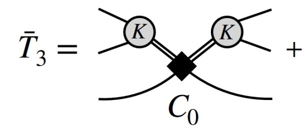
- A first look at $K^-K^-K^-$

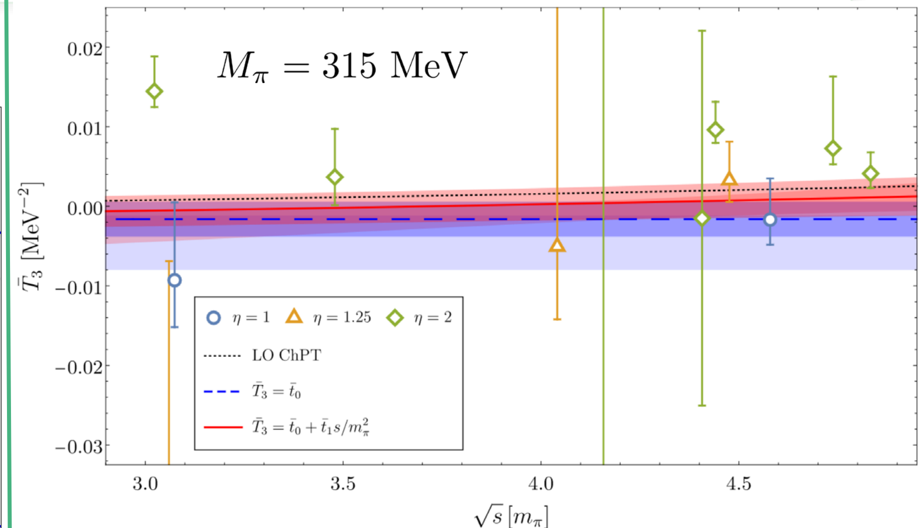
[Alexandru 2020]

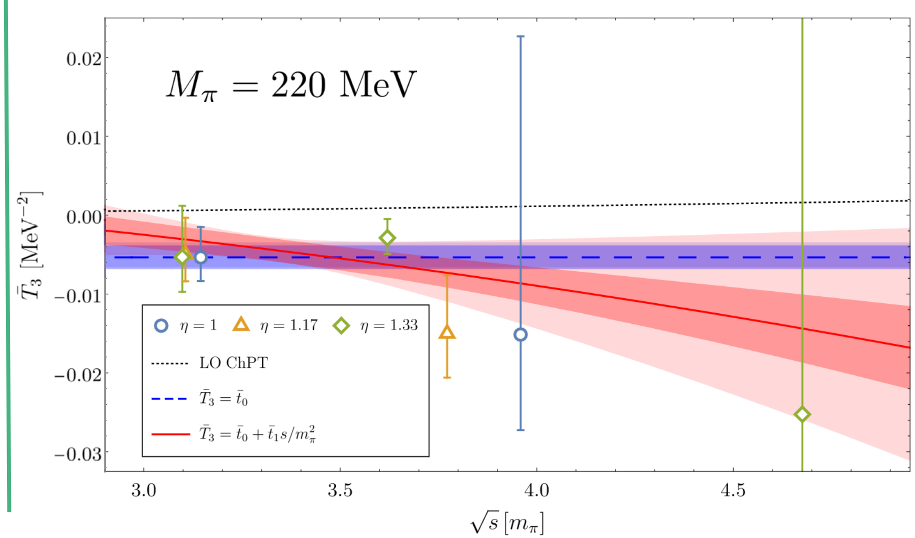


- Here: Mass dependence from NLO IAM
- Extraction of 3B force, unequal mass systems at max. isospin, … [Z. T. Draper et al. 2302.13587],
- Many more papers (F. Romero Lopez, ….)

- Extracting the 3B force [Brett, 2021]



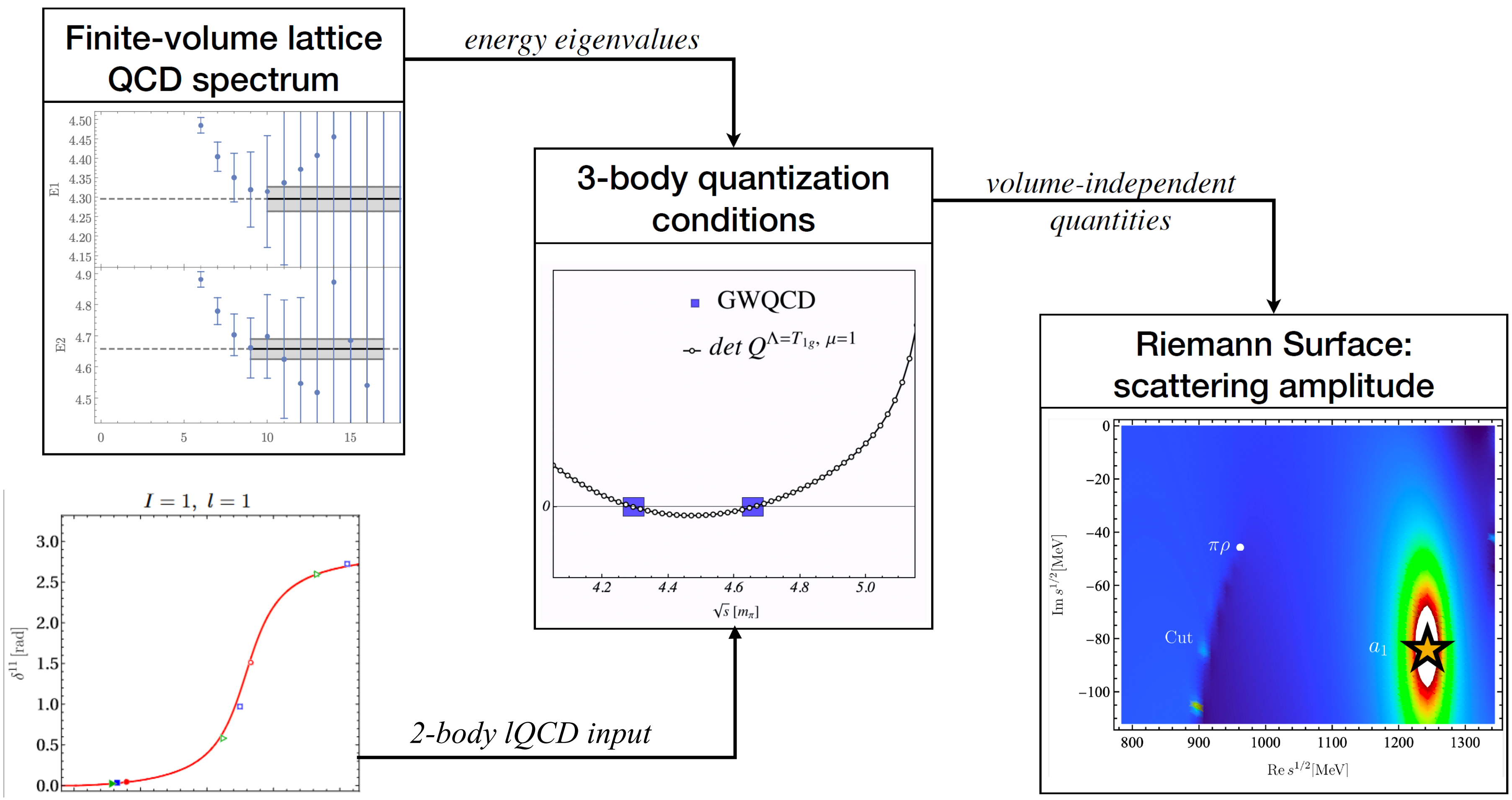




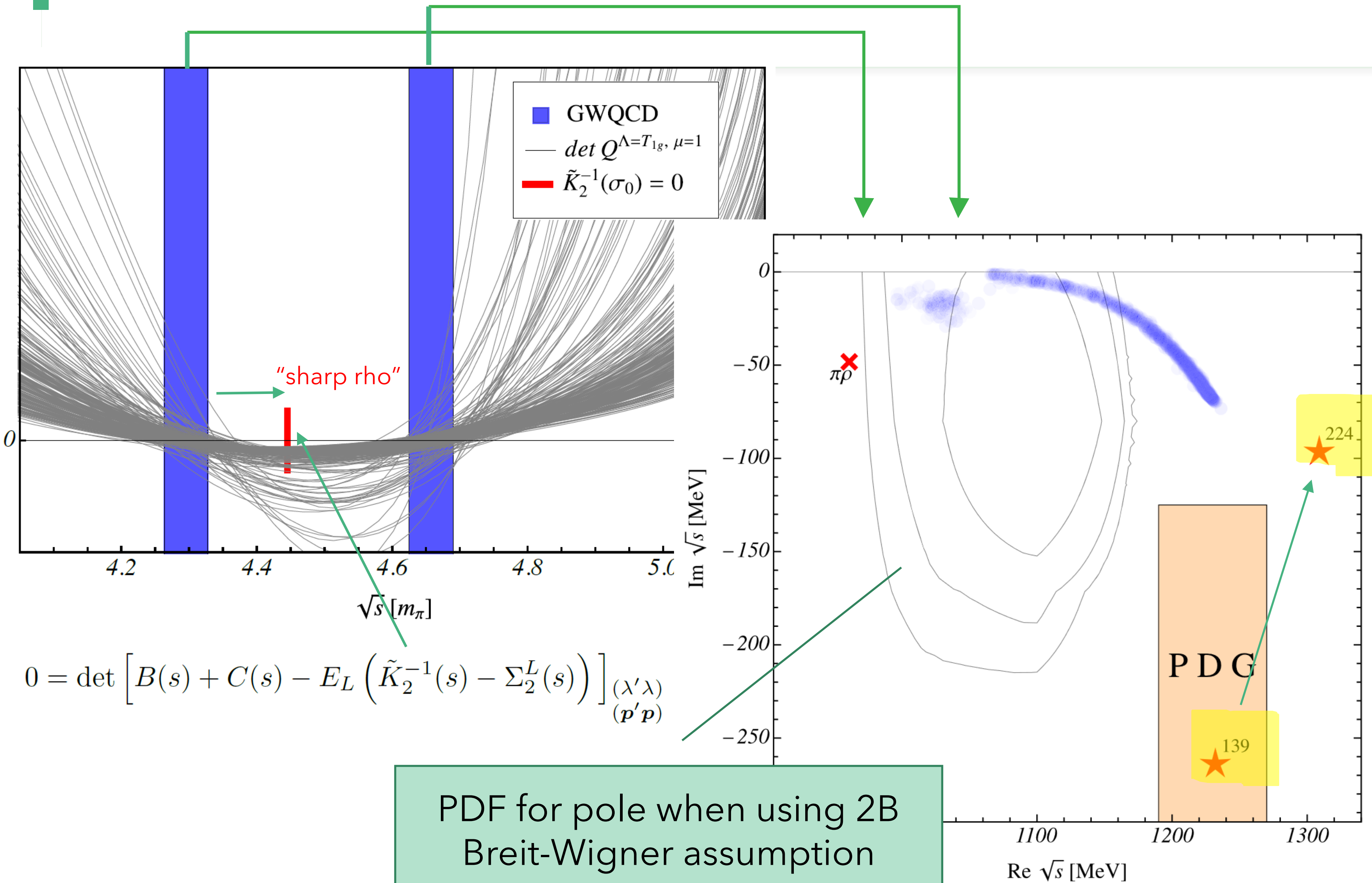
Extraction of $a_1(1260)$ from lQCD

[Mai/GWQCD, PRL 2021]

- First-ever three-body resonance from 1st principles (with explicit three-body dynamics).



Results – overview (4 parms)



$$0 = \det\left[B(s) + C(s) - E_L\left(\tilde{K}_2^{-1}(s) - \Sigma_2^L(s)\right)\right]_{(\lambda'\lambda)\atop(\boldsymbol{p}'\boldsymbol{p})}$$

PDF for pole when using 2B Breit-Wigner assumption

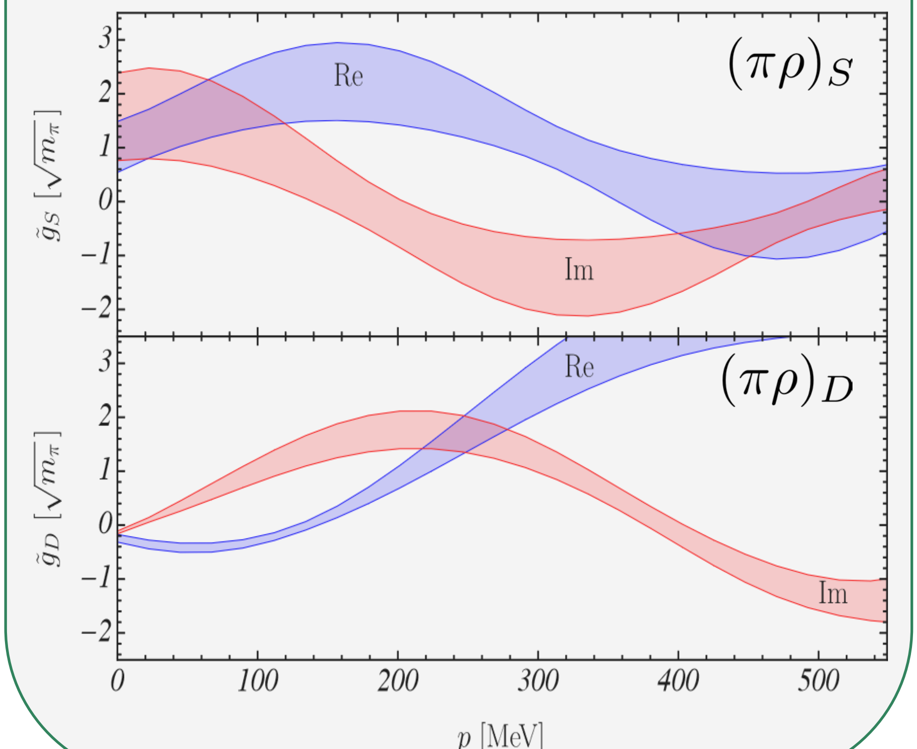
Branching ratios

- Calculate the residue at the pole:

$$\mathrm{Res}(T^{c}_{\ell'\ell}(\sqrt{s})) = \tilde{g}_{\ell'}\tilde{g}_{\ell}$$

- This result is not as reliable as pole position/existence of a_1
- More energy eigenvalues needed to better pin down the decay channels

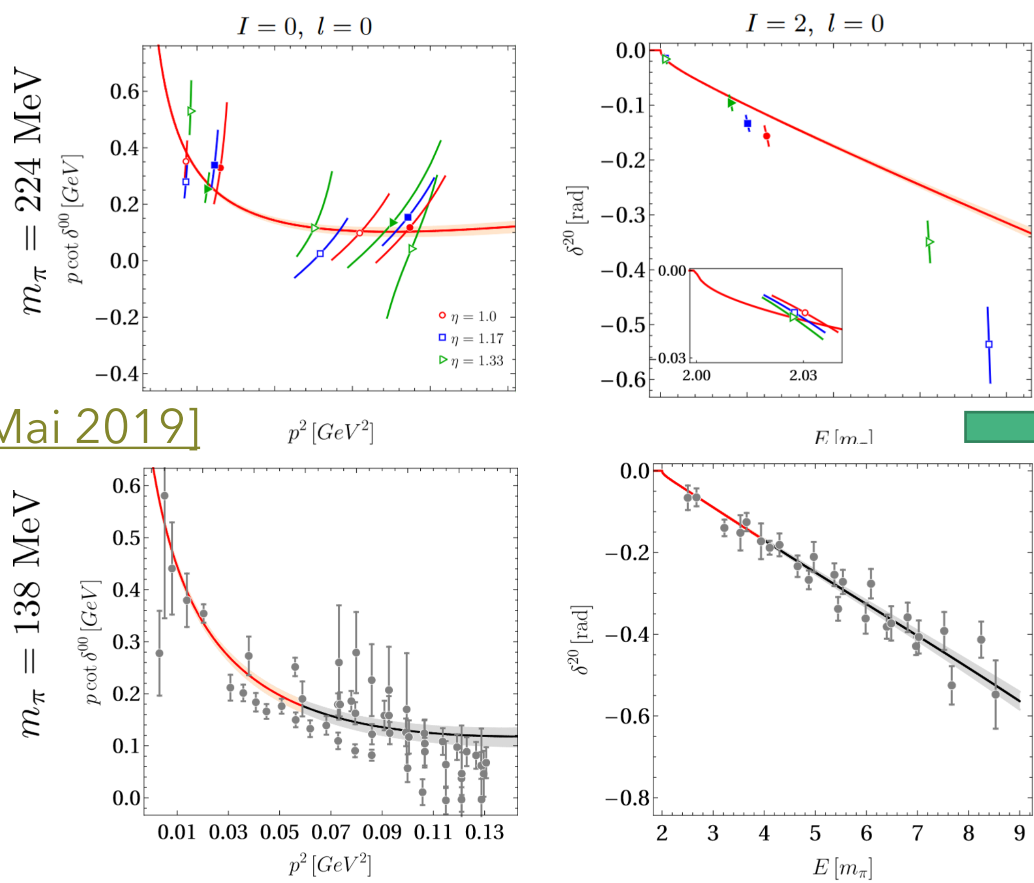
"**Branching ratios**" in 3B decays are momentum -dependent, complex pole residues



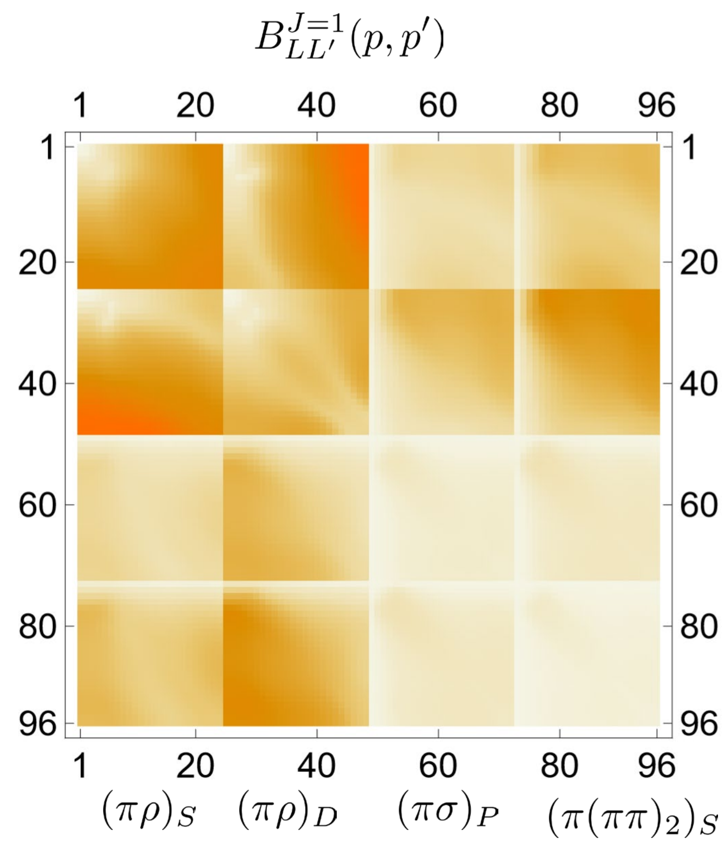
Outlook: 4+ coupled channels

$$a_1 \leftrightarrow (\pi\rho)_S \leftrightarrow (\pi\rho)_D \leftrightarrow (\pi\sigma)_P \leftrightarrow (\pi(\pi\pi)_{S,I=2})$$

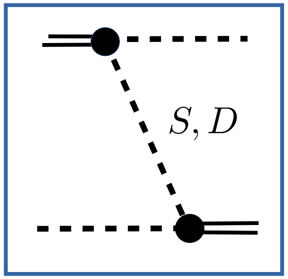
- Inclusion of all S- and P-wave isobars (from 2B lQCD input)
- Current status: physical point/inf. volume from experiment



[Mai 2019]



Summary



- Lattice QCD progress in determining the explicit dynamics of three-body systems:
  - Three pions at maximal isospin well understood (FVU, RFT, Peng,…)
  - First determination of existence and properties of a three-body resonance – the $a_1(1260)$ - in coupled channels, isobars with spin, and using three-body unitarity
- **Outlook:** More (isospin) channels; other physical systems
  - Lattice: more energy eigenvalues to assess uncertainties and put limits on decay properties. More pion masses to map out chiral trajectory
  - Phenomenology: Fit Dalitz plots instead of predicting them. Coupled-channel, unitary final-state interaction for data analysis (potentially GlueX)

Spare slides

Partial-wave decomposition

- Plane-wave basis

$$T_{\lambda'\lambda}(\boldsymbol{p},\boldsymbol{q}_1) = (B_{\lambda'\lambda}(\boldsymbol{p},\boldsymbol{q}_1) + C) + \sum_{\lambda''}\int \frac{d^3\boldsymbol{l}}{(2\pi)^3 2E_l}(B_{\lambda'\lambda''}(\boldsymbol{p},\boldsymbol{l}) + C)\,\tau(\sigma(l))T_{\lambda''\lambda}(\boldsymbol{l},\mathbf{q}_1)$$

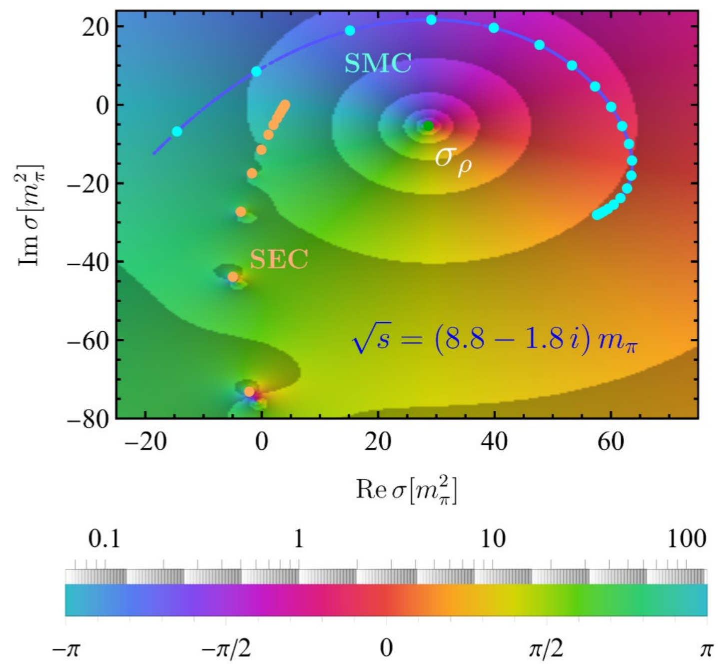
$$B^J_{\lambda\lambda'}(q_1,p) = 2\pi\int_{-1}^{+1}\mathrm{d}x\, d^J_{\lambda\lambda'}(x) B_{\lambda\lambda'}(\boldsymbol{q}_1,\boldsymbol{p}) \qquad B^J_{LL'}(q_1,p) = U_{L\lambda}B^J_{\lambda\lambda'}(q_1,p)U_{\lambda'L'}$$

- JLS basis:

$$T^J_{LL'}(q_1,p) = \left(B^J_{LL'}(q_1,p) + C_{LL'}(q_1,p)\right) + \int_0^\Lambda \frac{\mathrm{d}l\, l^2}{(2\pi)^3 2E_l}\left(B^J_{LL''}(q_1,l) + C_{LL''}(q_1,l)\right)\tau(\sigma(l))T^J_{L''L'}(l,p)$$

Analytic cont. 3-body

[Sadasivan (2021)]
[Doering (2009)]



SMC

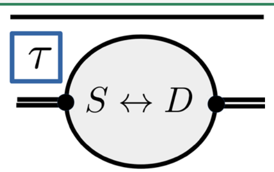
$$T^J_{LL'}(q_1,p) = \left(B^J_{LL'}(q_1,p) + C_{LL'}(q_1,p)\right) + \int_0^\Lambda \frac{dl\, l^2}{(2\pi)^3 2E_l} \left(B^J_{LL''}(q_1,l) + C_{LL''}(q_1,l)\right) \tau(\sigma(l))\, T^J_{L''L'}(l,p)$$

$$\tau^{-1}(\sigma) = K^{-1} - \Sigma\,,$$

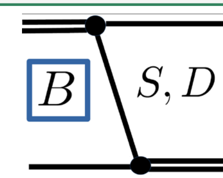
$$\Sigma = \int_0^\infty \frac{dk\, k^2}{(2\pi)^3} \frac{1}{2E_k} \frac{\sigma^2}{\sigma'^2} \frac{\tilde{v}(k)^* \tilde{v}(k)}{\sigma - 4E_k^2 + i\epsilon}$$

$$B_{\lambda\lambda'}(\boldsymbol{p},\boldsymbol{p}') = \frac{v^*_\lambda(P-p-p',p)\, v_{\lambda'}(P-p-p',p')}{2E_{p'+p}\left(\sqrt{s} - E_p - E_{p'} - E_{p'+p} + i\epsilon\right)}$$

SEC



Singularities

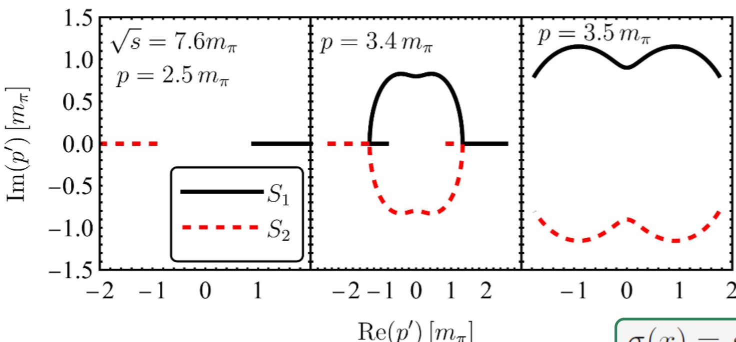


- Two contours (SMC and SEC)
- Deform both "adiabatically" to go to complex s
- Set of rules:
  - Contours cannot intersect with each others
  - Contours cannot intersect with (3-body) cuts
- Passing singularities left or right determines sheet

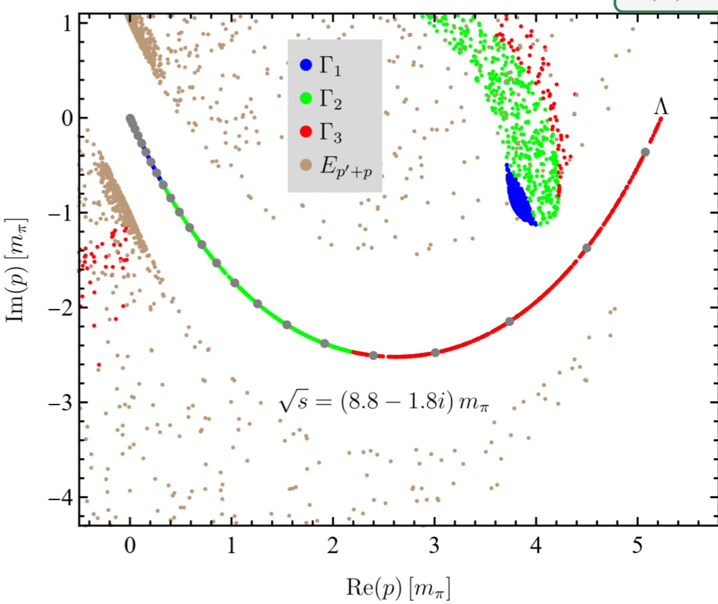
Analytic continuation 3-body (contd.)

- Three-body cuts

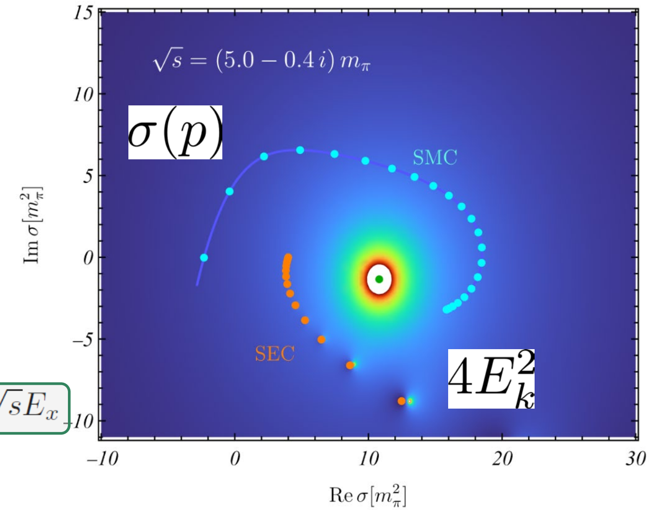
$$\sqrt{s} - E_p - E_{p'} - E_{p+p'} + i\epsilon = 0$$



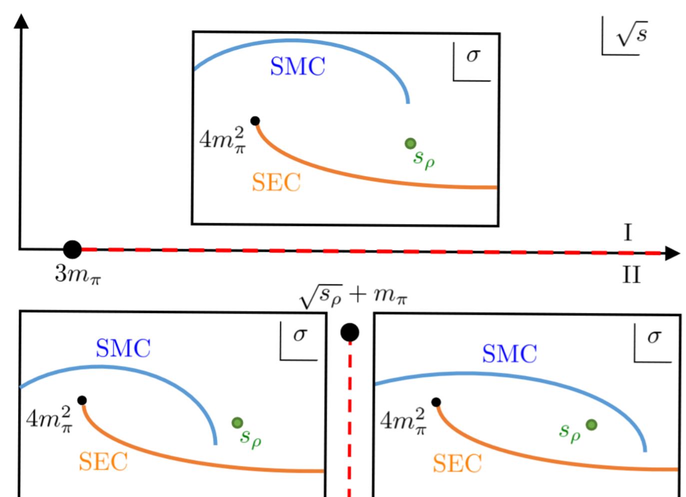
$$\sigma(x) = s + m_\pi^2 - 2\sqrt{s}E_x$$



- Complex branch points



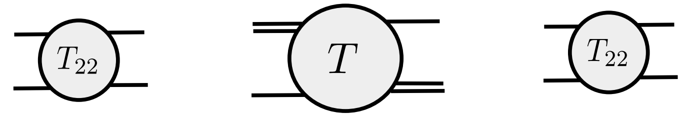
Integration limits at poles induce branch points

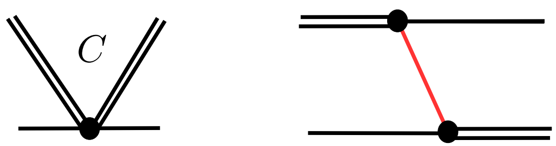


Scattering amplitude (Details)

Here: Version in which isobar rewritten in on-shell 2 → 2 scattering amplitude T_{22}

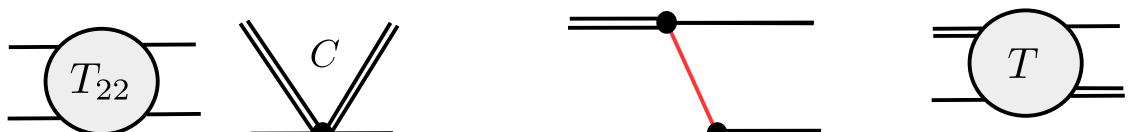
$$\langle q_1, q_2, q_3|\hat{T}_c(s)|p_1, p_2, p_3\rangle = \frac{1}{3!}\sum_{n=1}^{3}\sum_{m=1}^{3} T_{22}(\sigma(q_n))\langle q_n|T(s)|p_m\rangle T_{22}(\sigma(p_m))$$





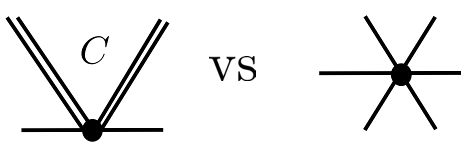
$$\langle q|T(s)|p\rangle = \langle q|C(s)|p\rangle + \frac{1}{m^2 - (P-p-q)^2 - i\epsilon}$$

$$-\int \frac{\mathrm{d}^3\boldsymbol{\ell}}{(2\pi)^3}\frac{1}{2E_\ell}T_{22}(\sigma(\ell))\left(\langle p|C(s)|\ell\rangle + \frac{1}{m^2-(P-p-\ell)^2-i\epsilon}\right)\langle \ell|T(s)|p\rangle$$



(S-wave)

Technical Detail:

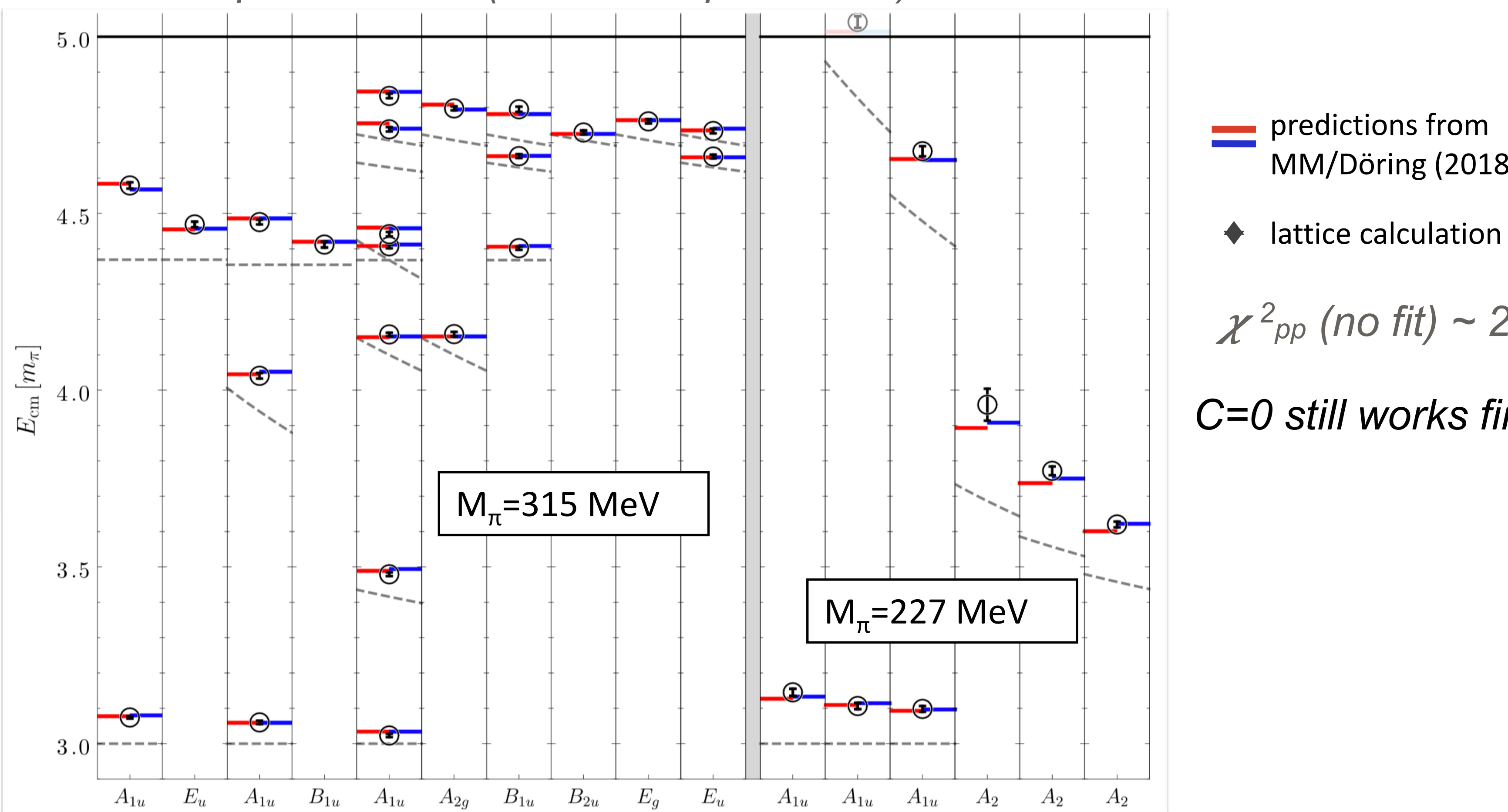


Scheme-dependent 3-body force requires a mapping [Brett (2021)]

GWUQCD data

Culver, MM, Brett, Alexandru, Döring (2019) PRD

- ***More recent*** *data is available*
  - *very dense spectrum from elongated boxes*
  - *different pion masses (chiral extrapolations?)*



χ^2_{pp} *(no fit) ~ 2*

C=0 still works fine

Plane-wave implementation of the C-term

- **Step 1**: JM-basis → Helicity basis
- **Step 2**: partial-wave basis → Plane-wave basis
- **Step 3**: C (and B, and 3B propagator) from plane-wave basis to irreps by suitable rotations

$$\mathcal{A}_{\lambda'\lambda}(s,\boldsymbol{p}',\boldsymbol{p}) = \sum_{M=-J}^{J} \frac{2J+1}{4\pi}\, \mathfrak{D}^{J*}_{M\lambda'}(\phi_{\boldsymbol{p}'},\theta_{\boldsymbol{p}'},0)\, \mathcal{A}^{J}_{\lambda'\lambda}(s,p',p)\, \mathfrak{D}^{J}_{M\lambda}(\phi_{\boldsymbol{p}},\theta_{\boldsymbol{p}},0)\,,$$

Step 2

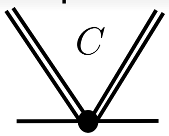
$$\mathcal{A}^{J}_{\lambda'\lambda}(s,p',p) = U_{\lambda'\ell'}\mathcal{A}_{\ell'\ell}(s,p',p)U_{\ell\lambda}\,,$$

$$U_{\ell\lambda} := \sqrt{\frac{2\ell+1}{2J+1}}(\ell 0 1\lambda|J\lambda)(1\lambda 0 0|1\lambda)) = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}, \\ \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}}, \end{pmatrix}\,,$$

Step 1

4 different fits to 2 energy eigenvalues

- Fitted isobar-spectator interaction (case 1, 2) for $|\boldsymbol{p}| \leq 2\pi/L|(1,1,0)| \approx 2.69\ m_\pi$



$$C_{\ell'\ell}(s,\boldsymbol{p}',\boldsymbol{p}) = \sum_{i=-1}^{\infty} c_{\ell'\ell}^{(i)}(\boldsymbol{p}',\boldsymbol{p})(s-m_{a_1}^2)^i$$

- a_1 can be generated as pole even though no built-in singularity

Non-zero coefficients	No of fit parameters	x^2
c_{00}^0 (no built-in pole)	1	9
c_{00}^0, c_{00}^1 (no built-in pole)	2	0.15
g_0, g_2, m_{a1}, c	4	10^{-7}

$$C_{\ell'\ell}(s,\boldsymbol{p}',\boldsymbol{p}) = g_{\ell'}\left(\frac{|\boldsymbol{p}'|}{m_\pi}\right)^{\ell'} \frac{m_\pi^2}{s-m_{a_1}^2} g_\ell \left(\frac{|\boldsymbol{p}|}{m_\pi}\right)^{\ell} + c\,\delta_{\ell'0}\delta_{\ell 0}$$

- In these cases, there is a built-in singularity, leading to resonance poles

Three kaons at maximal isospin

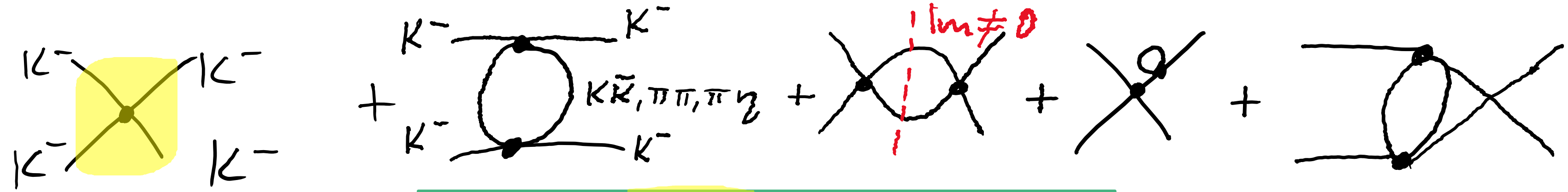
[Alexandru 2020]

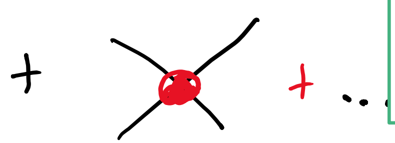
- First study of three kaons from lattice QCD with chiral amplitudes
- Other groups have improved on this in the meantime:
  - Max. isospin, non-identical masses ($\pi^+\pi^+K^+$, $\pi^+K^+K^+$)
    [Blanton 2021]
  - Pions and kaons at maximal isospin with unprecedented accuracy and no. of levels ($\pi^+\pi^+\pi^+$, $K^+K^+K^+$) [Blanton 2021]

- Two mass-degenerate light quarks (u,d); valence strange quark
- nHYP-smeared clover action
- quark propagation is treated using the LapH method with optimized inverters
- Lattice spacing determined from Wilson flow parameter w_0

Two kaons

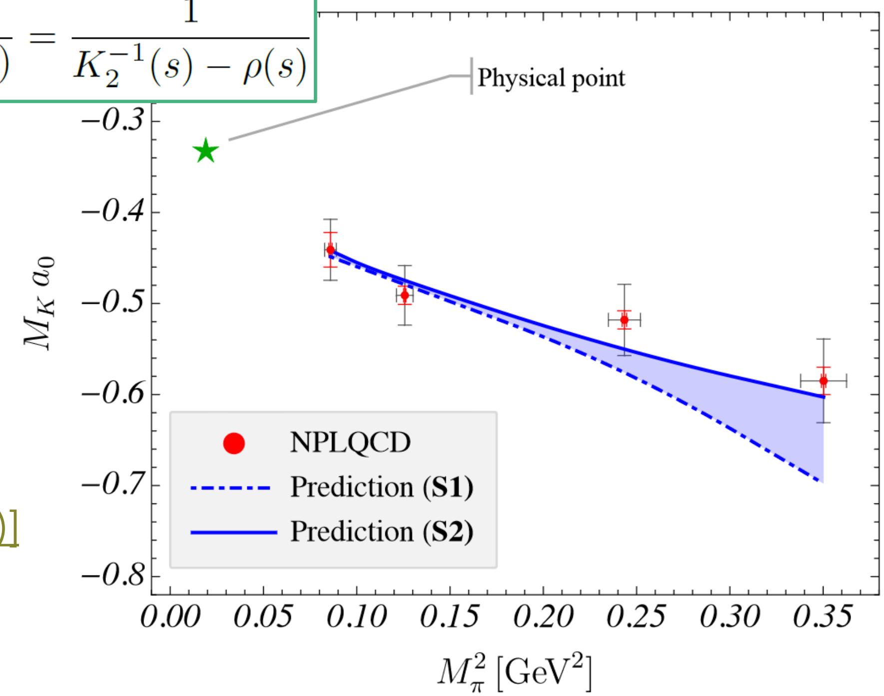
- Crossing symmetry allows to get the amplitude $K^- K^- \to K^- K^-$ from $K^+ K^- \to K^+ K^-$
- SU(3) CHPT unitarized with inverse amplitude method





$$T_2(s) = \frac{(T_{\rm LO}(s))^2}{T_{\rm LO}(s) - T_{\rm NLO}(s)} = \frac{1}{K_2^{-1}(s) - \rho(s)}$$

- Prediction of the I=1 KK scattering length: Data: [NPLQCD (2007)]



- LECs taken from most recent Global fit to lattice QCD [Molina (2020)]