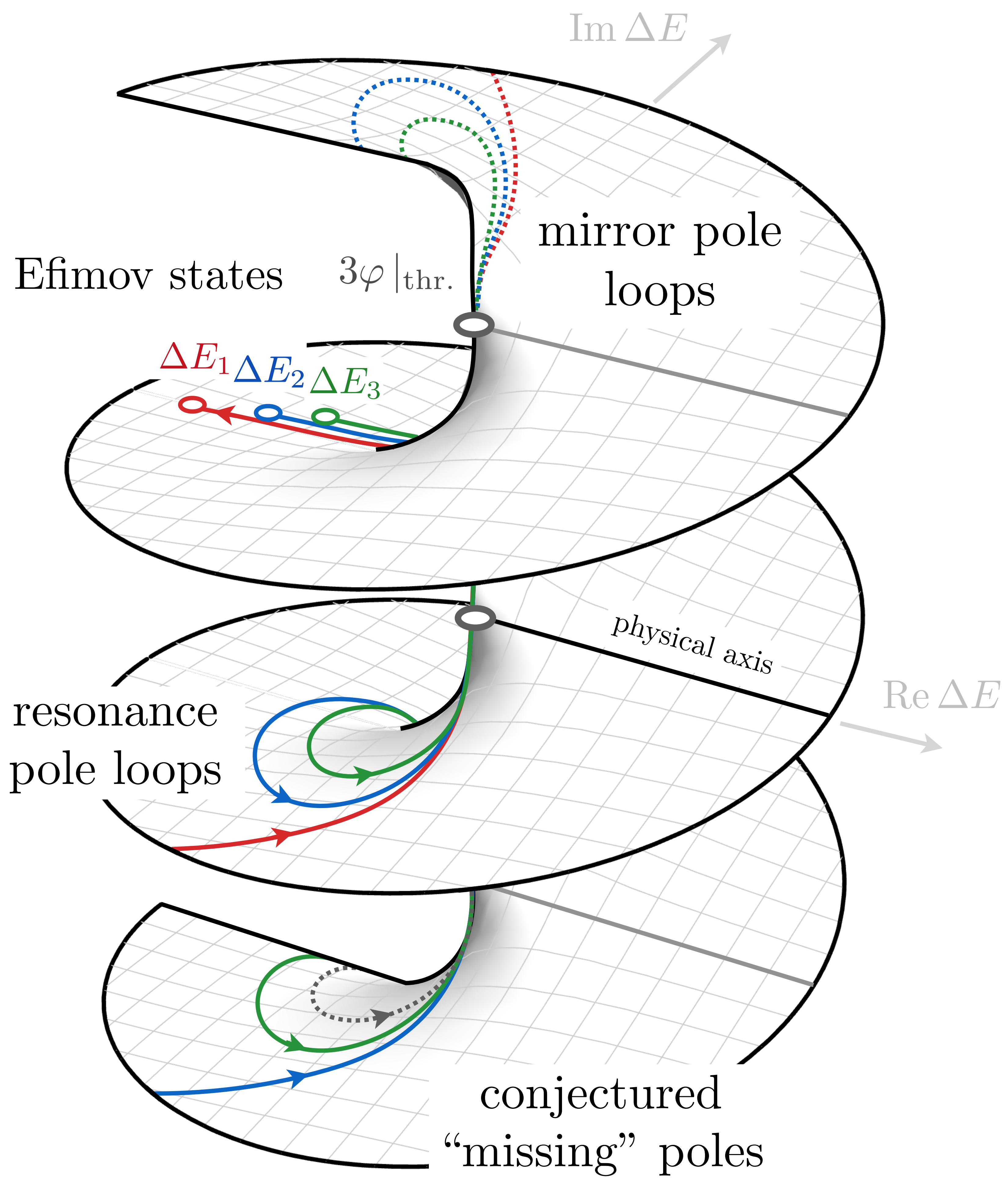
Near-threshold behavior of three-body scattering amplitudes

Sebastian M. Dawid



INDIANA UNIVERSITY
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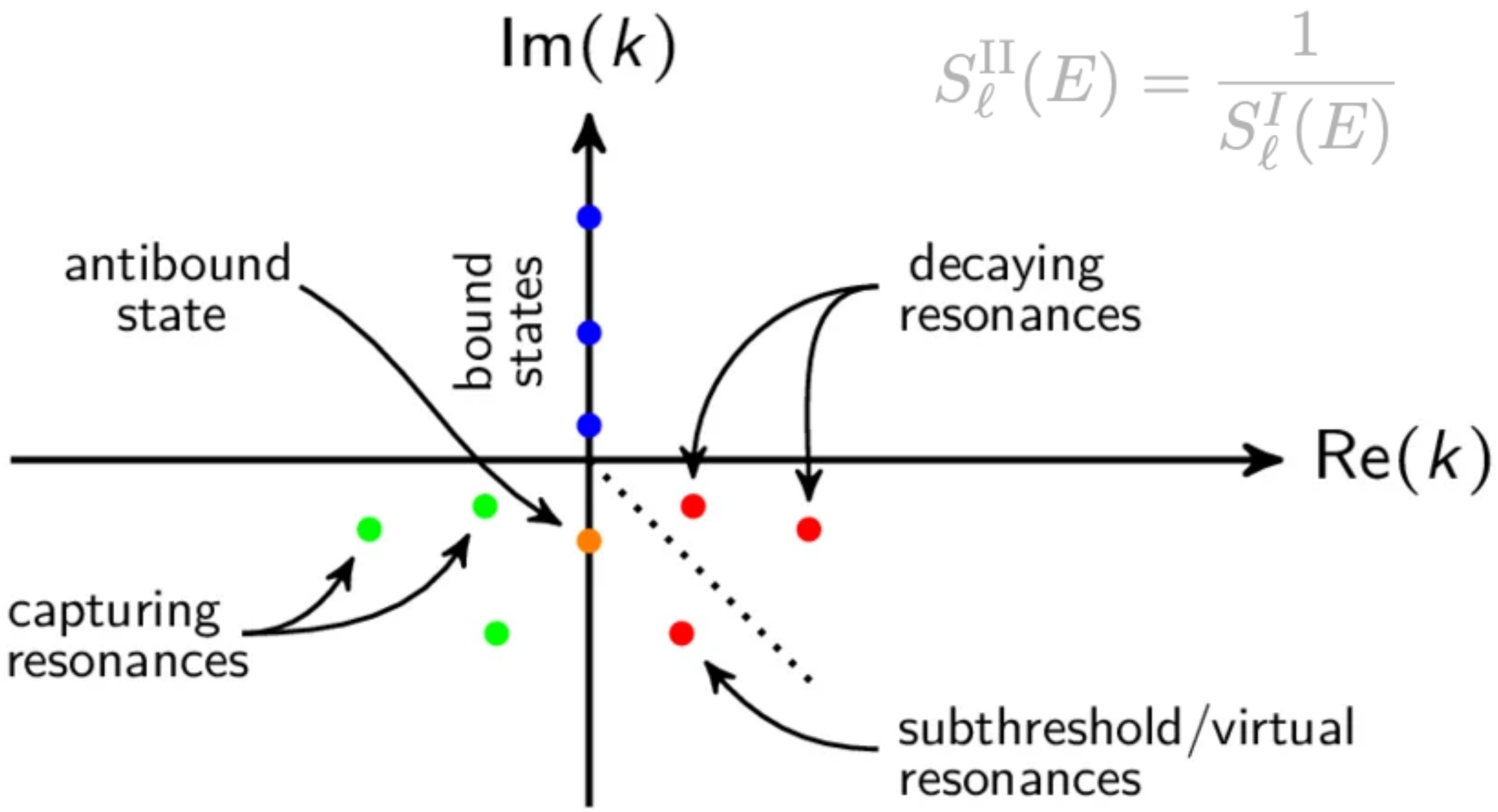
INT Program "*Nuclear Hamiltonians for Advancing Nuclear Physics and Beyond*", Seattle, 05.11.2026

ERE for three-body scattering

Dawid, Hoban, Chambers, (work in progress)
"A universal relation between three-body states"

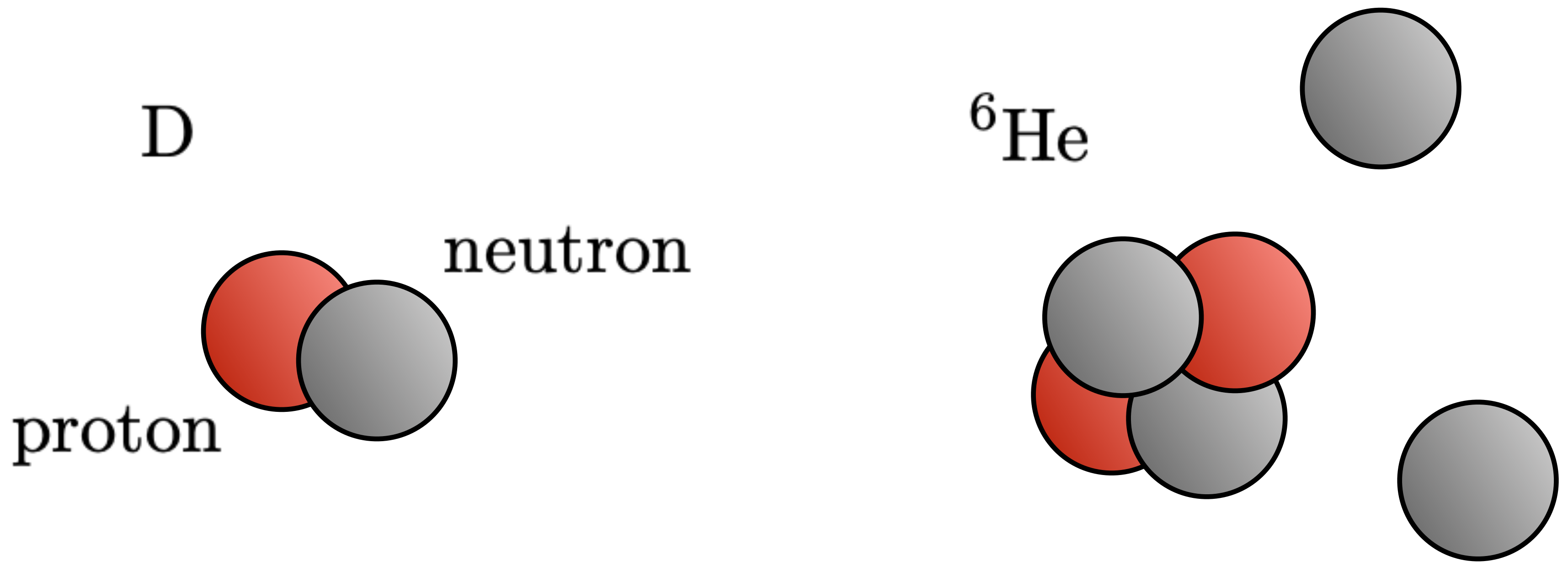
$$\mathcal{M}_2 = \frac{1}{k\cot\delta(k) - ik} \qquad k\cot\delta(k) = -\frac{1}{a} + \frac{1}{2}rk^2 + \dots$$



Johnson et al., ,**"White paper: From bound states to the continuum"**

Goal: A simple, algebraic, and analytic constraint of three-body scattering that could help with extracting/interpreting three-body reactions.

States at the threshold?



At threshold the phase space vanishes = the sheet dependence of pole positions disappears. Near the threshold, amplitudes are strongly constrained by unitarity.

S–matrix theory

S and T matrix

$$S_{ab}(E) = \langle \text{out}|\hat{\boldsymbol{S}}|\text{in}\rangle$$

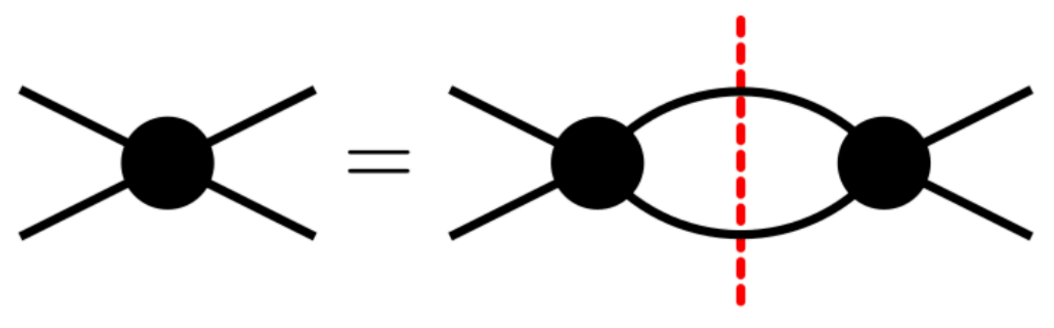
The transition operator T is

$$S = \mathbb{1} + iT$$

Disconnected ($\mathbb{1}$) — Connected (iT)

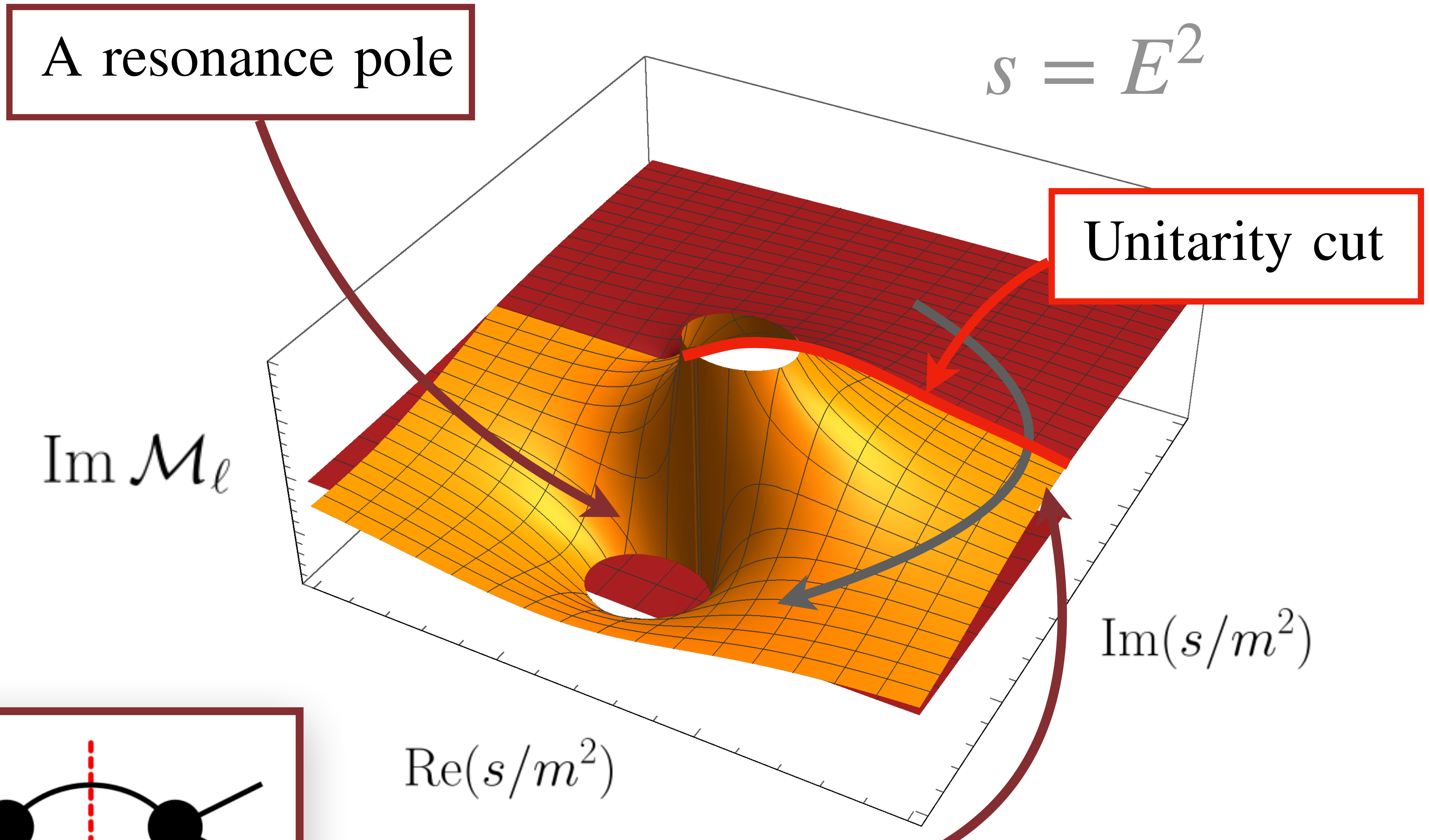
Unitarity of the S operator

$$T - T^\dagger = iTT^\dagger$$



Properties of the S matrix

- Analyticity (causality)
- Unitarity (probability conservation)
- Poincaré symmetry (frame independence)
- Crossing symmetry (particles–antiparticles)
- Internal symmetries (charge, isospin, G-parity)



K–matrix parametrization

Newton, **"Scattering theory of waves and particles"**

Unitarity implies (Cayley transform)

$$S = \frac{\mathbb{1} + iK}{\mathbb{1} - iK}$$

where the K operator is Hermitian.
If S is symmetric, K is real.

$$T = K + iKT$$

This is the K matrix parametrization

Principle of "nearby singularities"

- Analyticity on the first Riemann sheet
- Branch cuts correspond to open channels
- Bound-states & resonances correspond to poles

Two-body partial-wave amplitude

$$\mathcal{M}_\ell(s) = \frac{1}{\mathcal{K}_\ell^{-1}(s) - i\rho(s)}$$

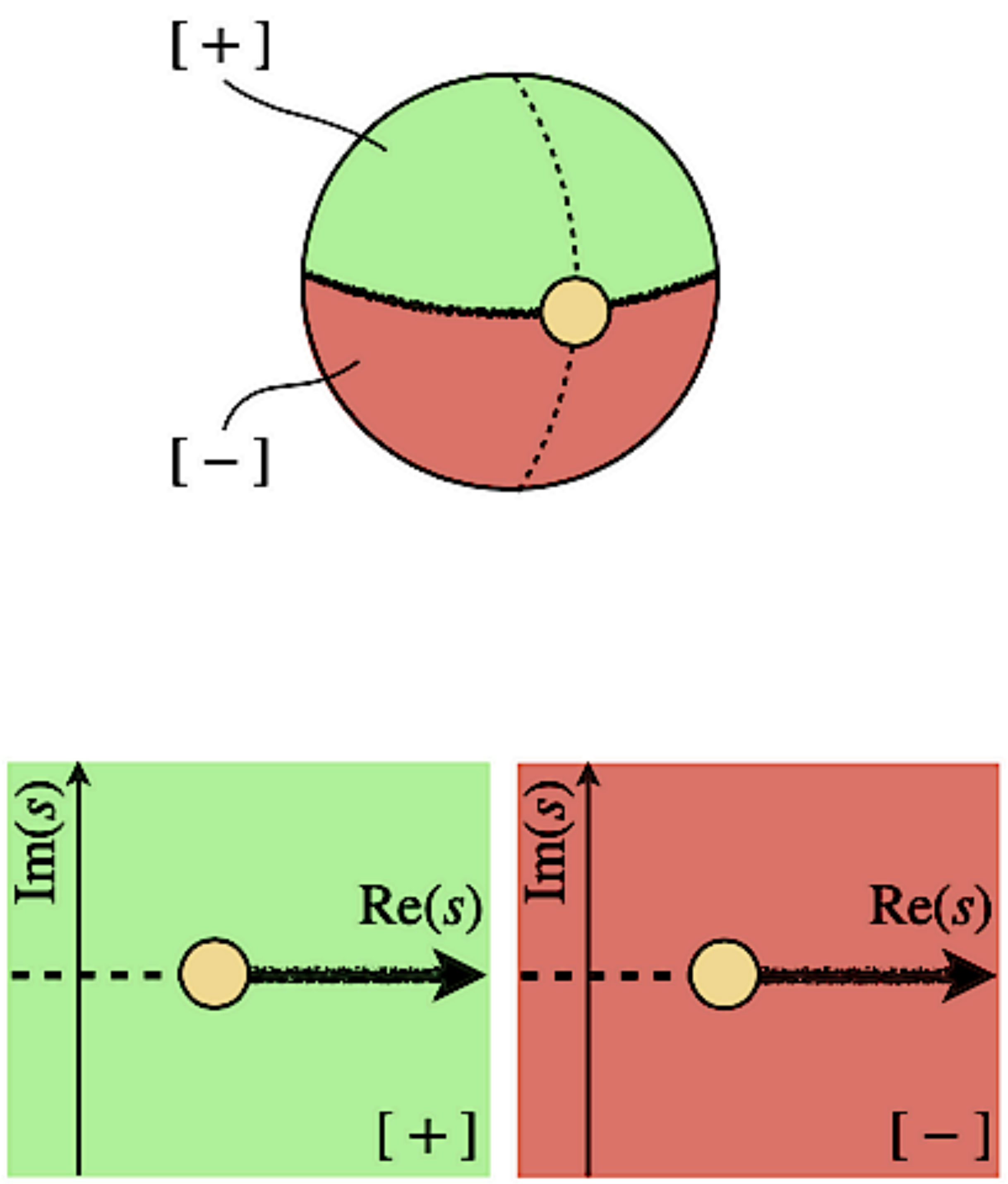
phase shift $\quad \mathcal{K}_\ell^{-1}(s) = \frac{q^\star}{8\pi\sqrt{s}} \cot(\delta_\ell(s))$

phase space $\quad \rho = \frac{1}{2\sqrt{s}}\sqrt{s - 4m^2}$

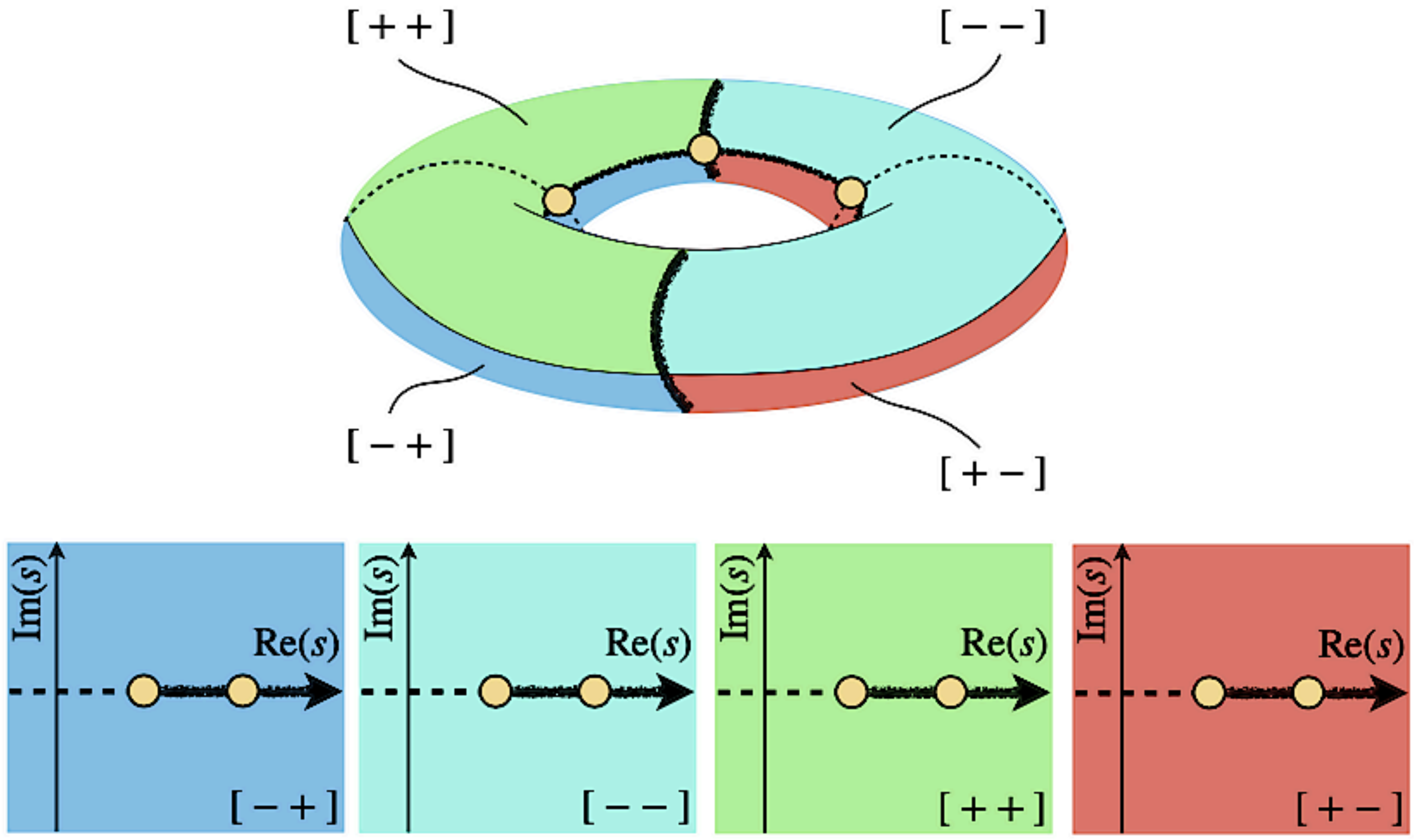
Riemann sheets

Mai, Meissner, Urbach, "**Towards a theory of hadron resonances**"

Singularities of the scattering amplitudes represent **physical phenomena.**



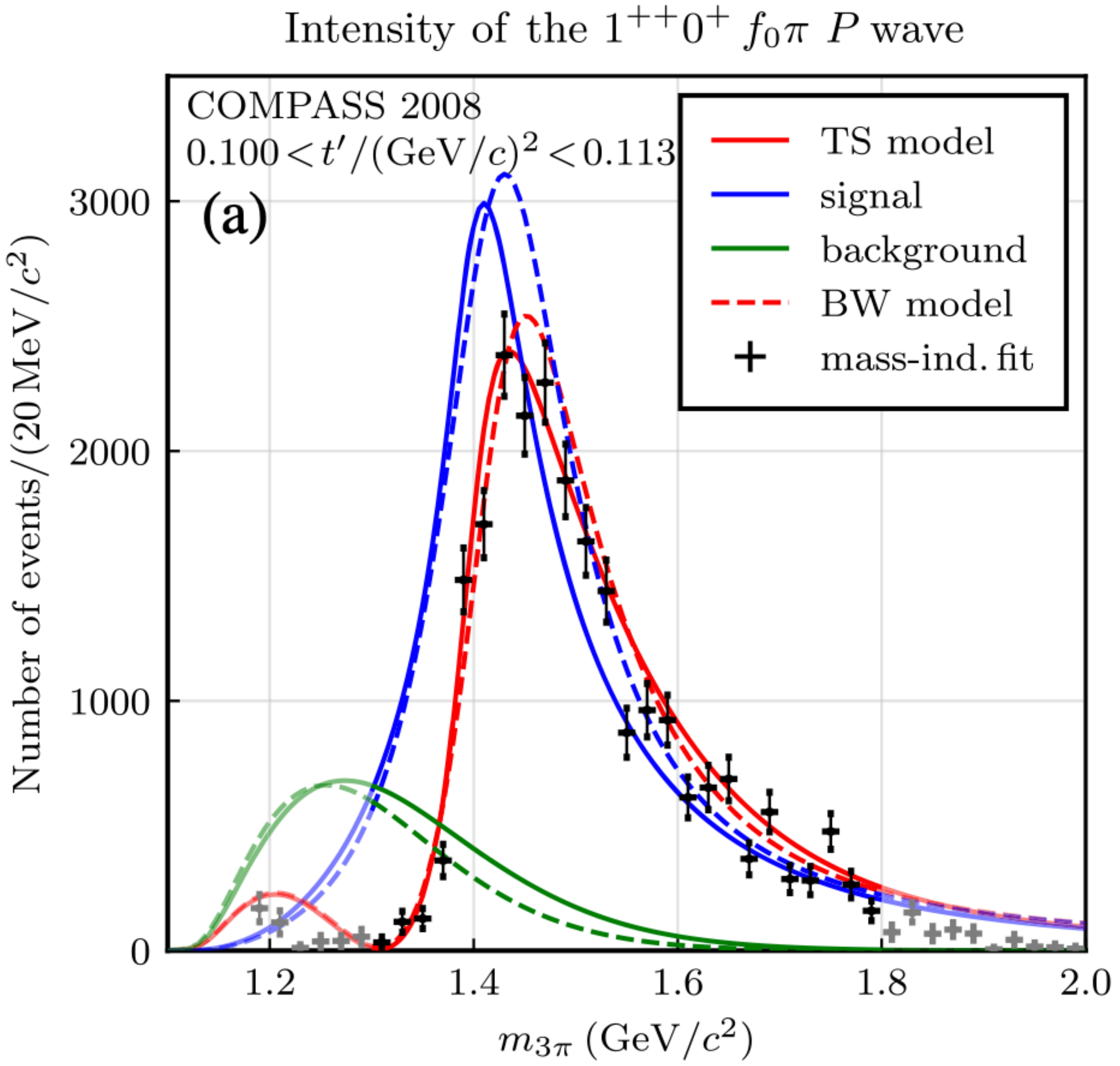
$$pK^- \to pK^-$$



$$pK^- \to pK^- \quad pK^- \to \Sigma^+\pi^-$$

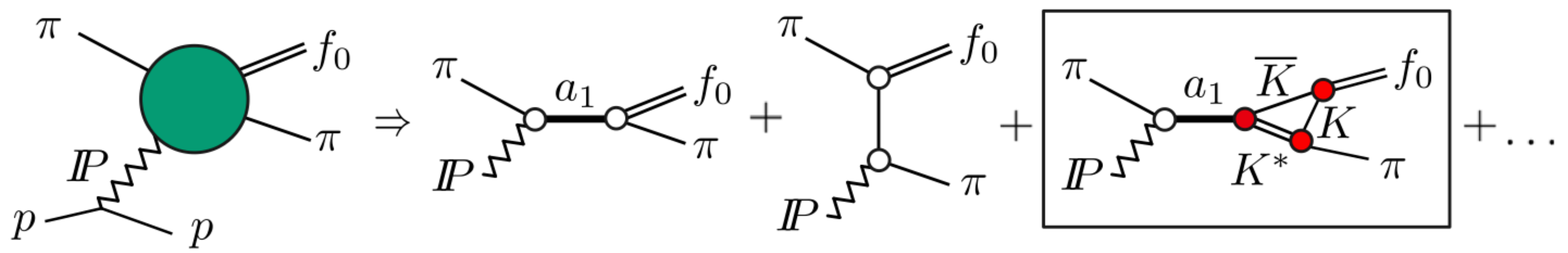
The problem of three–body forces

COMPASS, **"Triangle singularity as the origin of $a_1(1420)$"**



Is there only one mechanism generating bumps?

The case of $a_1(1420)$?



Most interesting states require inclusion of three-body channels

$\chi_{c1}(3872)$, $N^*(1440)$, $a_1(1260)$, $a_1(1420)$, $\pi_1(1600)$,…

Bumps in complicated reactions may correspond to **artificial enhancements** and not genuine resonances…

Three–particle scattering in LQCD

$$\det_{k,\ell,m_\ell}\left[\mathcal{K}_3^{-1}(E) + F_3(E,\boldsymbol{P},L)\right] = 0$$

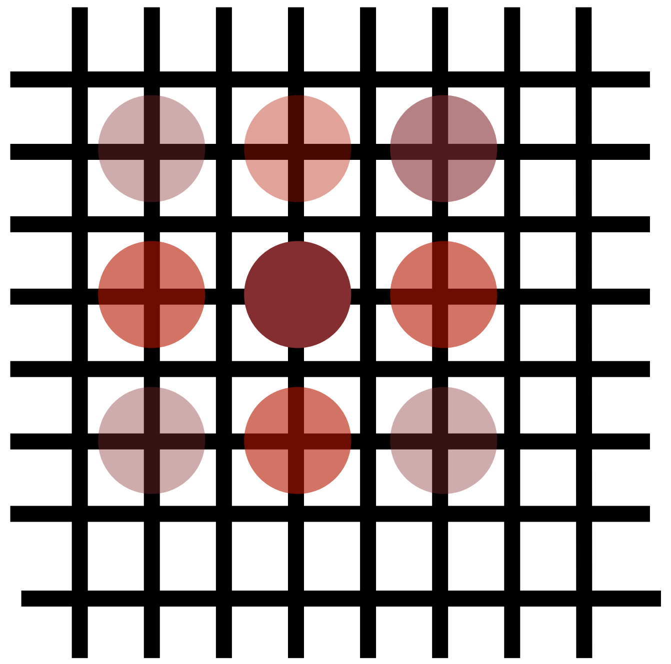
Hansen & Sharpe, "**Relativistic, model-independent, three-particle quantization condition**"

Mai & Döring, "**Three-body unitarity in finite volume**"

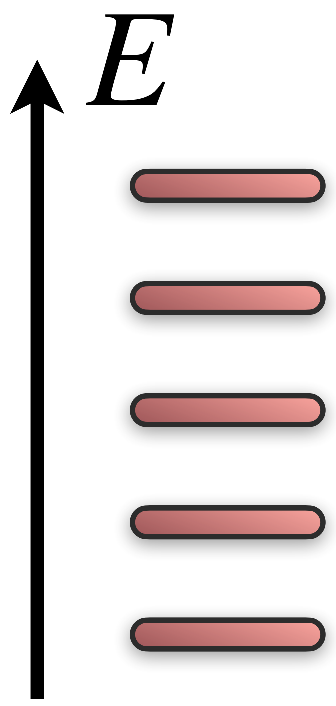
Müller, Pang, Rusetsky & Wu, "**Relativistic-invariant formulation of the NREFT three-particle quantization condition**"

Jackura, "**Three-body scattering and quantization conditions from S matrix unitarity**"

Finite Volume



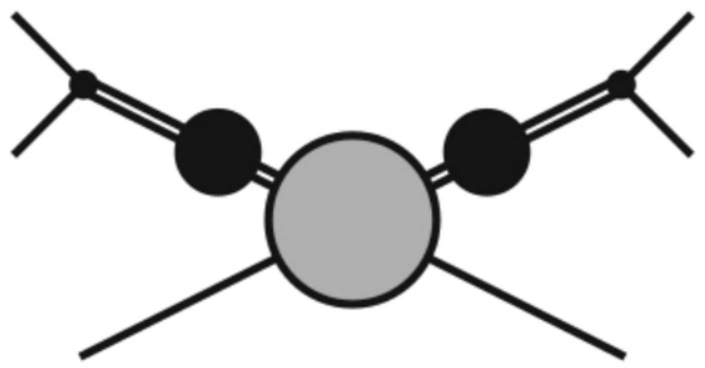
Lattice QCD



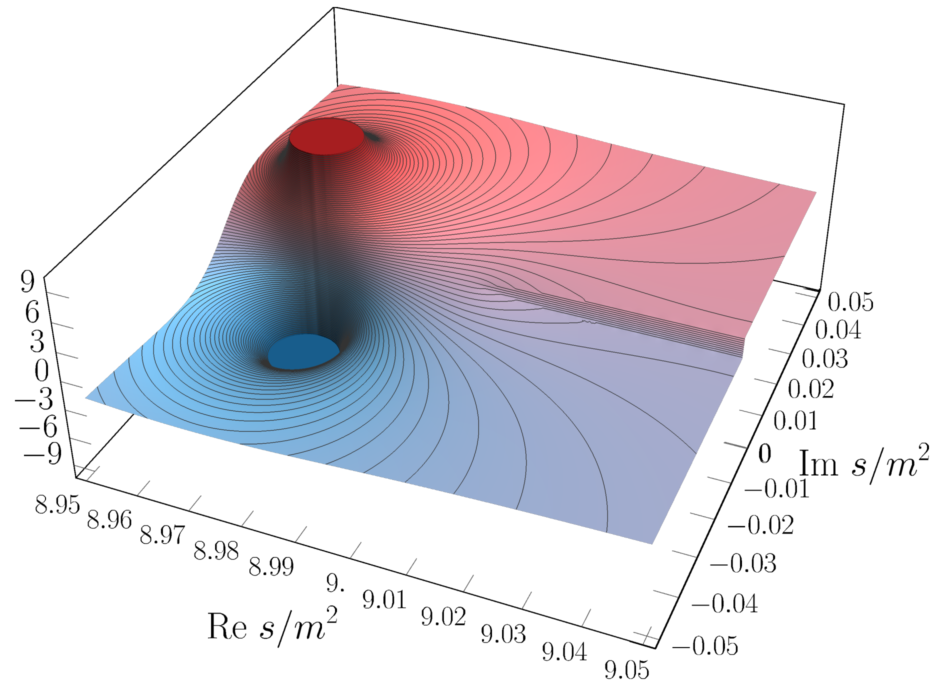
Spectrum

Quantization Condition

Infinite Volume



Amplitude



Resonances

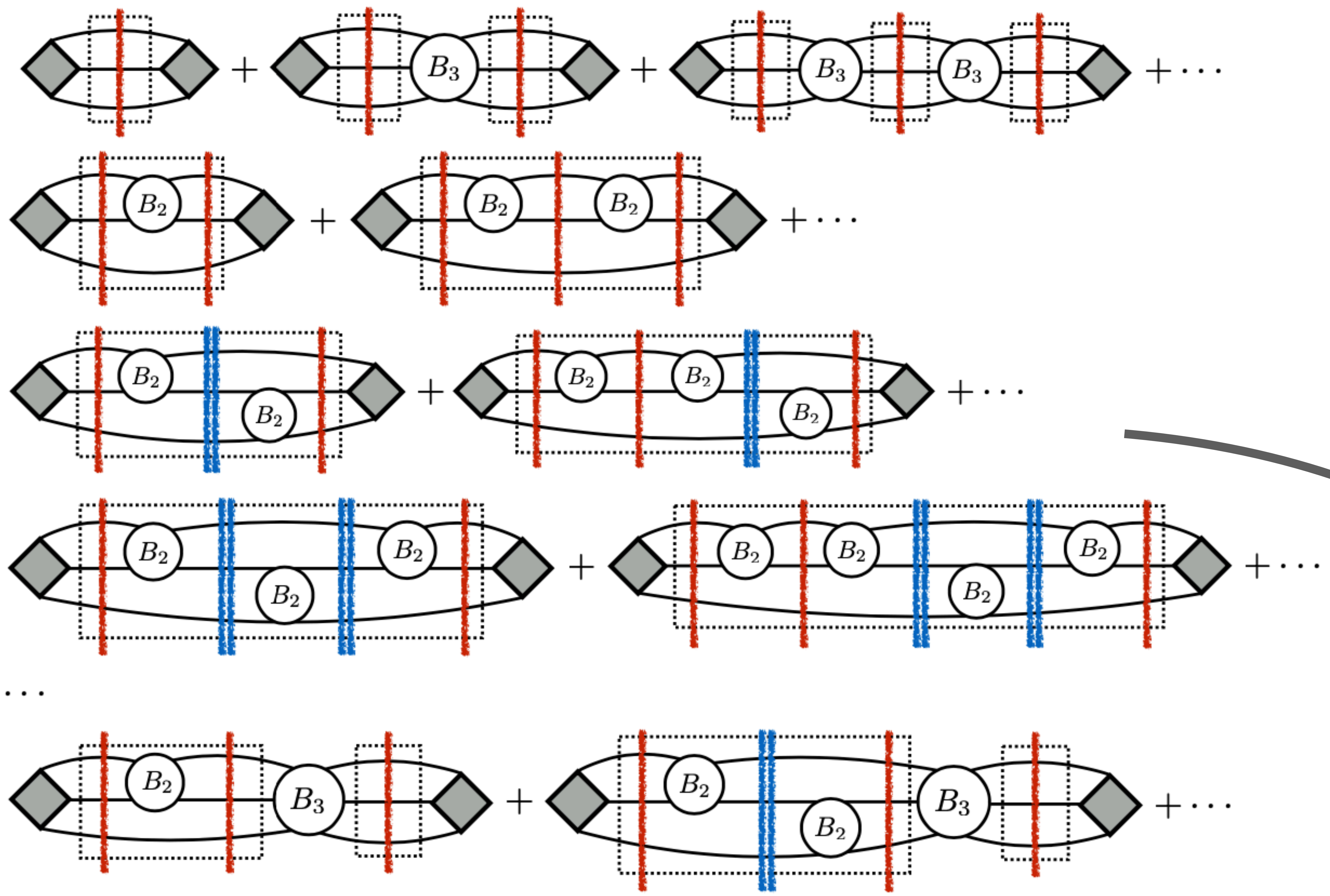
Three–body quantization condition

Bubna, Muller, Rusetsky, "**Finite-volume energy shift of the three-nucleon ground state**"

Draper, Hansen, Romero-López, Sharpe, "**Three relativistic neutrons in a finite volume**"

Schaaf & Sharpe, "**Implementing the three-neutron quantization condition**"

$C_L(E, \vec{P}) =$



Poisson summation formula

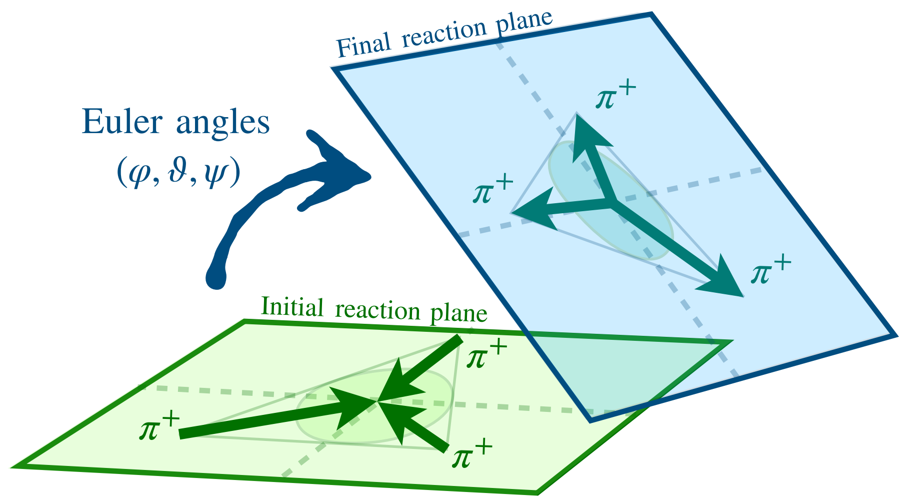
$$\left[\frac{1}{L^3}\sum_{\boldsymbol{k}} - \int \frac{d\boldsymbol{k}}{(2\pi)^3}\right] f(\boldsymbol{k}) = \frac{1}{L^3}\sum_{\boldsymbol{x}\neq 0} \tilde{f}(\boldsymbol{x})$$

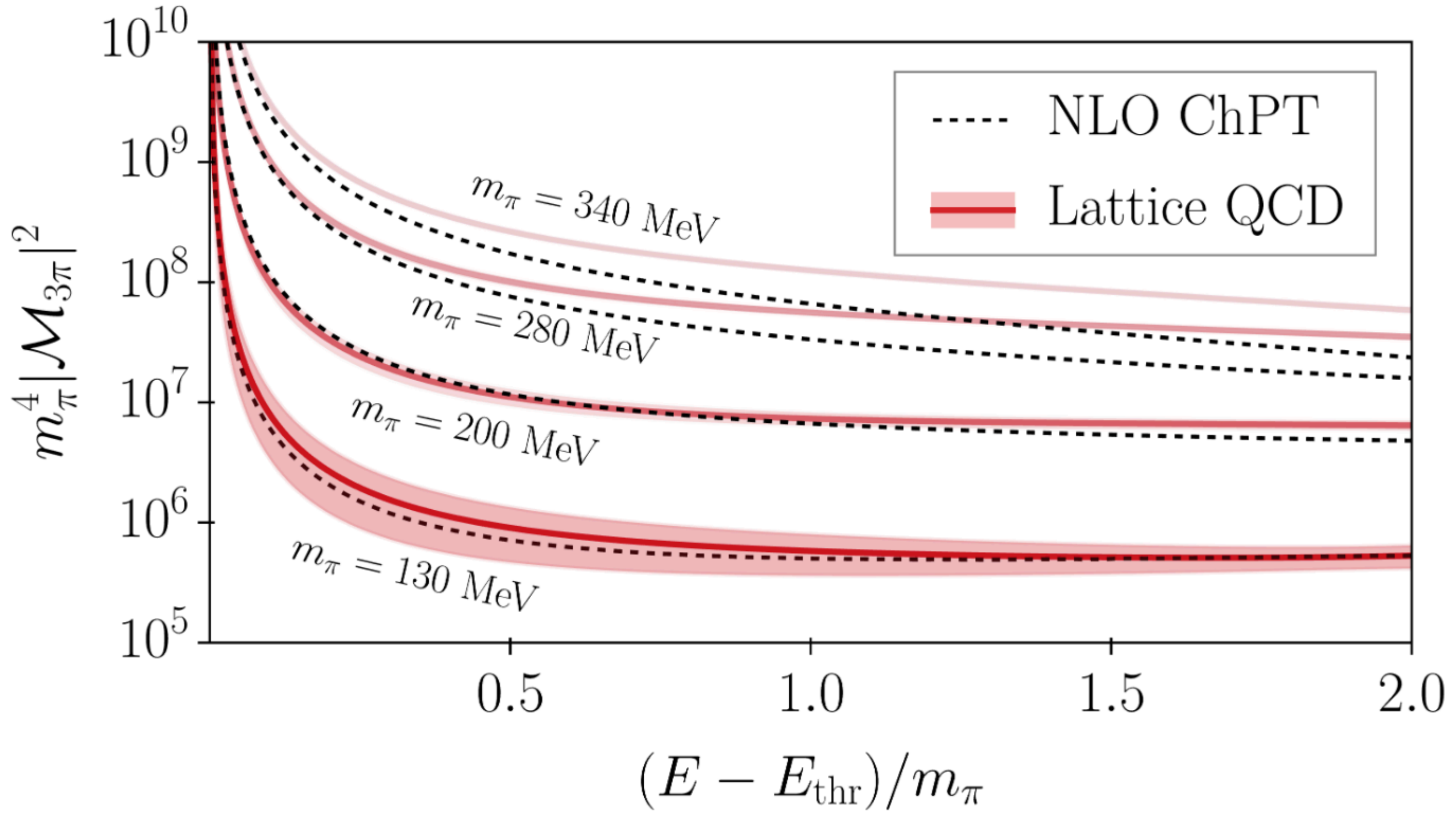
Hansen & Sharpe, "**Relativistic, model-independent, three-particle quantization condition**"

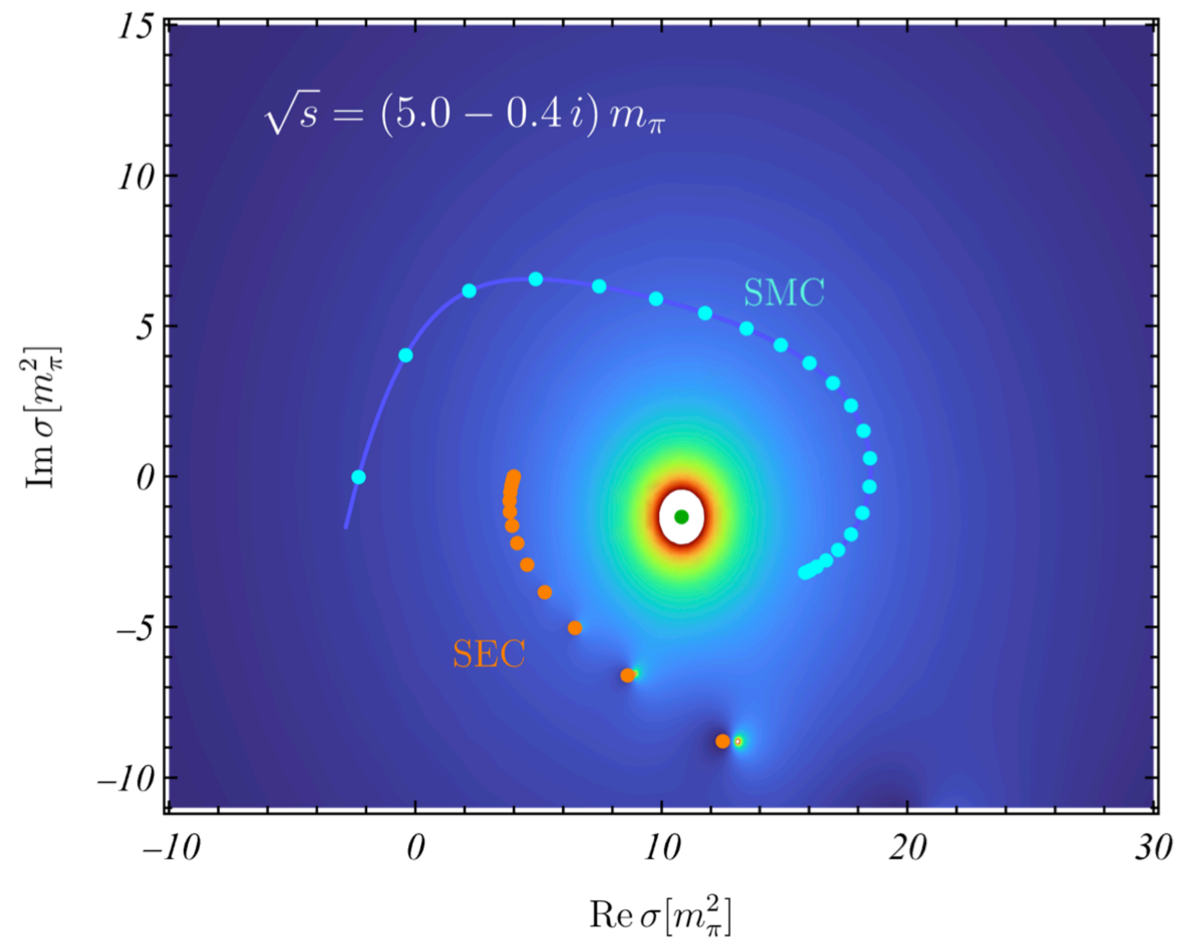
$$\det_{k,\ell,m}\left[\mathbb{1} - \boldsymbol{\mathcal{K}}_3(E^\star)\boldsymbol{F}_3(E, \boldsymbol{P}, L)\right] = 0$$

Overview of results

Dawid, Draper, Hanlon, Hörz, Morningstar, Romero-López, Sharpe, Skinner,
"Two- and three-meson scattering amplitudes at physical quark masses from lattice QCD"

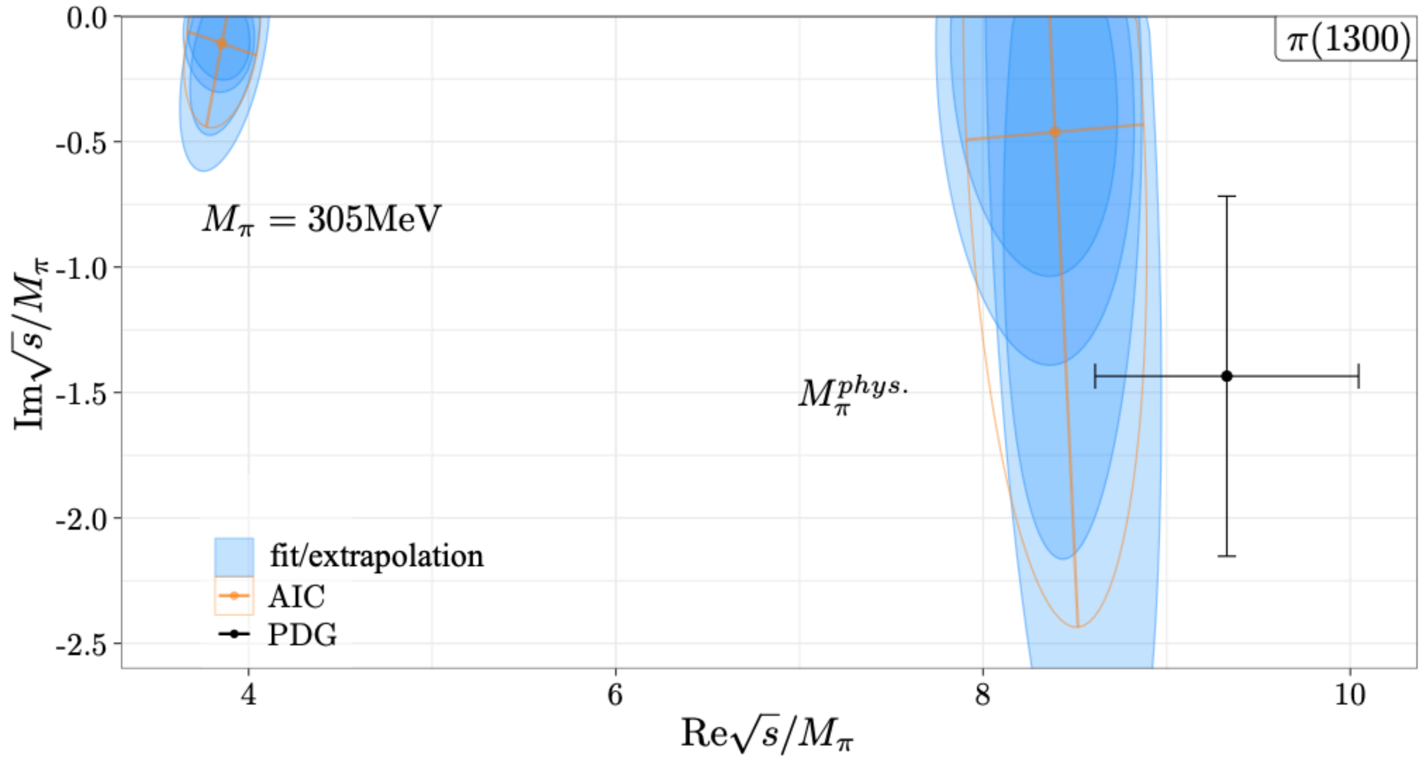




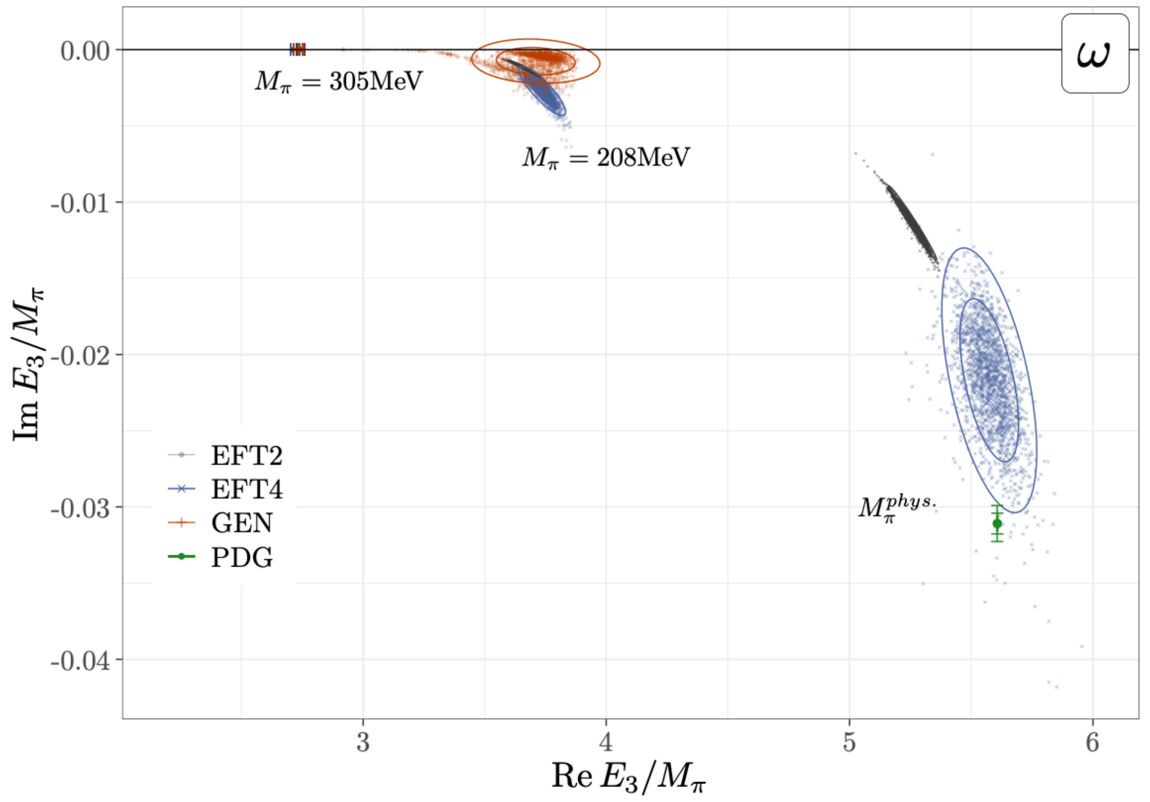


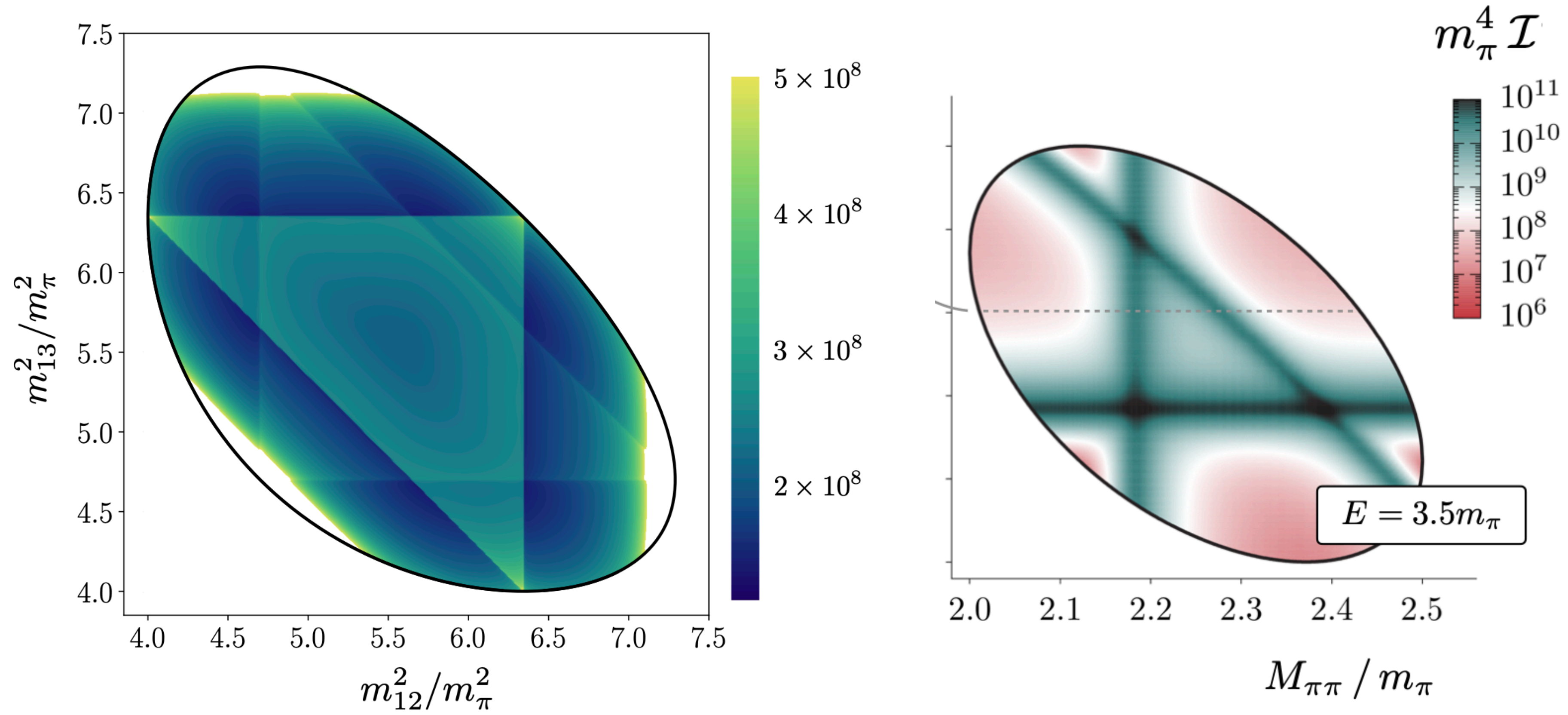
Mai et al. **"Three-body dynamics of the $a_1(1260)$ resonance from lattice QCD"**

Yan et al. "**Emergence of the π(1300) Resonance from Lattice QCD**"



Yan et al. "**ω Meson from lattice QCD**"



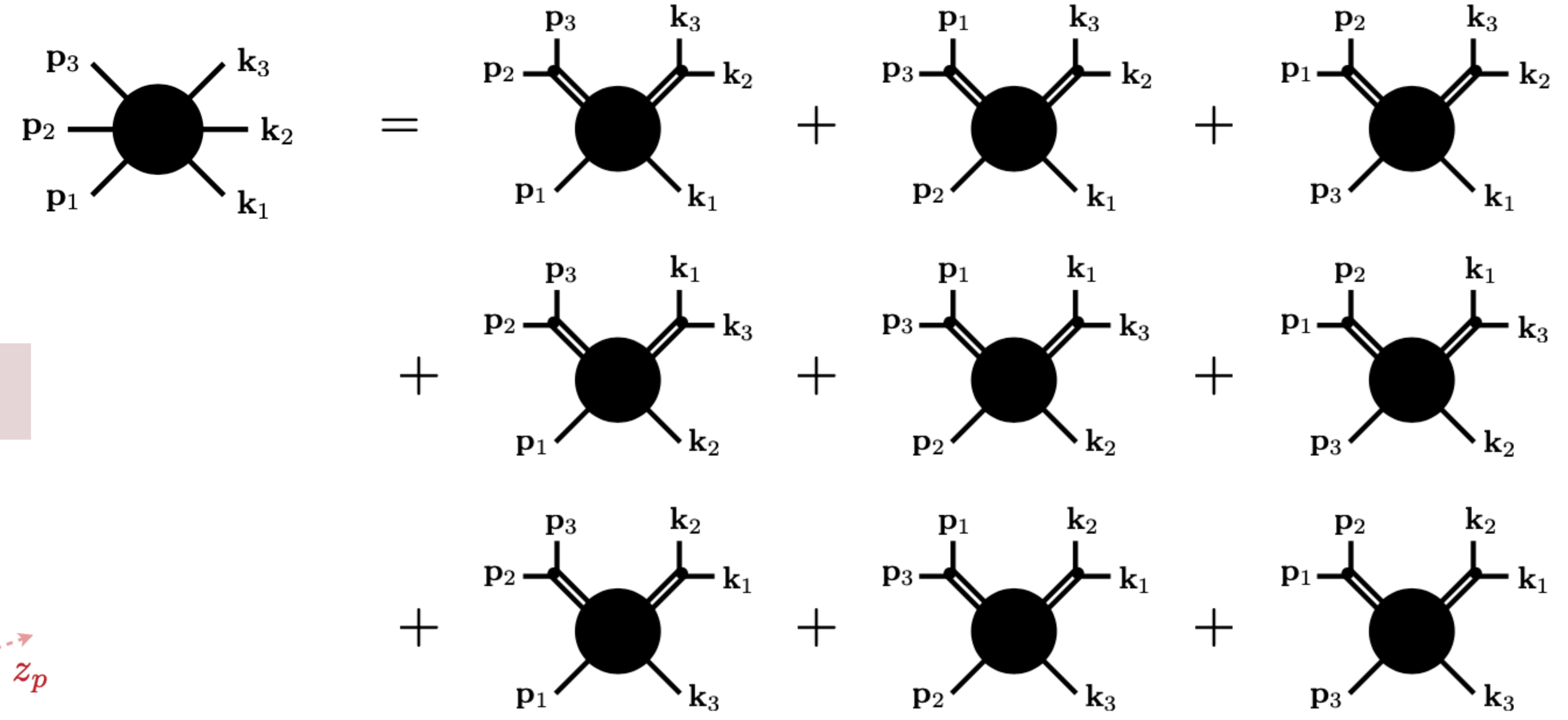


Briceno et al. (HadSpec) **"Isotensor πππ scattering with a ρ resonant subsystem from QCD"**

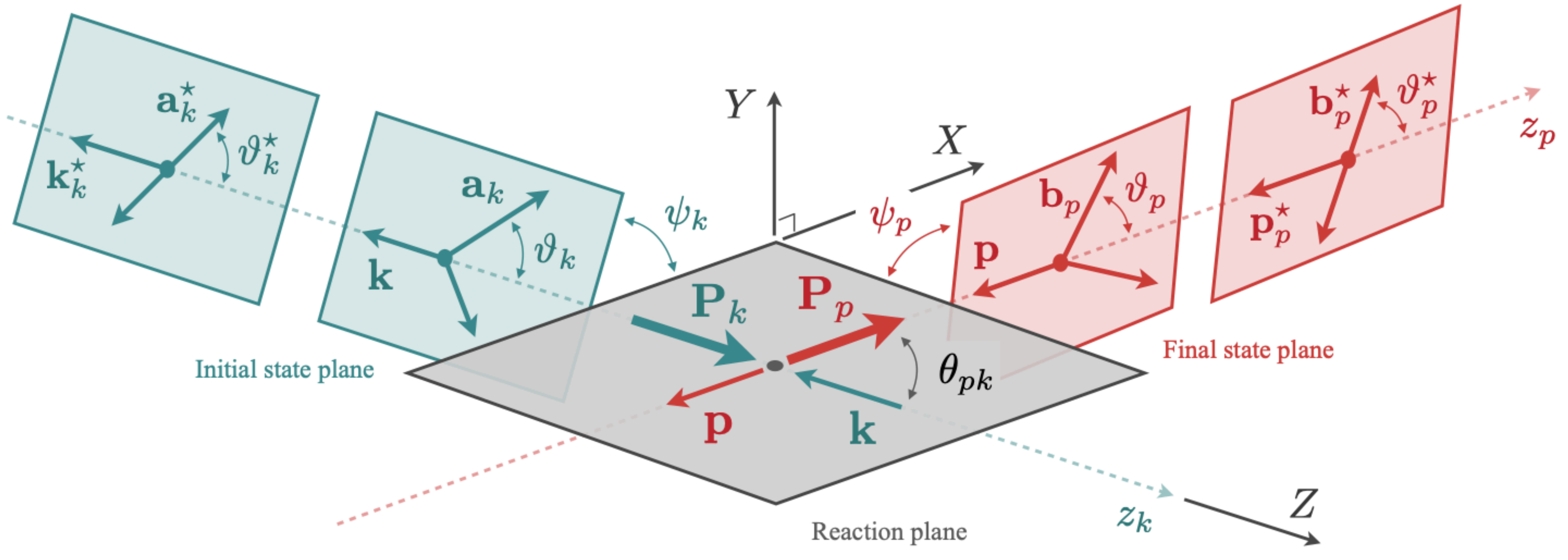
Hansen et al. (HadSpec) **"The energy-dependent π⁺π⁺π⁺ scattering amplitude from QCD"**

Partial-wave projection

$\mathcal{M}_3(\boldsymbol{p},\boldsymbol{k}) \;=$



Jackura & Briceño, **"Partial-wave projection of the one-particle exchange in three-body amplitudes"**

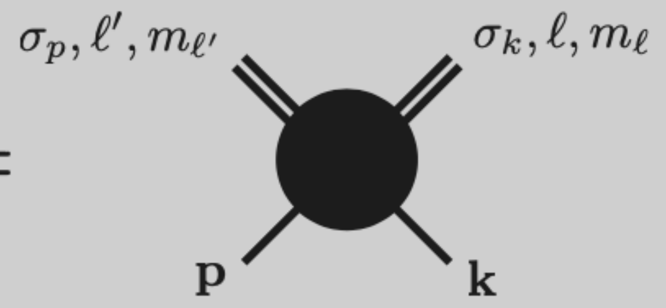


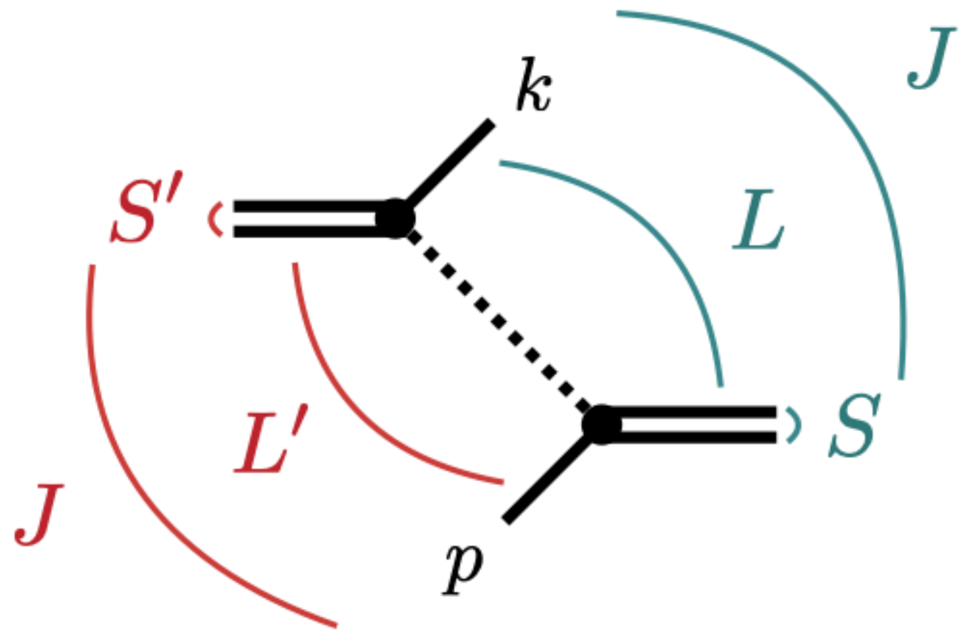
Kinematical variables

- two angles defining orientation of the initial-state pair
- two angles defining orientation of the final-state pair
- one angle defining orientation between spectators (pairs)
- invariant mass of the initial and final pair
- total energy

Pair – spectator amplitude

$$\mathcal{M}^{(u,u)}_{3;\ell' m_\ell';\ell m_\ell}(\boldsymbol{p},\boldsymbol{k}) \;=$$





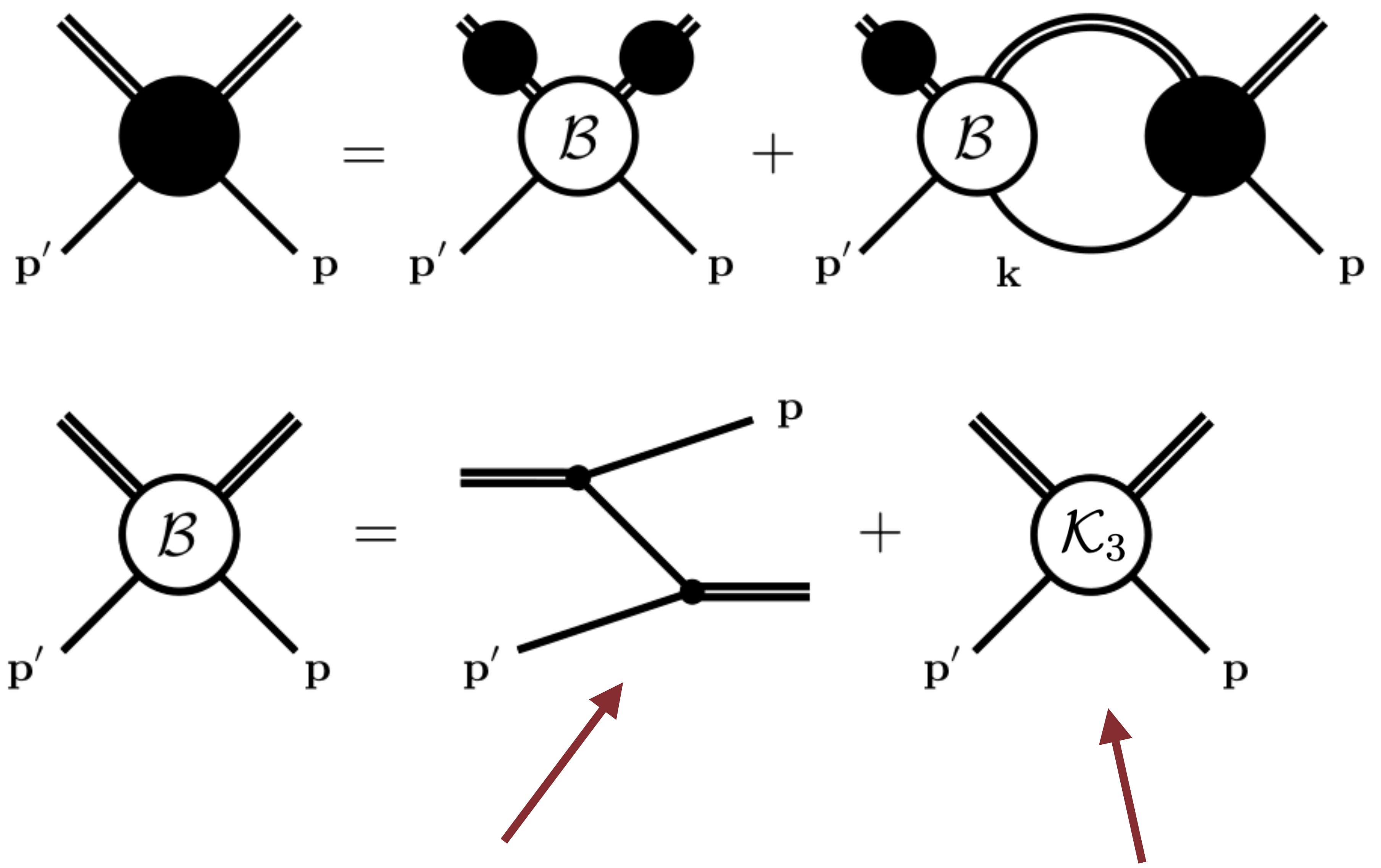
Three-to-three scattering

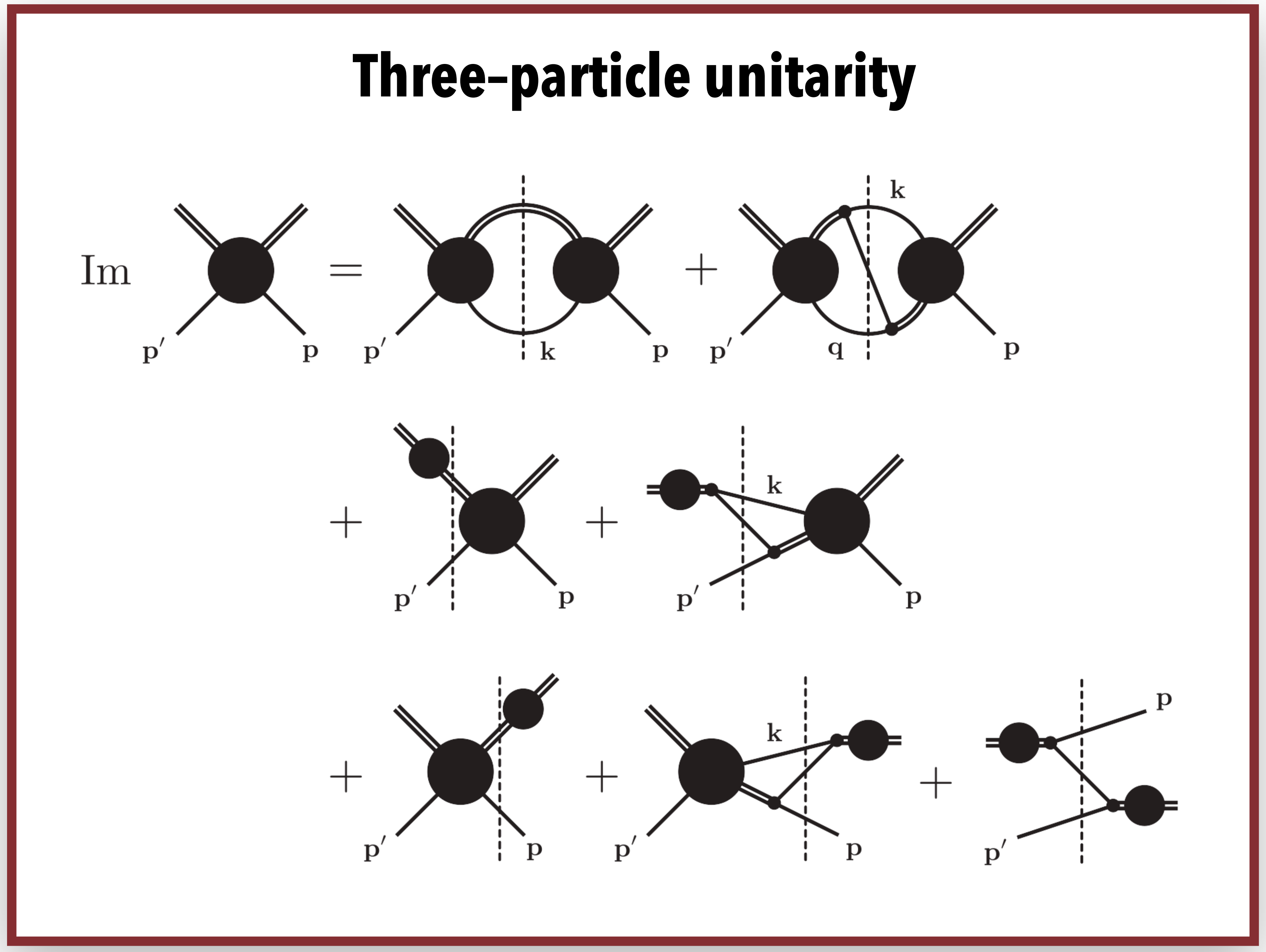
Mai et al., **"Three-body unitarity with isobars revisited"**

Jackura et al. (JPAC), "**Phenomenology of Relativistic 3-to-3 Reaction Amplitudes…"**

Jackura, Dawid, Fernandez-Ramirez, et al. (JPAC), **"Equivalence of three-particle scattering formalisms"**

Dawid, Szczepaniak, **"Bound states in the B-matrix formalism for the three-body scattering"**





Three-body amplitude as an int. equation

$$\mathcal{M}_3 = \mathcal{M}_2 \mathcal{B} \mathcal{M}_2 + \mathcal{M}_2 \int \mathcal{B}\, \rho_3\, \mathcal{M}_3$$

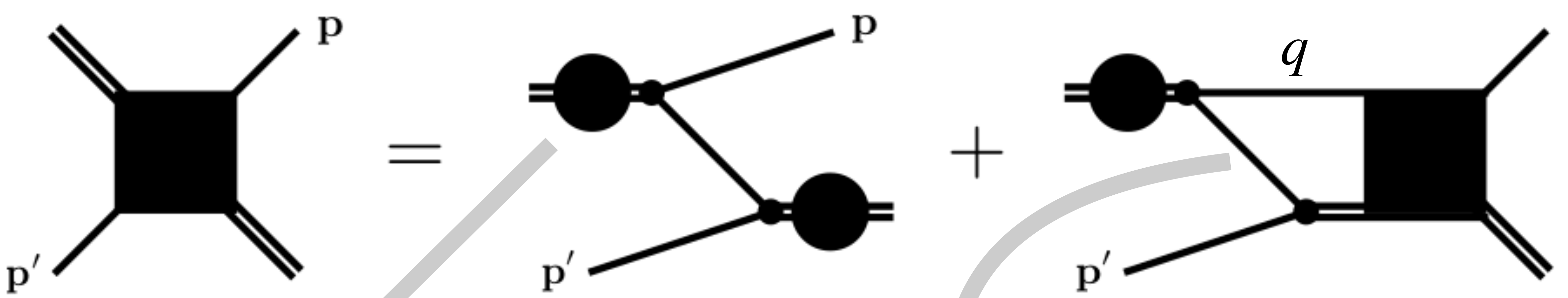
Analytic continuation

Dawid, Islam, Briceño,
"Analytic continuation of the relativistic three-body amplitudes"

Toward three-body resonances

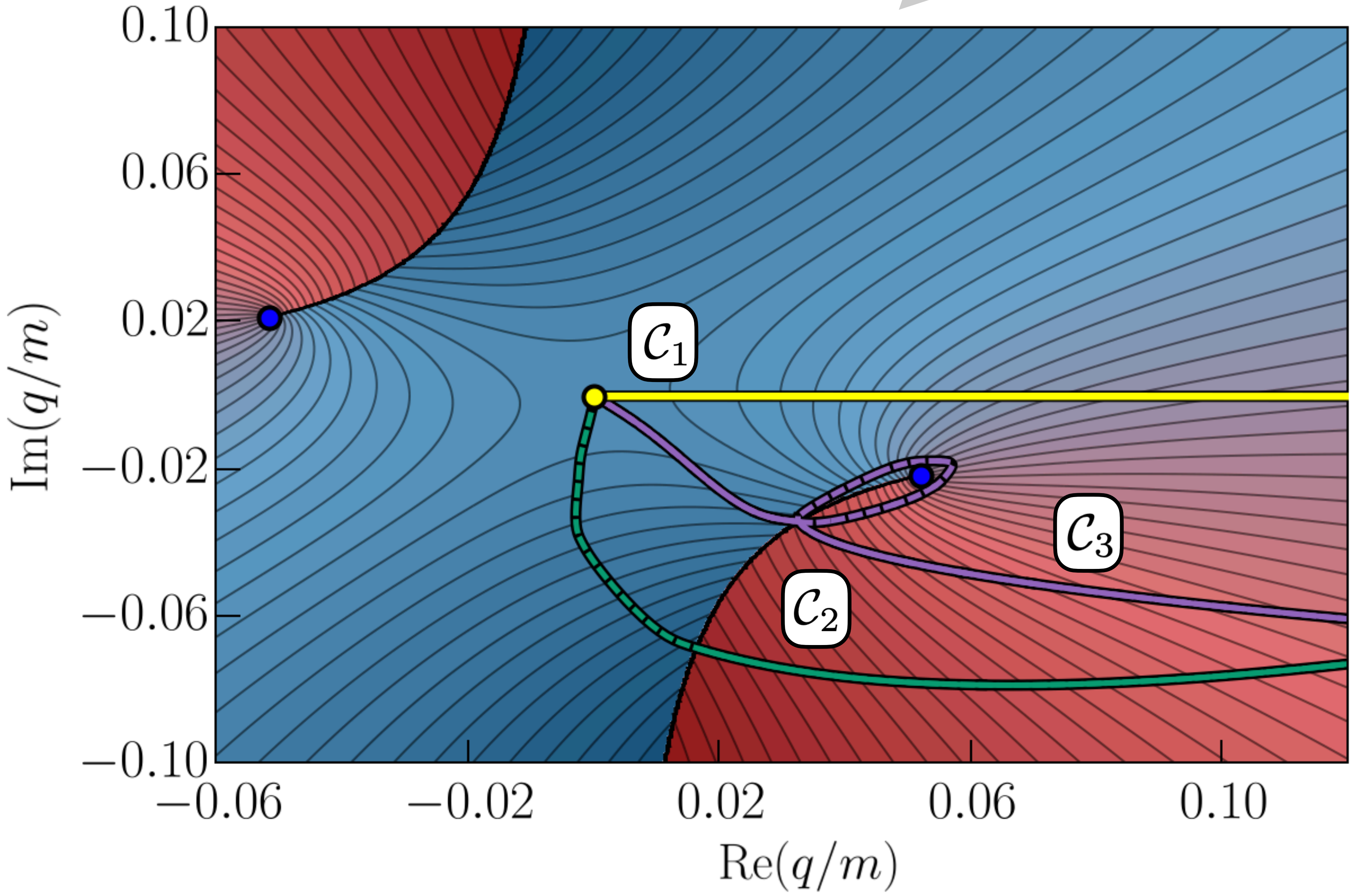
Higher Riemann sheets are accessed by **contour deformations.**

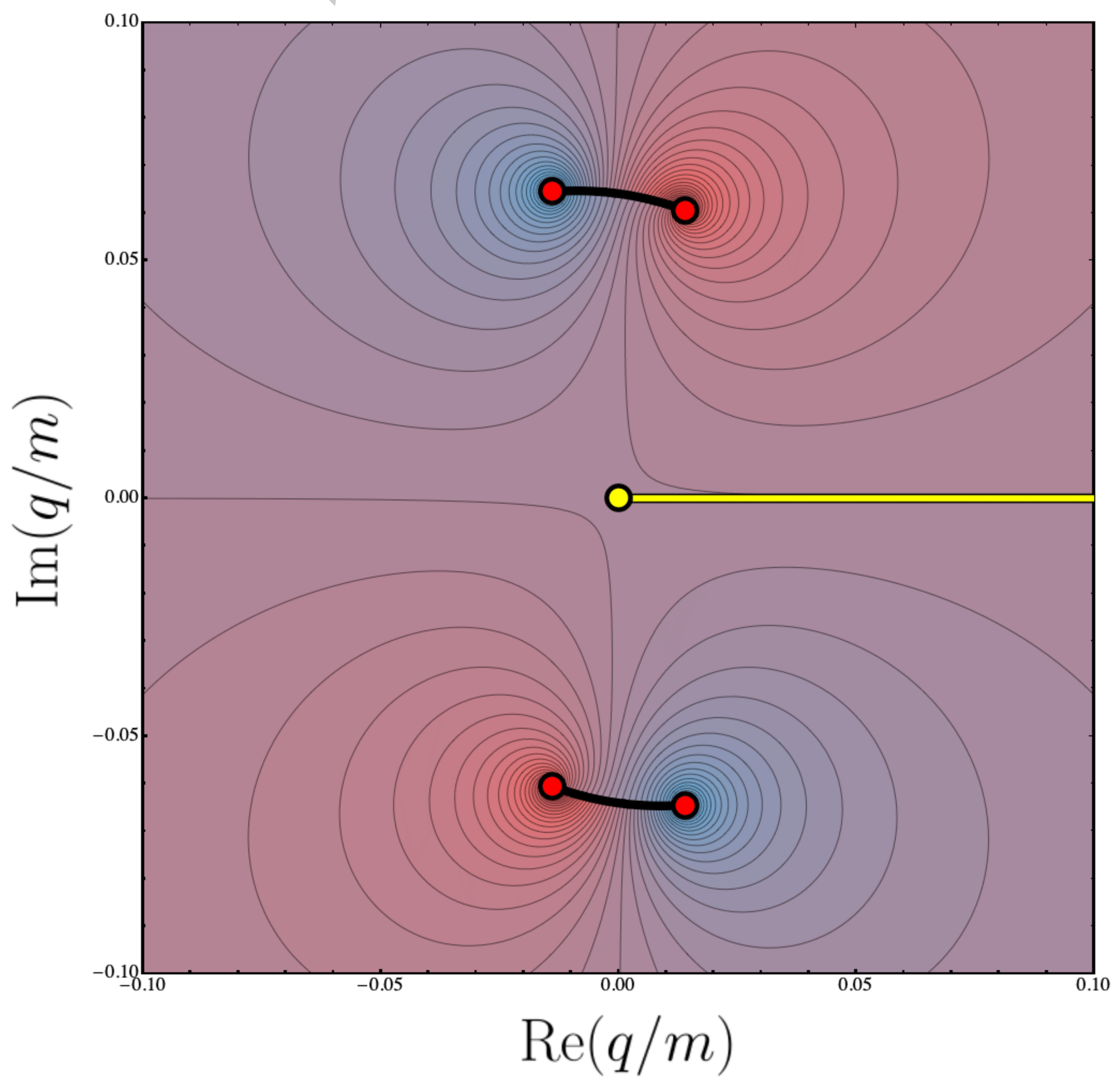
They are defined by a **monodromy:**
a number of windings around singularities



Two-body amplitude

One-particle exchange



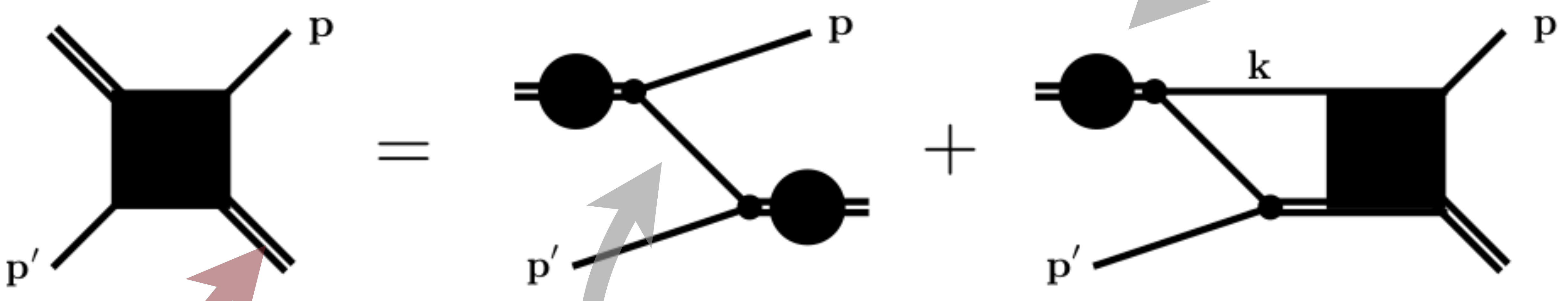


Jackura, Briceño, Dawid, Islam, McCarty, **Solving relativistic three-body integral equations in the presence of bound states**

Toy model with J=0

Two-body amplitude

$$\mathcal{M}_2^{-1} \sim -\frac{1}{a} - i\rho_2$$



Bound state of mass

$m_b = 1.732\, m \quad (ma = 2)$

$m_b = 1.996\, m \quad (ma = 16)$

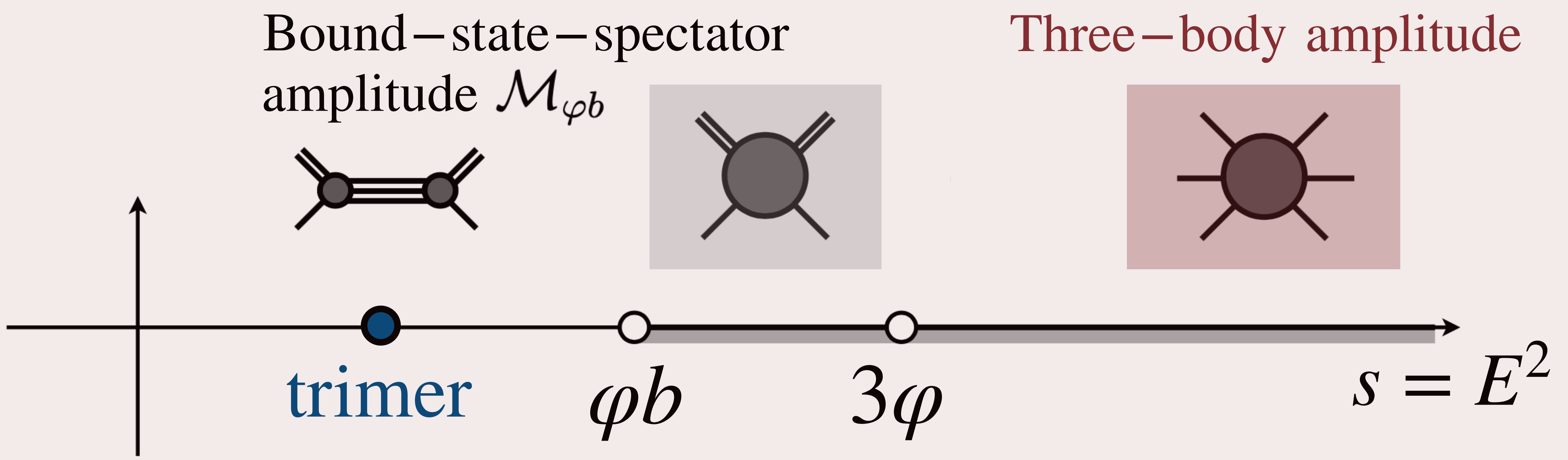
$$\mathcal{M}_2 \approx \frac{-g^2}{\sigma - m_b^2}$$

One-particle exchange

$$G(p', s, p) \propto \log\left(\frac{1 + z(p', s, p)}{1 - z(p', s, p)}\right)$$

LSZ reduction

$$\mathcal{M}_{\varphi b} = \lim_{\sigma', \sigma \to m_b^2} (\sigma' - m_b^2)\, \mathcal{D}\, (\sigma - m_b^2)$$

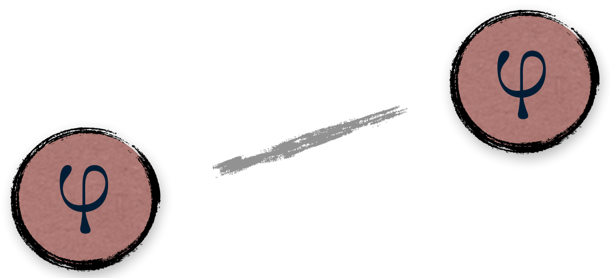


Efimov phenomenon

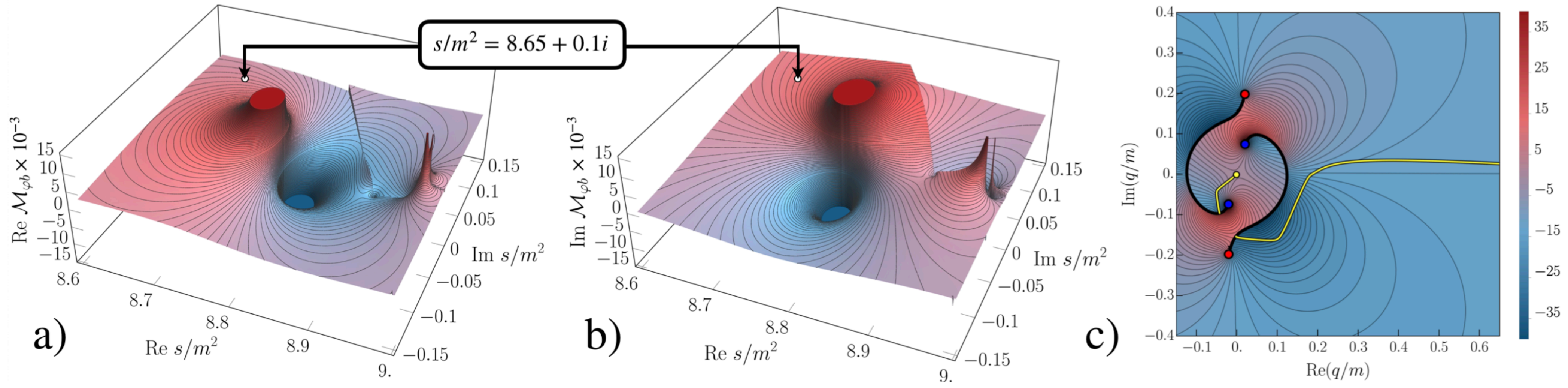
Efimov, **"Energy levels arising from resonant two-body forces in a three-body system"**

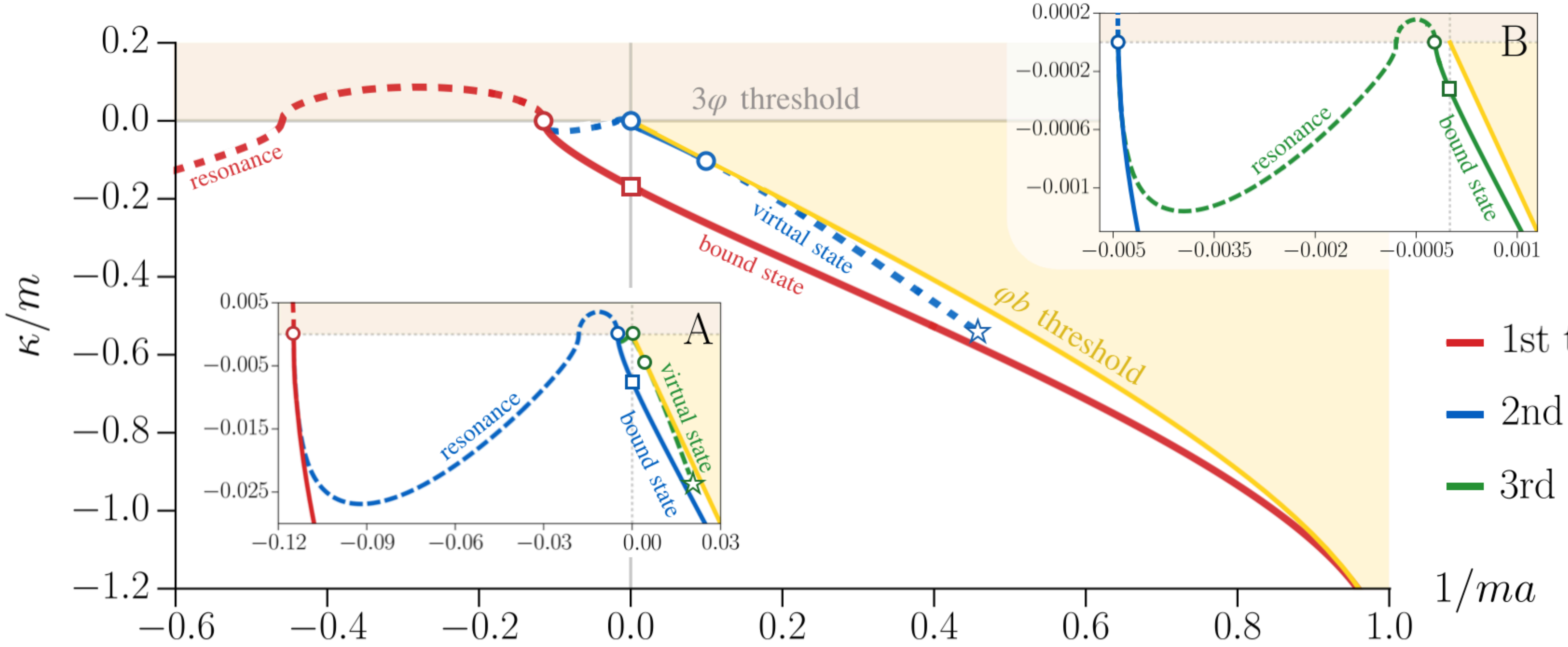
Dawid, Islam, Briceño, **"Analytic continuation of the relativistic three-body amplitudes"**

$$\mathcal{M}_2^{-1} \sim -\frac{1}{a} - i\rho_2$$

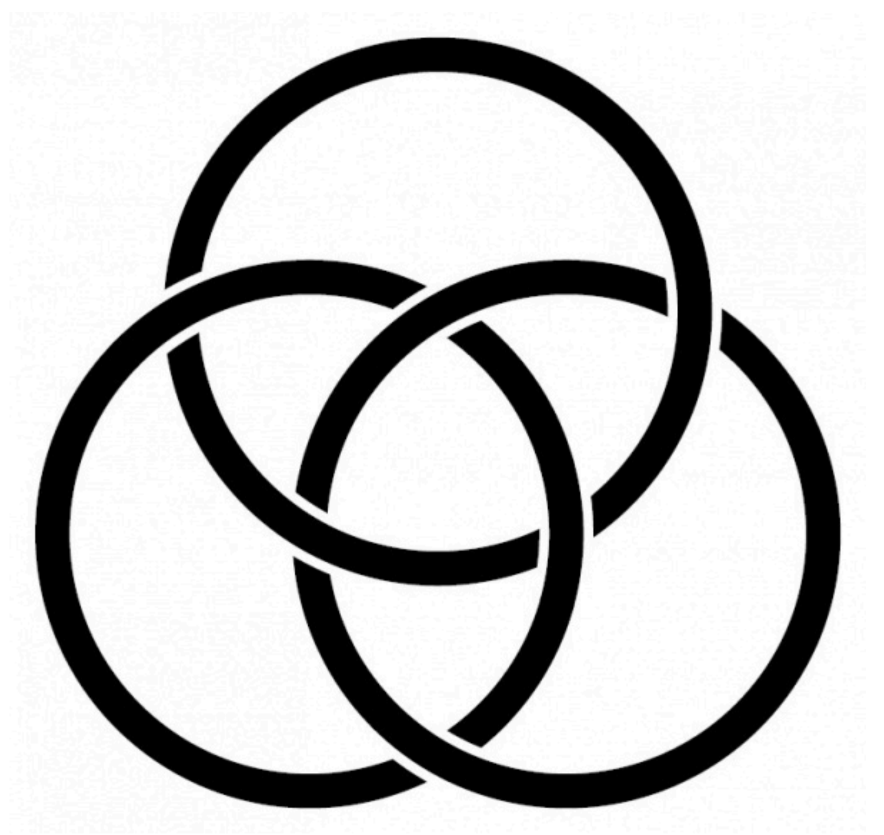


Weak, short-range potential





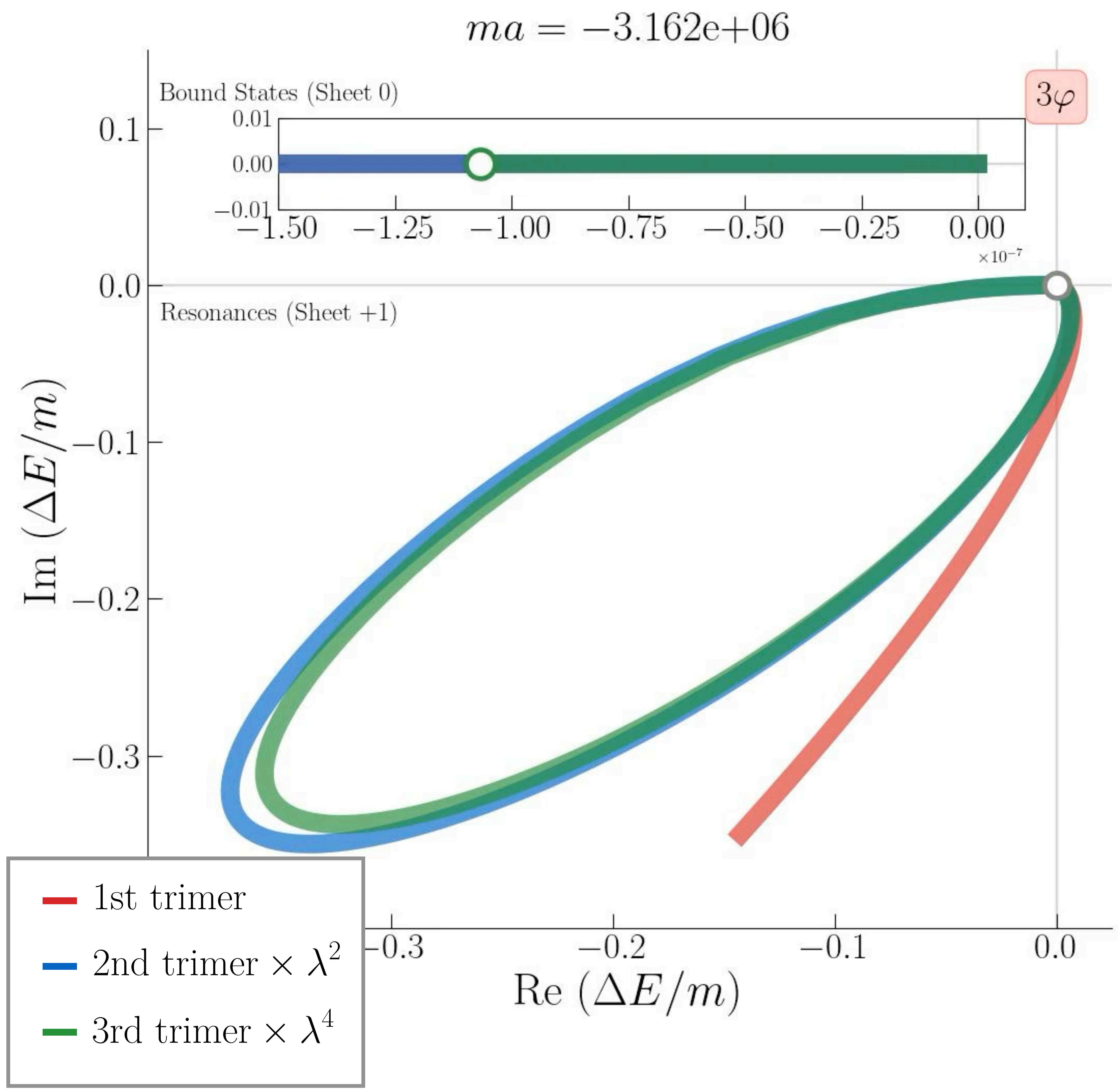
Borromean binding

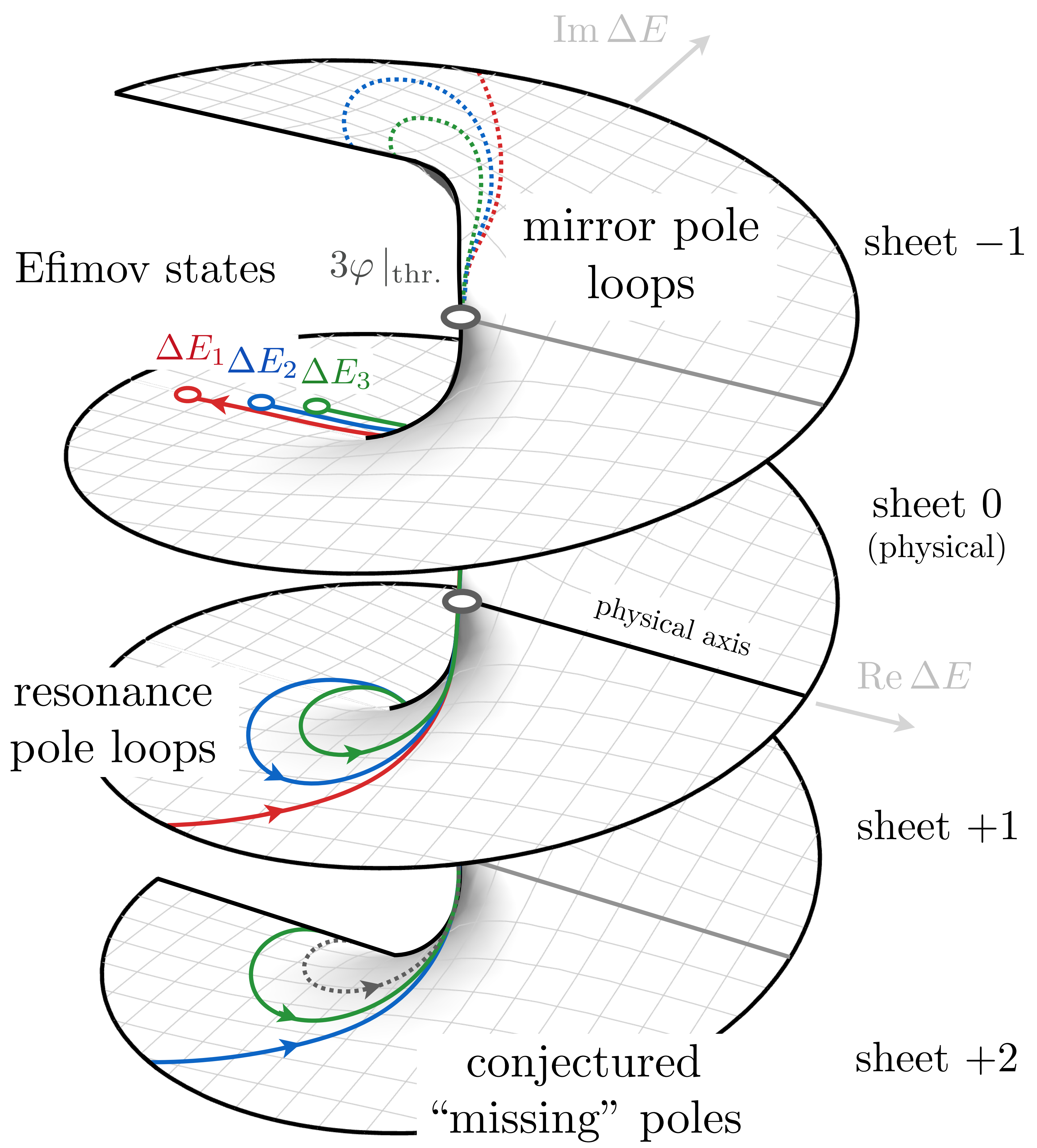


Efimov resonances

Dawid, Islam, Briceño, Jackura,
"Evolution of Efimov states"

Dawid, Hoban, Chambers,
"A universal relation between three-body states"





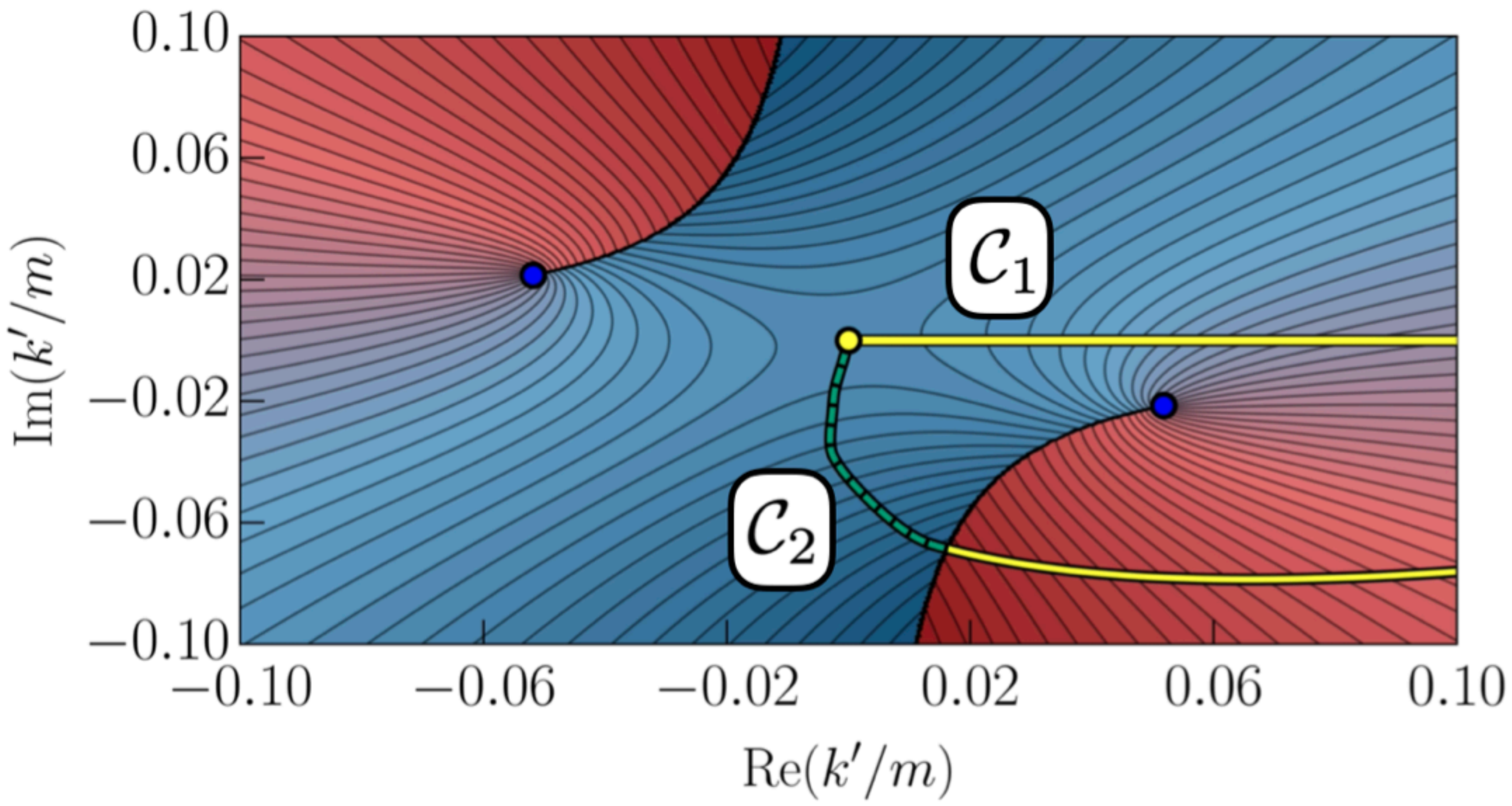
Trimers evolve on cyclic trajectories obeying the Efimov scaling.

Resonance pole N+1 appears exactly when N becomes bound!

Threshold behavior pt. 1

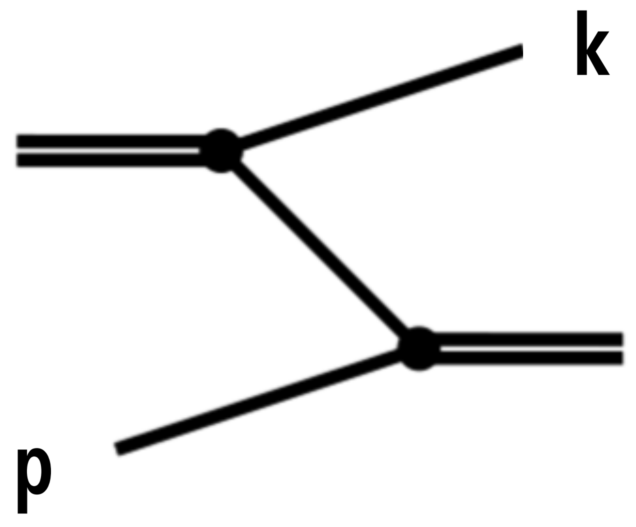
Dawid, Hoban, Chambers, (work in progress)
"A universal relation between three-body states"

$$\mathcal{M}_3(p,k) = -\mathcal{M}_2(p)\, B(p,k)\, \mathcal{M}_2(k) - \mathcal{M}_2(p) \int_0^{k_{\max}} \frac{\mathrm{d}k'\, k'^2}{(2\pi)^2 \omega_{k'}} B(p,k')\, \mathcal{M}_3(k',k)$$



Threshold branch point is generated as an endpoint singularity when two-body branch point collides with the origin of the complex spectator's momentum plane.

One-particle exchange (OPE) diverges at p,k=0 at the threshold



$$\propto \frac{1}{2pk} Q_0(z_{ij}(p,k))$$

OPE is Hilbert-Schmidt

$$\int_0^{k_{\max}} dp \int_0^{k_{\max}} dk\, |Q_0(z_{ij}(p,k))|^2 < \infty$$

Threshold behavior pt. 2

Dawid, Hoban, Chambers, (work in progress)
"A universal relation between three-body states"

Kernel of the integral equation can be uniformly approximated by a sum of separable kernels

$$B(p,k) = \sum_{i,j=0}^{\infty} \phi_i(p)\, \mathcal{B}_{ij}(p,k)\, \phi_j^*(k)$$

Solution of the integral equation

$$\mathcal{M}_3(p,k) = \sum_{i,j=1}^{\infty} \mathcal{M}_2(p)\, \phi_i(p) \left[\mathcal{B}^{-1} + \mathcal{I}\right]_{ij}^{-1} \phi_j^*(k)\, \mathcal{M}_2(k)$$

Three-body "phase space"

$$\mathcal{I}_{ij} = \int_0^{k_{\max}} \frac{dk'\, k'^2}{(2\pi)^2 \omega_{k'}}\, \phi_i^*(k')\, \mathcal{M}_2(k')\, \phi_j(k')$$

Three-body pole condition: $\det_{ij} \left[\mathcal{B}^{-1} + \mathcal{I}\right] = 0$

UV pieces of the integral are smooth, branch point is generated from the integration near k′=0.

$$\frac{k'^2}{(2\pi)^2\, \omega_{k'}}\, \phi_i^*(k')\, \mathcal{M}_2(k')\, \phi_j(k') \approx \frac{k'^2}{(2\pi)^2\, m}\, \phi_i^*(0)\, \frac{16\pi(\Delta E + 2m - 3k'^2/4m)}{-\frac{1}{a} + \sqrt{\frac{3}{4}k'^2 - \Delta E m}}\, \phi_j(0)$$

Threshold behavior pt. 3

Dawid, Hoban, Chambers, (work in progress)
"A universal relation between three-body states"

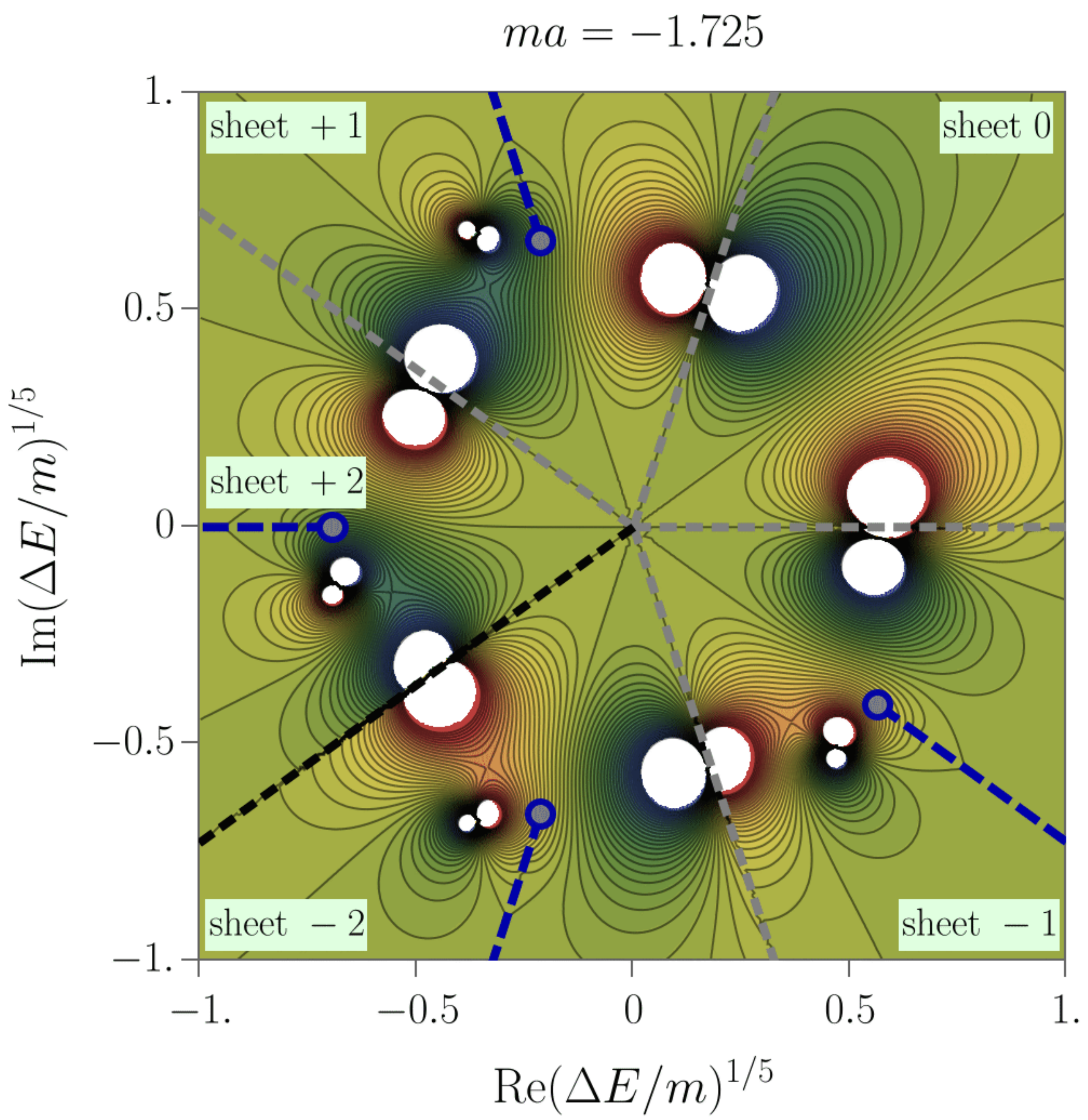
The threshold singularity is a "mild" logarithm

$\Lambda = k_{\max}$

Threshold branch point

$$\mathcal{I}_{ij} = -(\phi_i^*(0)\phi_j(0))\left(\frac{4a^2}{3\sqrt{3}\pi}\right)(m\Delta E)^2 \log\left(-\frac{m\Delta E}{\Lambda^2}\right) + \mathcal{R}_{ij}$$

$\Delta E = E - 3m$



A threshold pole on sheet 0 is accompanied by threshold poles on every sheet!

Analytic continuation

$$\log\left(-\frac{m\Delta E}{\Lambda^2}\right) \to \log\left(-\frac{m\Delta E}{\Lambda^2}\right) + 2\pi i n\,.$$

$$\det[\boldsymbol{P} + \boldsymbol{Q}\cdot\log(\Delta E)] = 0$$

Universal quantization condition

Open questions

- We make a strong claim: when a pole appears at the three-body threshold, it is accompanied by infinitely many poles on all Riemann sheets (regardless of the microscopic details of the interactions). As a consequence, when a three-body bound-state pole appears near a threshold, there should also be a companion three-body virtual state or a resonance. This seems to work well for halo nuclei. Are there known counterexamples?
- Is there any use for the states on higher Riemann sheets: are they experimentally detectable, or mostly too far displaced from the threshold to be of any physical interest?
- The K–matrix parametrization of the amplitude is very general: what does it imply for the universal near-threshold behavior of general N-body states?
- Can the logarithmic threshold behavior be cast into a useful approximate algebraic parametrization of three-body scattering amplitudes?
- Is there a useful way to connect these S-matrix theoretic considerations to nuclear Hamiltonians?

Thank you!

Acknowledgments

All the work described in this talk was done in collaboration with these most kind and honorable people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Andrew Hanlon (KSU)


Declan Hoban (UC Berkeley)

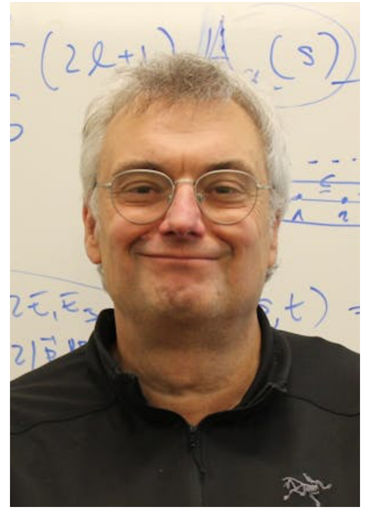

Ben Hörz (Intel)


Colin Morningstar (CMU)


Fernando Romero-López (U. Bern)


Stephen Sharpe (UW)


Sarah Skinner (CMU)


Adam Szczepaniak (IU)