Qubitization Strategies for Sigma Models and Gauge Theories

INT Workshop:
Tensor Networks in Many Body and Quantum Field Theory

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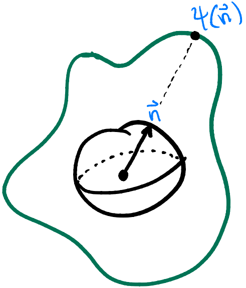

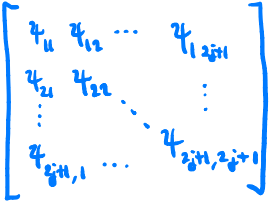

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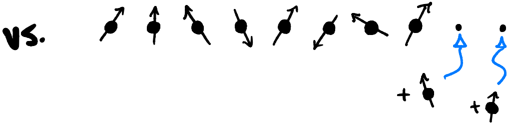
arXiv:2109.07500
arXiv:2209.00098

Quantum Simulation

The promise of quantum computers: nonperturbative real-time QFT — a problem too difficult for classical computers...

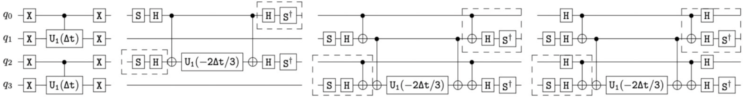
$$n \quad \text{2-state particles} \quad \Rightarrow \quad H = 2^n \times 2^n \text{ matrix: exponential growth}$$



(Adding one more qubit to the computer... IF this is not too difficult, presents a huge advantage.)

HOWEVER, to be practical, the circuit complexity cannot be excessive either:

$$\textit{gates} \times \text{qubits}$$



Simulation of Field Theories

For bosonic field theories, an extra hurdle: 1-site Hilbert space is ∞-dimensional, and requires **truncation**.

What's worse: we're ultimately interested in continuum physics...

- One expects we must *remove* the truncation $n_{\max} \to \infty$
- And then take the continuum limit $a \to 0$

But removal of the truncation may increase circuit complexity very quickly!

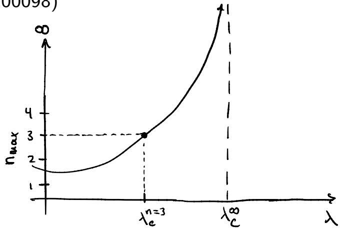
$\ell_{\max}$	CNOTs
1	60
2	3826
3	11826

(sigma model; arXiv:2209.00098)

Likely necessary to formulate theories such that the truncation need *not* be removed.

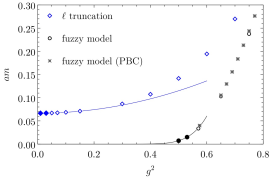
UNIVERSALITY to the rescue?

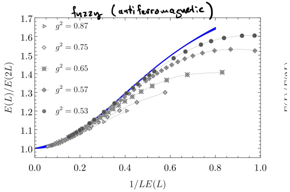
- If the system has a quantum critical point, and is in the right universality class ...

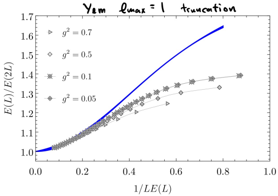


Outline / Preview

1. Sigma models
   - Laplacian truncations
   - Fuzzification
   - Fuzzy O(N) Models
2. Checking universality
   - Matrix Product States
   - Scaling Curves
   - Results
3. Gauge theories
   - Laplacian truncation
   - Fuzzy SU(2) gauge theory
   - Rudimentary numerical comparisons
4. Summary, and the Future...







Sigma Models

A 1+1d lattice $O(N)$ sigma model has Hamiltonian operator

$$\hat{H} = \frac{g^2}{2} \sum_x \hat{\boldsymbol{L}}^2(x) - \frac{1}{g^2} \sum_x \boldsymbol{n}(x) \cdot \boldsymbol{n}(x+1)$$

where $\boldsymbol{n}(x) \in S^{N-1}$ are unit vectors and $\hat{L}_k(x)$ are "orbital angular momentum" operators.

Example: $O(3)$ model

$$\hat{L}_k = -i\epsilon_{klh} n_l \frac{\partial}{\partial n_h}, \quad \text{so} \quad \hat{\boldsymbol{L}}^2 = -\Delta = \text{Laplacian on } S^2$$

The spectrum of $\hat{\boldsymbol{L}}^2$ is $\ell(\ell+1)$ for $\ell = 0, 1, 2, \ldots$, so the $|\ell, m\rangle$ basis is infinite-dimensional.

Natural candidate for qubitization: choose some $\ell_{\max}$ — this preserves the symmetry properties [1].

$$n_k \longmapsto (Y_k)_{\ell,m;\ell',m'} = \langle \ell, m | n_k | \ell', m' \rangle, \quad \ell, \ell' \leq \ell_{\max}$$

$$\sum_k \hat{L}_k^2 \longmapsto (h_0)_{\ell,m;\ell',m'} = \delta_{\ell,\ell'} \delta_{m,m'} \cdot \ell(\ell+1)$$

But operators get severely truncated, e.g. $\boldsymbol{n}^2 = 1$ but $\sum_k Y_k^2 \neq \mathbb{1}$.

We have tested this qubitization in a numerical simulation (*spoiler: doesn't look good!*)

Fuzzy 2-Sphere

Recall: target space of the full theory is the S^2 manifold: ∞-dim operators / states

There exists a mathematical construction — *the fuzzy sphere* (Madore '93) — which replaces the Cartesian coords n_i by **MATRICES** such that

$$\boldsymbol{n}^2 = 1 \quad \Rightarrow \quad \sum_k \mathbb{J}_k^2 = \mathbb{1}$$

but the fuzzy coords no longer commute.

SU(2) spin-j generators were up to the task: $\mathbb{J}_k \propto S_k^{(j)}$, and they preserve the rotation property of n_k:

$$\mathrm{e}^{i\theta_k \frac{\sigma_k}{2}} \mathbb{J}_i \, \mathrm{e}^{-i\theta_k \frac{\sigma_k}{2}} = R_{ij}(\theta)\mathbb{J}_j, \quad R \in O(3)$$

Functions on the sphere get mapped as

$$\psi(\boldsymbol{n}) = \psi_0 + \psi_i n_i + \tfrac{1}{2}\psi_{ij} n_i n_j + \cdots$$

$$\longmapsto \Psi(\mathbb{J}) = \psi_0 \mathbb{1} + \psi_i \mathbb{J}_i + \frac{1}{2}\psi_{ij}\mathbb{J}_i\mathbb{J}_j + \cdots$$

For fixed j, Ψ is a $(2j+1)\times(2j+1)$ matrix (*the series truncates!*).

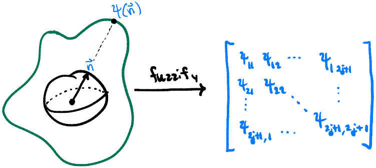
Non-commutation of fuzzy coordinates:

$$[\mathbb{J}_i, \mathbb{J}_j] = i\epsilon_{ijk}\mathbb{J}_k$$

Fuzzy Qubitization (Alexandru, 2019)

States of the fuzzy quantum system are now finite matrices $\Psi = (2j+1) \times (2j+1)$.

The Hilbert space is determined from the inner product of matrices: $\langle \Psi | \Phi \rangle := \mathrm{tr}\left[\Psi^\dagger \Phi\right]$



From the rotation of $\mathbb{J}_i$ we infer a rotation on arbitrary fuzzy states:

$$\Pi_R \Psi := \mathrm{e}^{-i\theta_k \frac{\sigma_k}{2}} \Psi \, \mathrm{e}^{i\theta_k \frac{\sigma_k}{2}}$$

One may then define a "canonical momentum" operator conjugate to $\mathbb{J}_i$,

$$\mathscr{L}_k \Psi := i \frac{\mathrm{d}}{\mathrm{d}\theta} \left[\mathrm{e}^{-i\theta \frac{\sigma_k}{2}} \Psi \, \mathrm{e}^{i\theta \frac{\sigma_k}{2}} \right] \Big|_{\theta=0} = \frac{1}{2} [\sigma_k, \Psi]$$

The $\mathbb{J}_i$ and momenta satisfy "canonical commutators"

$$[\![\mathbb{J}_i, \mathscr{L}_j]\!] \Psi = i \epsilon_{ijk} \mathbb{J}_k \Psi, \quad \forall \, \Psi$$

($[\![\bullet, \bullet]\!]$ is a commutator on the *fuzzy* Hilbert space)

The fuzzy algebra

The fuzzy theory shares much structural similarity with the field space rep of the sigma model.

Full sigma model:

states: $\psi(\boldsymbol{n})$

Leibniz rule:

$$\widehat{L}_k(\phi\psi) = (\widehat{L}_k\phi)\psi + \phi(\widehat{L}_k\psi)$$

operator algebra:

$$[n_i, \widehat{L}_j] = i\epsilon_{ijk} n_k$$
$$[\widehat{L}_i, \widehat{L}_j] = i\epsilon_{ijk}\widehat{L}_k$$
$$[n_i, n_j] = 0$$

Note: The truncation is removed as $j \to \infty$:

$\lim_{j\to\infty} \frac{1}{2j+1}\mathrm{tr}[\Psi^{(j)}] = \frac{1}{4\pi}\int_{S^2}\psi(\boldsymbol{n})\sin\theta \mathrm{d}\theta\mathrm{d}\phi$

where $\Psi^{(j)} = (2j+1)\times(2j+1)$

Fuzzy model (w/ $\mathscr{L}_k = [\mathbb{J}_k, \bullet]$):

states: $\Psi(\mathbb{J})$

Leibniz rule:

$$\mathscr{L}_k(\Phi\Psi) = (\mathscr{L}_k\Phi)\Psi + \Phi(\mathscr{L}_k\Psi)$$

operator algebra:

$$[\![\mathbb{J}_i, \mathscr{L}_j]\!] = i\epsilon_{ijk}\mathbb{J}_k$$
$$[\![\mathscr{L}_i, \mathscr{L}_j]\!] = i\epsilon_{ijk}\mathscr{L}_k$$
$$[\![\mathbb{J}_i, \mathbb{J}_j]\!] = i\epsilon_{ijk}\mathbb{J}_k$$

$\Rightarrow$ commutativity of the n_i is lost, but symmetry-related commutators are intact.

$j = 1/2$ Fuzzy Representation

For $j = 1/2$, one has $\mathbb{J}_k = \sigma_k/\sqrt{3}$, with σ_k the usual Pauli matrices. Wave functions are 2×2 matrices of the form

$$\Psi = \psi_0 \mathbb{1} + \psi_k \mathbb{J}_k,$$

so the Hilbert space of a single fuzzy spin is 4-dimensional.

The fuzzy Laplacian operator

$$-\Delta_{S^2} \longmapsto \kappa \sum_k [\mathbb{J}_k, [\mathbb{J}_k, \bullet]]$$

reproduces the $l = 0, 1$ subspace of the full $-\Delta_{S^2}$ eigenspace when $\kappa = 3/4$.

The Hamiltonian operator for N fuzzy spins is obtained by fuzzifying the sigma model Hamiltonian:

$$\hat{H} \mapsto \sum_x \left(\frac{\kappa g^2}{2} \sum_k [\mathbb{J}_k(x), [\mathbb{J}_k(x), \bullet]] \pm \frac{\kappa}{g^2} \sum_k \mathbb{J}_k(x) \mathbb{J}_k(x+1) \right)$$

We allow for both signs $\pm$ in the neighbor term; turns out ferromagnetic and anti-ferromagnetic cases differ.

This Hamiltonian is invariant under $O(3)$ rotations, similar to the $\ell_{\max}$ truncation. (For $\ell_{\max} = 1$, the 1-site Hilbert space is also 4-dimensional.)

How to Simulate the Fuzzy Model

A basis $|a\rangle$, $a = 1, 2, 3, 4$, more suitable for computations is given by

$$\langle a|b\rangle := \mathrm{tr}\,(\mathbb{T}_a^\dagger \mathbb{T}_b) = \delta_{ab}, \quad \text{where} \quad \{\mathbb{T}_a\} = \{\tfrac{i}{\sqrt{2}}\mathbb{1}, \sqrt{\tfrac{3}{2}}\mathbb{J}_k\},$$

leading to a 4×4 matrix representation of all 1-site operators.

In this basis, the Hamiltonian is given by [2]

$$\hat{H} = \sum_x \left[g^2 h_0(x) \pm \frac{\kappa}{g^2} \sum_k j_k(x) j_k(x+1) \right]$$

where h_0, j_k are 4×4 matrices given by

$$(h_0)_{ab} = \frac{\kappa}{2} \sum_k \mathrm{tr}\big(\mathbb{T}_a^\dagger [\mathbb{J}_k, [\mathbb{J}_k, \mathbb{T}_b]]\big), \qquad (j_k)_{ab} = \mathrm{tr}\big(\mathbb{T}_a^\dagger \mathbb{J}_k \mathbb{T}_b\big)$$

One can attempt using MCMC to study the model by using Boltzmann weights $\langle \boldsymbol{a}|\mathrm{e}^{-\epsilon \hat{H}}|\boldsymbol{b}\rangle$, for small timesteps ϵ, but has a **sign problem**!

Instead of a MC simulation, we have studied the system using the machinery of **matrix product states**, which does *not* have a sign problem (the trade-off: you're restricted to low-dimensional systems, e.g. 1+1, 2+1).

Matrix Product States (MPS)

The state vectors of an N-site quantum chain with 1-site Hilbert space $\mathscr{H}_1$ of dimension d are of the form

$$|\psi\rangle = \sum_{i_1,\ldots,i_N=1}^{d} c_{i_1\ldots i_N} |i_1,\ldots,i_N\rangle,$$

and they span a d^N-dimensional vector space $\mathscr{H}_N$.

An (open boundary) **matrix product state** has the form [3]

$$|\psi_A\rangle = \sum_{i_1,\ldots,i_N=1}^{d} A_1^{i_1} A_2^{i_2} \cdots A_{N-1}^{i_{N-1}} A_N^{i_N}\, |i_1,\ldots,i_N\rangle,$$

where each $A_x^{i_x}$ is a $D_x \times D_x$ complex matrix, except for the two ends which are row and column vectors. Any state in $\mathscr{H}_N$ can be written as an MPS, for D_x's large enough.

There is reason to believe, however, that ground states of *gapped, local* systems are well-approximated by an MPS, for D quite small (i.e. not exponentially large in N).

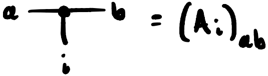
For example, the ground state of the spin-1 AKLT model with Hamiltonian

$$H = \sum_x \left[\widehat{\boldsymbol{S}}(x) \cdot \widehat{\boldsymbol{S}}(x+1) + \frac{1}{3} \left(\widehat{\boldsymbol{S}}(x) \cdot \widehat{\boldsymbol{S}}(x+1) \right)^2 \right]$$

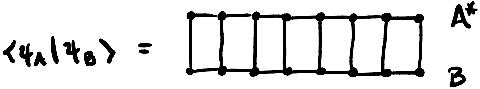
is known to be an MPS with $D = 2$, for any size N.

MPS Diagrams

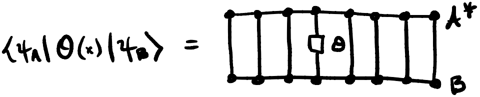
An MPS $|\psi_A\rangle$ can be represented as a *tensor network* with diagram



Inner products of two states $|\psi_A\rangle$, $|\psi_B\rangle$ are represented by fully contracted networks,



Expectation values with a 1-site operator insertion $\hat{\mathcal{O}}(x)$ with components $\langle i|\hat{\mathcal{O}}|j\rangle$ can likewise be drawn as



Variational MPS Algorithm (a.k.a. DMRG)

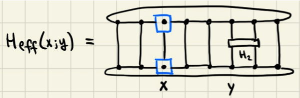
Putting MPS to use ...

The idea is to minimize the expectation value of H with respect to the coefficients $(A_x^{i_x})_{ab}$ of the MPS matrices [4, 5], subject to the constraint that $\langle\psi_A|\psi_A\rangle = 1$. Thus one minimizes

$$\mathcal{L} = \langle\psi_A|\hat{H}|\psi_A\rangle - \lambda(\langle\psi_A|\psi_A\rangle - 1)$$

The minimization with respect to the matrices A_x^i at site x is equivalent to solving a generalized eigenvalue problem,

$$(H_{\text{eff}})_{IJ}(A_x)_J = \lambda(N_{\text{eff}})_{IJ}(A_x)_J$$



One performs the minimization 1 (or 2) sites at a time, moving across the chain in sequence. The eigenvalue λ converges towards the ground state energy as this is iterated.

Once converged, you end up with an estimate for the ground state energy E_0 and an MPS approximation of the ground state itself (via the A_x^i).

Testing Equivalence to the Sigma Model [arXiv:2109.07500,arXiv:2209.00098]

We want to compare the physics of the two qubitizations above with the full σ-model:

$$S(\boldsymbol{n}) = \frac{1}{2g^2} \int \mathrm{d}^2 x \; \partial_\mu \boldsymbol{n}(x) \partial_\mu \boldsymbol{n}(x)$$

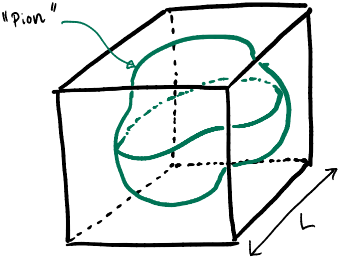
Phenomenology: the first excited state of the continuum theory is an $O(3)$-triplet of particles of mass m (think *pion mass* for the QCD analogy); they have relativistic dispersion relations. The mass is *nonperturbative* in the coupling g^2:

$$ma \sim \frac{\mathrm{e}^{-b/g^2}}{g^2}$$

The theory is **asymptotically free** (like QCD): the coupling g^2 of the theory grows at large distances, and decreases at short distances.

In a finite box of size L, the "pion mass" is deformed to some value $M(L)$.

So we can compare the finite-volume mass gap $M(L)$ at various mL values. Since $p \propto 1/L$ in a box, $p/m \sim 1/mL$ gives us a sense for what energy scale we are probing, relative to m.



Determining $M(L)$

Using variational MPS we get estimates of the finite-volume energy gap

$$a\Delta(L) = a\hat{E}_1(L) - a\hat{E}_0(L)$$

However, the *energy* gap may not coincide with the *mass* gap in the Hamiltonian framework: $a\Delta(L) \neq aM(L)$, generally. A renormalization is necessary to restore relativistic covariance [6].

Procedure: obtain the infinite-volume energy gap $\Delta = \Delta(\infty)$, and the infinite-volume mass gap by $am = \xi^{-1}$. Then form the ratio

$$\eta = \frac{am}{a\Delta},$$

and rescale all energies $\hat{E}_k$ by η. Then $M(L) = \eta\Delta(L)$. Note $\eta = \eta(g^2)$.

We measure *am* from correlators evaluated in the MPS ground states $|\psi_0\rangle$,

$$C(z) = \langle\psi_0|\mathbb{J}_3(x)\mathbb{J}_3(x+z)|\psi_0\rangle$$

and fitting to a Bessel function $K_0(z)$.

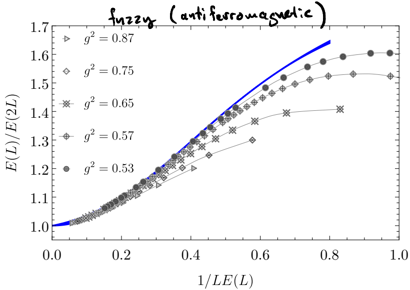
Finite-Size Scaling Curves

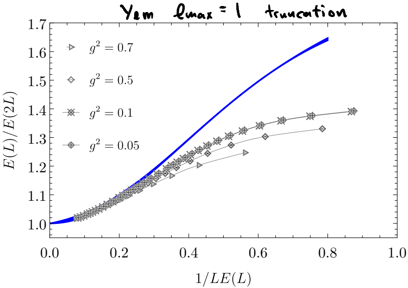
A convenient way to visualize the comparison of our models to the sigma model is to plot the FSS curves of each model:

$$\frac{M(L)}{M(2L)} \quad \text{vs.} \quad 1/M(L)L$$

This curve is universal and known for the σ-model with periodic boundary conditions [7, 8]. For open boundaries, we have to do a MC simulation of the sigma model with appropriate boundary conditions.

Results [9, 10]! (**Blue** curve is the scaling curve from MC: the full sigma model. **Gray** points are from MPS.)





thanks to GW's Pegasus HPC. (and Andy Sheng)

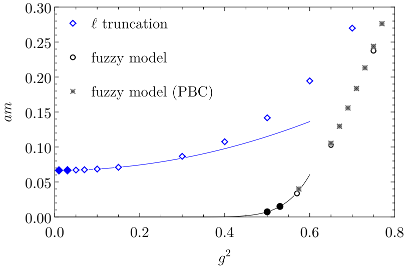
More FSS Curves; Mass Gaps

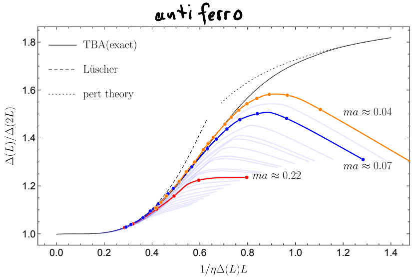
Right: Continuum limits?

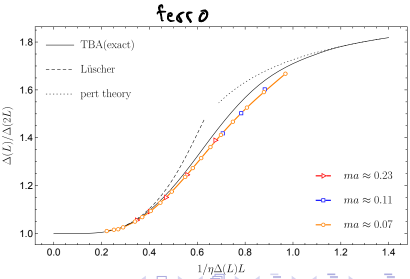
The $\ell_{\max} = 1$ truncation does not have a continuum limit: the mass freezes out as $g^2 \to 0$.

Strong evidence that the Fuzzy model *does* have a continuum limit.

Below: Ferro- vs. Antiferro-magnetic fuzzy models (periodic case).







Why does it work? Ideas...

Wilsonian RG perspective: if it has the right symmetries, then there's "only a few options" for what theory sits at the critical point.

Fuzzy reps may guarantee the existence of a critical point:

- Recall: the field space rep. nearest-neighbor term has a highly degenerate ground state that gets split for $g^2 \neq 0$.

  $$V = -\frac{1}{g^2}\sum_x \boldsymbol{n}(x)\cdot\boldsymbol{n}(x+1)$$

  Lattice critical point: $\hat{\xi}^{-1} = 0$. All physical state energies merge into the degenerate ground state energy (in lattice units) as $g^2 \to 0$.

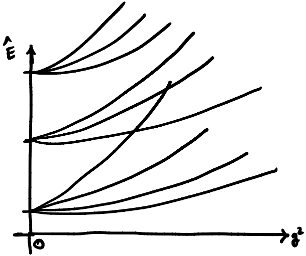
- The fuzzy neighbor operator has a ground state degeneracy that's exponential in the volume, e.g. 4^N for fuzzy O(3). The kinetic op breaks this degeneracy for $g^2 \neq 0$.

  $$V = \pm\frac{1}{g^2}\sum_{x,k} \mathbb{J}_k(x)\mathbb{J}_k(x+1)$$

  The degeneracy can be understood by going to a "product basis"

  $$\Psi = \psi_{ab}\, \mathrm{e}_a \mathrm{e}_b^\top, \quad V = \Big(\sum_{x,k}\sigma_k(x)\sigma_k(x+1)\Big)_a \otimes \mathbb{1}_b$$

  - Yields an equivalence to the Heisenberg comb [11] of Bhattacharya, et al. 2020
  - $\ell_{\max}$ truncation *does not* have this degeneracy



Finite Gauge Theories: History

There's a long history of proposing quantum gauge theories with finite local Hilbert space:

- Horn, 1981: SU(2), $\dim \mathscr{H}_\ell = 5$
- Hamer, 1981: SU(2), $\dim \mathscr{H}_\ell = 5, 14, \ldots$
- Orland & Rohrlich, 1989, "gauge magnets": SU(2), $\dim \mathscr{H}_\ell = 4, 10, \ldots$
- Brower, Chandrasekharan, & Wiese, 1997, "quantum links": U(N), SU(N), $\dim \mathscr{H}_\ell = 2N$
- ...

All of which are possible candidates to qubitize a lattice gauge theory.

Hamer's method is the generalization of $\ell_{\max}$ truncation to SU(2); we'll discuss this next.

Then we'll see that O.R.'s model is suitable for fuzzification.

Gauge theories & Laplacian truncation

Consider a lattice gauge theory in the field space representation: states are wave functions

$$\psi(U) \in \mathbb{C}$$

Left/right gauge transformations rotate the argument of ψ by some $g \in SU(N)$:

$$\Pi_g^L \psi(U) = \psi(g^\dagger U), \quad \Pi_g^R \psi(U) = \psi(Ug)$$

These are generated by the "canonical momenta"

$$\widehat{\mathcal{L}}_a \psi(U) = \Big(-(t_a U)_{ij} \frac{\partial}{\partial U_{ij}} + (t_a U)^*_{ij} \frac{\partial}{\partial U^*_{ij}} \Big) \psi(U)$$

$$\widehat{\mathcal{R}}_a \psi(U) = \Big((U t_a)_{ij} \frac{\partial}{\partial U_{ij}} - (U t_a)^*_{ij} \frac{\partial}{\partial U^*_{ij}} \Big) \psi(U)$$

Kinetic energy operator (per link) is

$$K(\ell) = \sum_a \left(\widehat{\mathcal{L}}_a^2 + \widehat{\mathcal{R}}_a^2 \right)$$

$N = 2$ case: K is the Laplacian on SU(2), eigenstates are Wigner matrices:

$$\sum_{k=1}^{3} \widehat{\mathcal{L}}_k^2 \, \mathscr{D}^j_{mm'}(U) = j(j+1) \, \mathscr{D}^j_{mm'}(U) \;\Rightarrow\; \textit{truncate to } j \leq j_{\max}$$

Fuzzy SU(2) gauge theory

Orland & Rohrlich (1989) noticed that 4×4 Gamma matrices satisfy the SU(2) gauge theory algebra. Define

$$\mathscr{U}_{ij} = \delta_{ij}\Gamma_4 - i\sum_k (\sigma_k)_{ij}\Gamma_k$$

$$\Sigma_k^L = \begin{pmatrix} \sigma_k & 0 \\ 0 & 0 \end{pmatrix}, \quad \Sigma_k^R = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_k \end{pmatrix}$$

Then

$$\begin{aligned}
[\mathscr{U}_{ij}, \Sigma_k^L] &= (\sigma_k \mathscr{U})_{ij}, \\
[\mathscr{U}_{ij}, \Sigma_k^R] &= -(\mathscr{U}\sigma_k)_{ij}, \\
[\Sigma_i^L, \Sigma_j^L] &= 2i\epsilon_{ijk}\Sigma_k^L \\
[\Sigma_i^R, \Sigma_j^R] &= 2i\epsilon_{ijk}\Sigma_k^R \\
[\Sigma_i^L, \Sigma_j^R] &= 0
\end{aligned}$$

O.R.'s gauge magnet has a 4d 1-link Hilbert space.

Fuzzy interpretation: We think of $\mathscr{U}_{ij}$ as noncommutative coordinates corresponding to U_{ij}. The $\mathscr{U}_{ij}$ satisfy fuzzy SU(2) manifold equations:

$$U_{ij}U_{kj}^* = \delta_{ik} \longrightarrow \tfrac{1}{2}\{\mathscr{U}_{ij}, \mathscr{U}_{kj}^\dagger\} = \delta_{ik}\mathbb{1}$$

$$U_{11}U_{22} - U_{12}U_{21} = 1 \longrightarrow \tfrac{1}{2}\left(\{\mathscr{U}_{11}, \mathscr{U}_{22}\} - \{\mathscr{U}_{12}, \mathscr{U}_{21}\}\right) = \mathbb{1}$$

Fuzzy SU(2) gauge theory

The fuzzy states are 4×4 matrices $\Psi(\mathscr{U})$: 16d 1-link Hilbert space. They transform as

$$\Pi_g^L \Psi = \Theta_g^L \Psi \, \Theta_g^{L\dagger}$$

A conjugate momentum to $\mathscr{U}_{ij}$ can be defined as the generator:

$$\mathscr{L}_k \Psi := i \frac{\mathrm{d}}{\mathrm{d}\omega} \left(\mathrm{e}^{-i\omega \Sigma_k^L} \Psi \, \mathrm{e}^{i\omega \Sigma_k^L} \right) \Big|_{\omega=0} = -[\Sigma_k^L, \Psi]$$

likewise for right-generators. These satisfy a Leibniz rule:

$$\mathscr{L}_k(\Psi\Phi) = (\mathscr{L}_k \Psi)\Phi + \Psi(\mathscr{L}_k \Phi)$$

The operator algebra then follows from the Leibniz rule & O.R. commutators:

$$\begin{aligned} [\![\mathscr{U}_{ij}, \mathscr{L}_k]\!] &= -(\sigma_k \mathscr{U})_{ij}, \\ [\![\mathscr{U}_{ij}, \mathscr{R}_k]\!] &= (\mathscr{U} \sigma_k)_{ij}, \\ [\![\mathscr{L}_i, \mathscr{L}_j]\!] &= -2i\epsilon_{ijk}\mathscr{L}_k \\ [\![\mathscr{R}_i, \mathscr{R}_j]\!] &= -2i\epsilon_{ijk}\mathscr{R}_k \\ [\![\mathscr{L}_i, \mathscr{R}_j]\!] &= 0 \end{aligned}$$

Thus the fuzzy rep closely parallels the the field space rep, except that $[\![\mathscr{U}_{ij}, \mathscr{U}_{kl}]\!] \neq 0$.

Hamiltonian

The simplest option for the Hamiltonian is by analogy with Kogut and Susskind:

$$H = \frac{g^2}{2} K \pm \frac{1}{g^2} V$$

where

$$K = \sum_{\ell} \sum_{k} \left(\mathscr{L}_k^2(\ell) + \mathscr{R}_k^2(\ell) \right)$$

and

$$V = \sum_{x} \mathscr{U}_{ij}(\ell_1) \mathscr{U}_{jk}(\ell_2) \mathscr{U}_{lk}^{\dagger}(\ell_3) \mathscr{U}_{il}^{\dagger}(\ell_4)$$

The plaquette operator has a highly degenerate ground state, as it does in field space.

- In the product basis $\Psi = \psi_{ab} \mathrm{e}_a \mathrm{e}_b^{\top}$, the a-sector is an O.R. gauge magnet.
- The j_{max} truncation does not have a degenerate ground state for V.

Another symmetric 1-link term is

$$K_2 = \sum_{\ell} [\mathscr{U}_{ij}(\ell), [\mathscr{U}_{ij}^{\dagger}(\ell), \bullet]]$$

$K, \ K_2$ break the degeneracy of V in different ways.

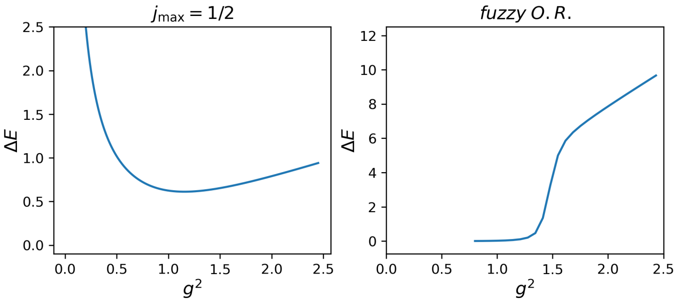
Simulation and Costs

Simulating this theory is difficult – 16d Hilbert space and 2 space dimensions.

Naive implementation with open MPS is sketchy; needs PEPS, and the gauge constraint needs to be implemented in some way ...

... So universality remains to be tested.

Some rudimentary small-volume gap comparisons: (a single plaquette; exact diagonalization)



However, the circuit complexity for this theory is quite low! Per plaquette:

- Fuzzy Orland-Rohrlich: 236 CNOTs
- Compare: $j_{\max} = 1/2$ and the Horn model (w/ 5d Hilbert spaces): $\sim 17,000$ CNOTs [12]

Summary / work in progress

- Suitably regularizing bosonic QFTs for quantum computers is a subtle task.
- Rather than removing the Hilbert space regulator "manually" (e.g. $\ell_{\max} \to \infty$), one must appeal to universality and tune to a quantum critical point.
- The fuzzy sphere construction and the Heisenberg comb yield better "coverage" of the full sigma model than the $\ell_{\max} = 1$ truncation, all of which have a 4-dimensional 1-site Hilbert space.
- MPS can be used to construct the scaling curves used to assess universality.
- Generalization of fuzzification to $SU(2)$ gauge theory is worked out, but viable simulation methods are still being explored / sought.

Thanks for listening!

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Spin Chains and the Sigma Model

The spin-1 Heisenberg model is defined by the Hamiltonian

$$\hat{H} = \pm J \sum_{x} \sum_{k=1}^{3} S_k(x) S_k(x+1)$$

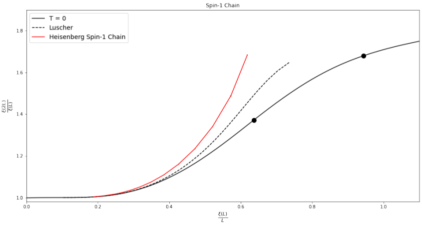
where the $\pm$ sign refers to anti-ferromagnetic and ferromagnetic cases, respectively, and the S_k are spin-1 matrices.

Haldane's Conjecture (1982): The low-energy (long-distance) physics of the *anti*-ferromagnetic spin-1 Heisenberg chain is described by the long-distance physics of the $1+1d$ $O(3)$ σ-model [13].

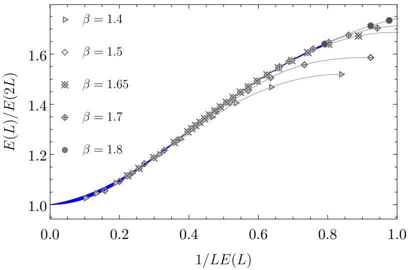
"Long-distance" means energy scales $E \ll m$ where m is the intrinsic energy scale of the theory, i.e. the mass of the lightest particle in the σ-model spectrum.

So one can think of the spin-1 Heisenberg chain as a (poor) regularization of the σ-model.

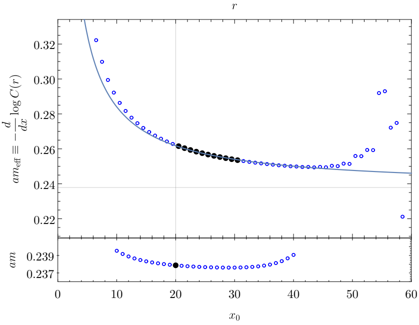
But we are interested in whether one can obtain a regularization that reproduces σ-model physics even at scales $E \gg m$.



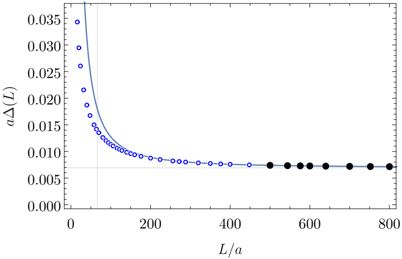
Extra plots



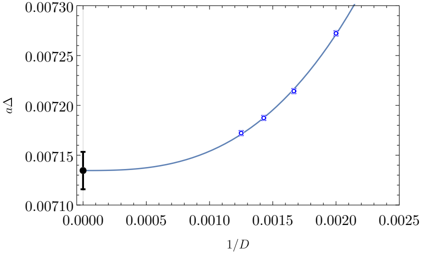
$\beta = 1.4$
$\beta = 1.5$
$\beta = 1.65$
$\beta = 1.7$
$\beta = 1.8$
$E(L)/E(2L)$
$1/LE(L)$



r
$am_{\mathrm{eff}} \equiv -\frac{d}{dx}\log C(r)$
am
x_0



$a\Delta(L)$
L/a



$a\Delta$
$1/D$