Importance of observing critical fluctuations in heavy-ion collisions for the QCD Equation of State

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September 4, 2025

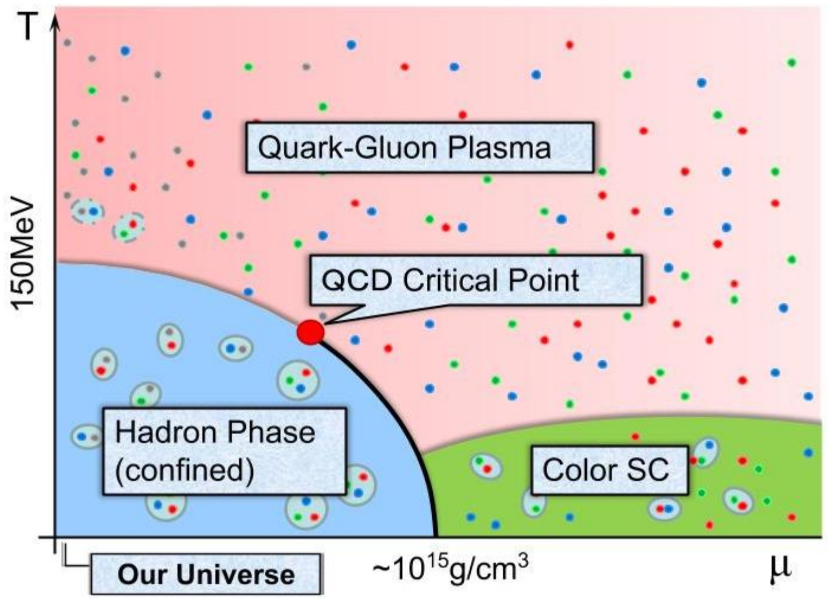
INT Program INT-25-2b “From Colliders to the Cosmos”

Extreme QCD in heavy-ion collisions and the cosmos

Understanding the dynamics of the strong interaction under extreme conditions of temperature and density!

Important questions:

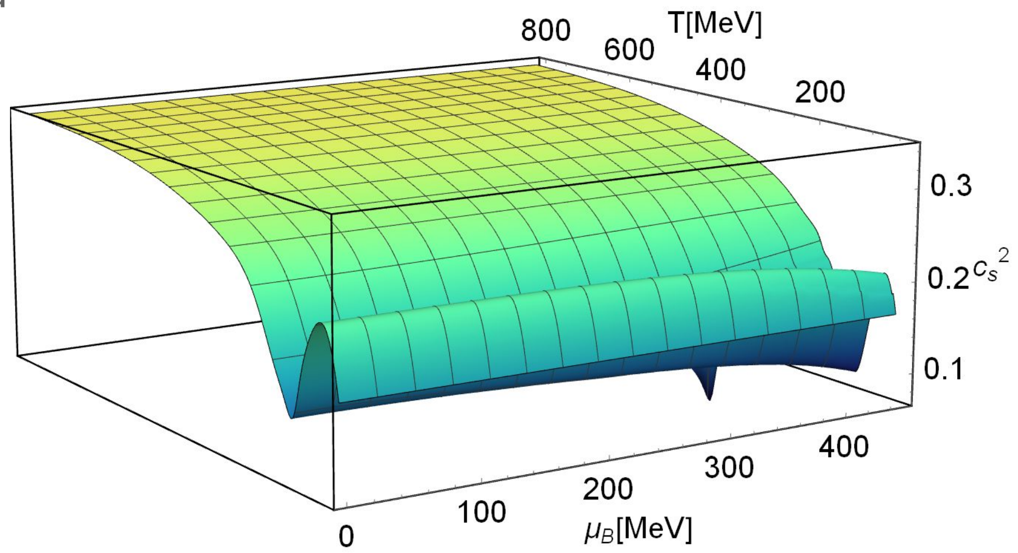
- Onset of deconfinement and chiral symmetry restoration?
- Properties of the strongly coupled QGP?
- Existence of a phase transition with critical point?
- What are the dof in the core of compact stars?



Connect first-principle QCD calculations with experimental observables via a realistic dynamical modeling of heavy-ion collisions **and astrophysical events!**

BEST Collaboration EoS with a critical point

- Based on lattice QCD results for the EoS in a Taylor-series expansion up to 4th-order which includes a CP EoS within the 3d Ising model static universality class. Hohenberg, Halperin, Rev. Mod. Phys. 49 (1977)
- Concerted effort of BEST Topical Collaboration to provide a flexible EoS including a CP to practitioners of hydrodynamic simulations related to Beam Energy Scan at RHIC.
- Most sensitive to CP are derivatives of the EoS (speed of sound, susceptibilities).



P. Parotto, MB et al., PRC 101 (2020)

How to observe the critical point in HIC

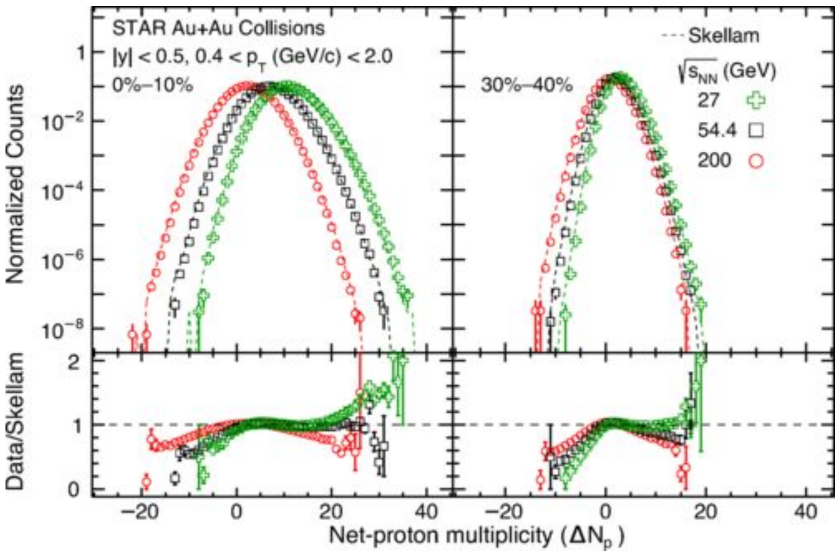
- At a Critical Point, the correlation length ξ diverges and so do the fluctuations.

- Observable in higher-order cumulants (C_n) of net-baryon number (related to susceptibilities X_n).

- To 0th order in volume fluctuations:

$\frac{\chi_2}{\chi_1} = \frac{\sigma^2}{M}$	$\frac{\chi_3}{\chi_2} = S\sigma$	$\frac{\chi_4}{\chi_2} = \kappa\sigma^2$
variance	Skewness	Kurtosis

- At a CP the τ_{relax} diverges with ξ, which leads to critical slowing down



STAR, PRL 127 (2021)

Interesting deviations from the baseline in the experimental data... are they due to the critical point of QCD?

Need a dynamical model!

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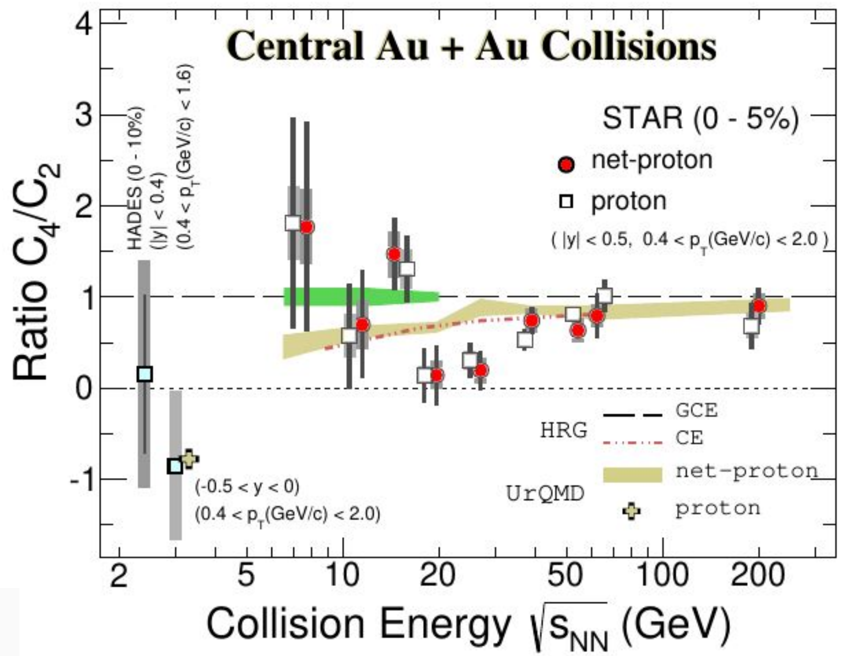

STAR, PRL 128 (2022)

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Need a dynamical model!

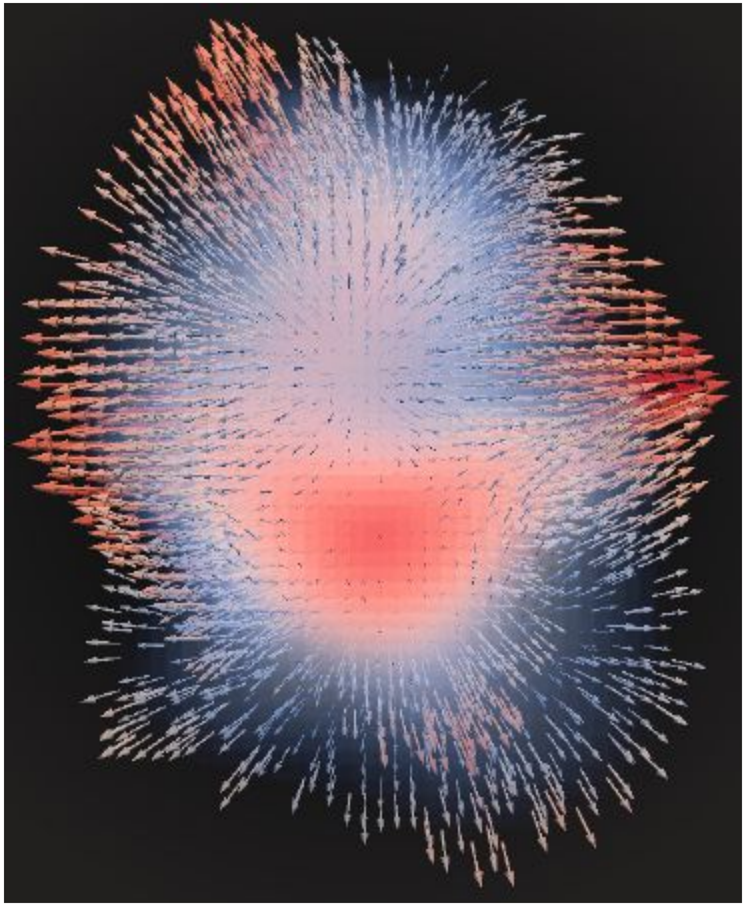
Importance of dynamical modeling

In a grand-canonical ensemble the system is...

- in thermal equilibrium (= long-lived)
- in equilibrium with a particle heat bath
- spatially infinite
- and static

Systems created in a heavy-ion collision are

- short-lived
- spatially small
- inhomogeneous
- and highly dynamical!

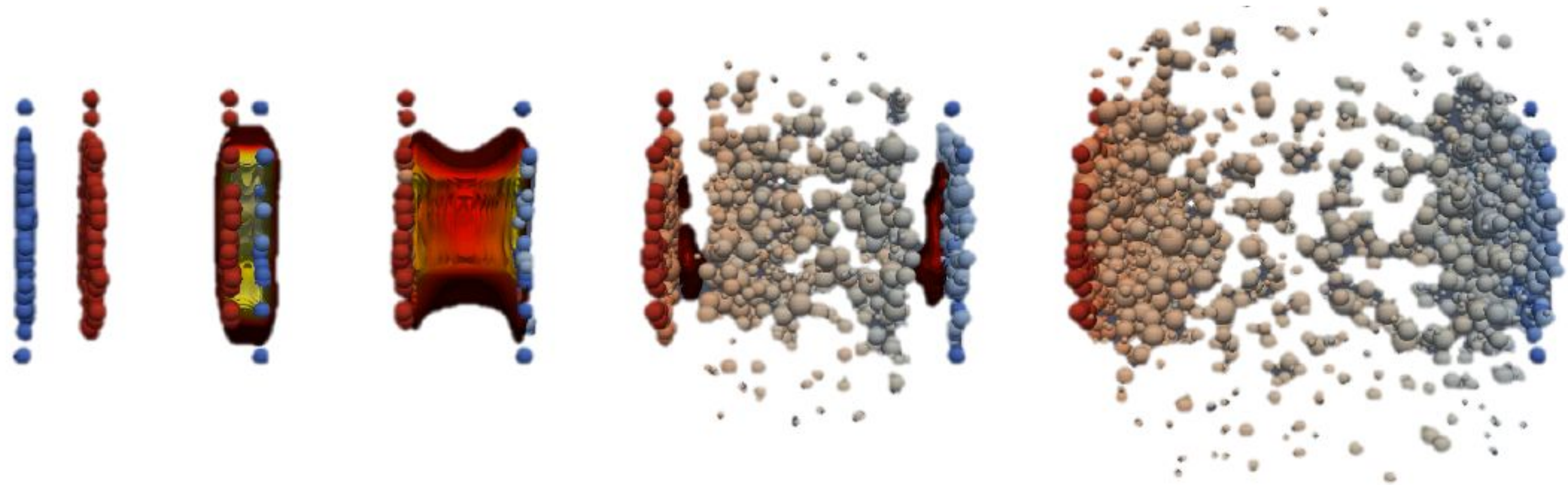


madai.us

Solution: Develop dynamical models to describe the phase transition in heavy-ion collisions
Event-by-event dynamical modeling allows us in addition to study different particle species, experimental cuts, hadronic final interactions, etc.

EMMI Rapid Reaction Task Force: MB, A. Kalweit, M. Nahrgang et al. NPA 1003 (2020), 2001.08831

Fluctuations all along the way



- Initial state fluctuations due to quantum mechanical fluctuations and multiplicity fluctuations
- Thermal fluctuations, including the formation and dynamics of critical fluctuations
- Fluctuations due to the hadronization process
- Fate of fluctuations in the hadronic phase
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Net-baryon diffusion: model B in 1+1 dimensions

- In the long-time, equilibrium limit the net-baryon density is the slowest mode near the CP.
- For baryonic matter that decouples from the energy flow of the system (model B of Hohenberg, Halperin), the diffusive dynamics follows the minimization of the free energy F

$$\partial_t n_B(t,x) = \kappa \nabla^2 \left(\frac{\delta \mathcal{F}[n_B]}{\delta n_B} \right) + \nabla \mathbf{J}(\mathbf{t}, \mathbf{x})$$

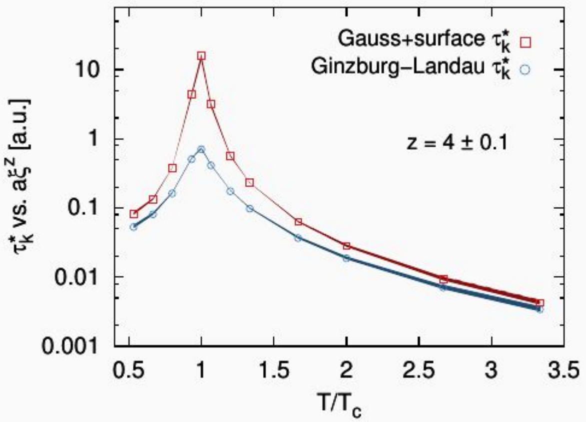
with the stochastic current (Gaussian, white noise)

$$J(t,x) = \sqrt{2T\kappa}\,\zeta(t,x), \quad \kappa = \frac{Dn_c}{T}$$

- The fluctuation-dissipation balance is respected.

…studied in a static box…

- Dynamical structure factor for Gaussian model in continuum:
  $S(k,t) = S(k)\exp(-t/\tau_k)$ with $\tau_k^{-1} = \frac{Dm^2}{n_c}\left(1 + \frac{K}{m^2}k^2\right)k^2$
- Analyze ξ-dependence of relaxation time for modes with $k^* = 1/\xi$:



for both models: $\tau_k^* = a\,\xi^z$ with

$$z = 4 \pm 0.1$$

$$a = \frac{n_c \xi_0}{D(1+\tilde{K})}$$

$\Rightarrow$ Simulations reproduce scaling of model B!

M. Nahrgang and MB, PRD 99 (2019) and PRD 102 (2020)

For the full dynamics of a HIC, couple to fluctuations in $T^{\mu\nu}$ → model H

T. Schaefer et al., PRL 133 (2024)

Hohenberg, Halperin, Rev. Mod. Phys. 49 (1977)

...and in expanding (boost-invariant) systems

Choose an appropriate coordinate system for the geometry of a HIC

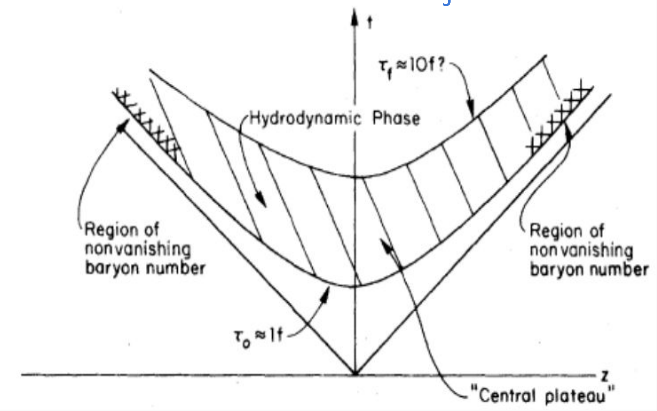
$$\tau = \sqrt{t^2 - z^2}$$

$$y = \frac{1}{2} \ln \left(\frac{t+z}{t-z} \right)$$

$$\frac{n(y,\tau)}{\tau} = n(x,t)$$

Fluctuations in expanding background, e.g. J. Kapusta et al, PRC 85 (2012); Y. Akamatsu et al. PRC 95 (2017), M. Martinez et al, PRC 99 (2019)

J. Bjorken PRD 27 (1983)



- The nonlinear stochastic diffusion equation transforms as:

$$\partial_\tau n_B = \frac{D n_c}{\tau \chi_2(\tau)} \partial_y^2 n_B - \frac{D n_c K(\tau)}{\tau} \partial_y^4 n_B + \frac{D n_c}{6\,\tau \chi_4(\tau)} \partial_y^2 {n_B}^3 - \partial_y \xi \,.$$

D: diffusion coefficient of the baryon current

In Gauss limit: M. Sakaida et al PRC 95 (2017);
Nonlinear (only critical): M. Kitazawa, G.Pihan, N. Touroux, MB, M. Nahrgang NPA 1005 (2021)
G. Pihan, MB, M. Kitazawa, T. Sami, M. Nahrgang, PRC 107 (2023)

Singular and regular susceptibilities

- Parametrize the susceptibilities $\chi_2(\tau)$ and $\chi_4(\tau)$ with a regular part using the argument in

M. Asakawa, U. Heinz, B. Müller, PRL 85 (2000)

$$\chi_n(\tau) = \frac{\langle \Delta N_B^n \rangle}{S}\Big|_{\mathrm{QGP/HRG}} = \frac{\chi_\mathrm{B}^n}{s/T^3}\Big|_{\mathrm{QGP/HRG}}$$

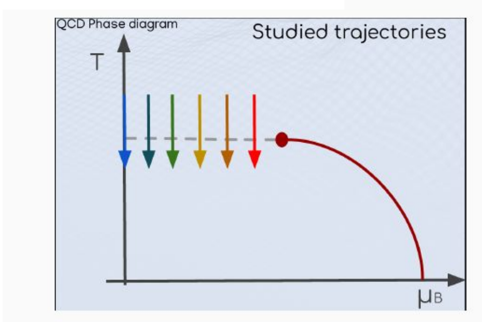
With $\chi_B{}^n$ and the entropy fixed to lattice results at T=280 MeV for the QGP and T=130 MeV for the HRG, matched via tanh function.

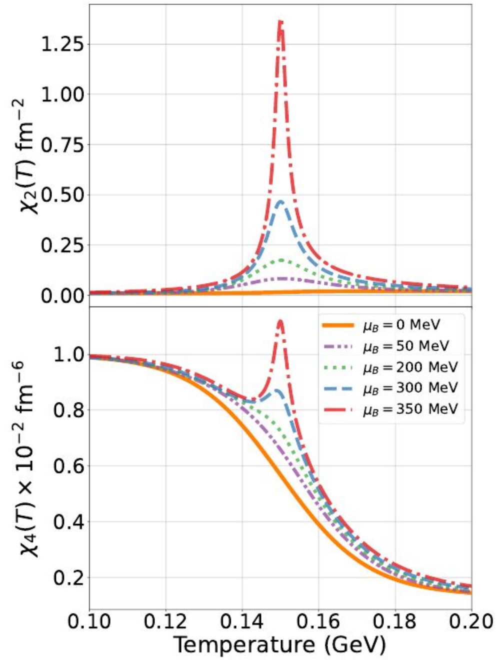
- Couple with the singular contribution (3D Ising) via

$$\chi_n(T) = \chi_n^{sing}(T) + \chi_n^{reg}(T)$$

- Match to the coefficients in the expansion of the free energy density functional.

$$\chi_n(\tau) = \tau \left(\frac{\delta^n \mathcal{F}}{\delta n_B^n}\Big|_{\Delta n_B = 0} \right)^{-1}$$





- Investigate several trajectories in the QCD phase diagram.

G. Pihan, MB, M. Kitazawa, T. Sami, M. Nahrgang, PRC 107 (2023)

Validating the linear (Gauss) model

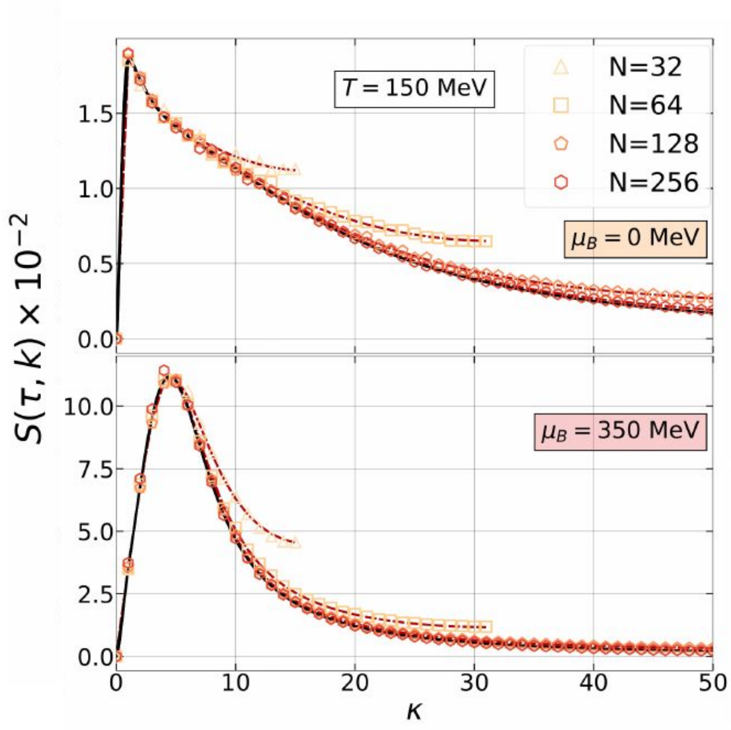
only here
$X_4 = 0$!

Important step for all fluctuation codes:

validation of the appropriate linear model:

- Structure factor and equal-time correlation function are well reproduced
- Approach to continuum as resolution is increased.
- Lower wavenumbers well described with the maximal resolution chosen for this work.
- Enhancement of fluctuations with low wavenumbers at T_c = 150 MeV.
- Discretization and baryon conservation effects under control.

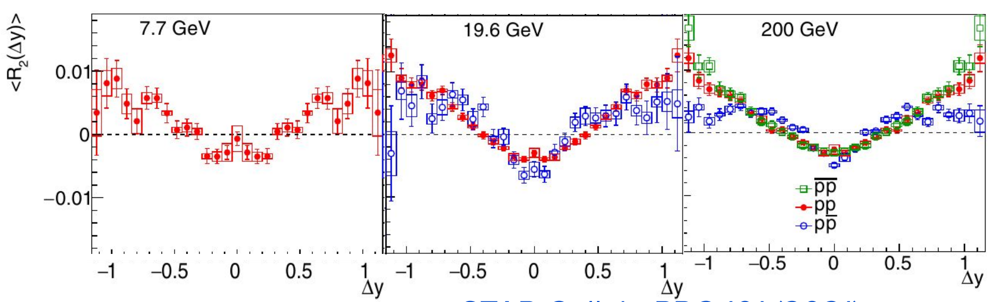
Structure factor Sk



G. Pihan, MB, M. Kitazawa, T. Sami, M. Nahrgang, PRC 107 (2023)

Anticorrelations as a signal for the critical point

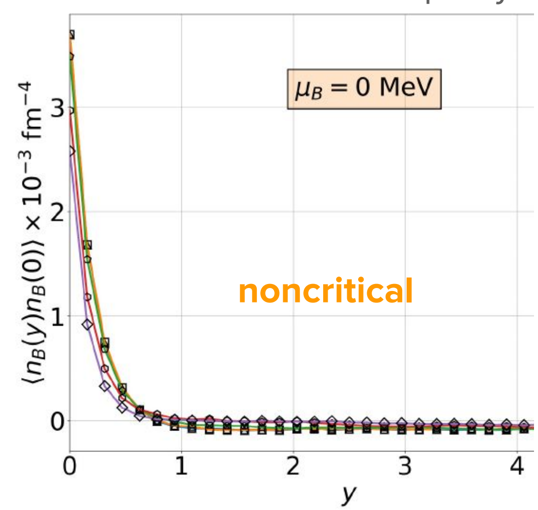
- Large fluctuations are balanced by large anti-correlations (net-baryon conservation)
- Due to the dynamics these anti-correlations cannot diffuse fast enough
- Approaching Tc they are visible at y ∼ 1-2
- At lower T the minimum becomes smaller and moves to larger y
- → Possible detection depends crucially on T_{FO}
- Interesting experimental data:
  (STAR AuAu, 30-40% most central, 0.4<pT<2GeV)

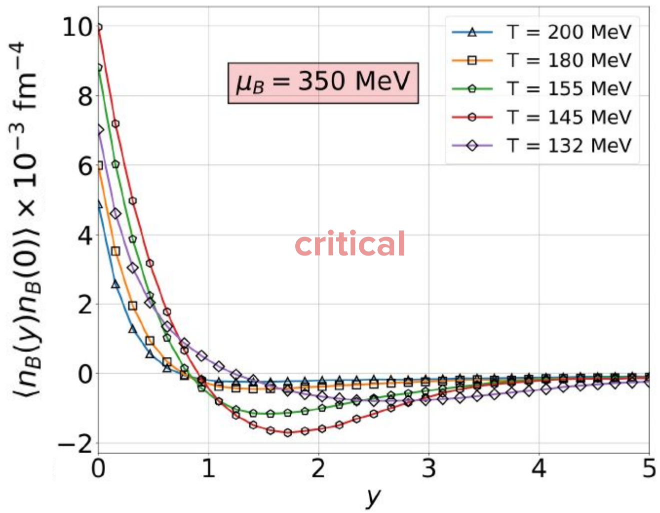


STAR Collab, PRC 101 (2021)

Word of caution: not yet an apple-to-to apple comparison possible!

correlation function in rapidity

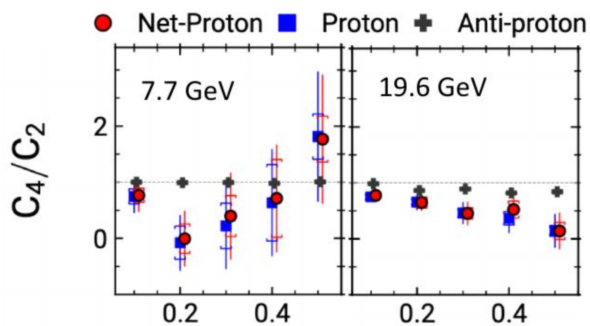




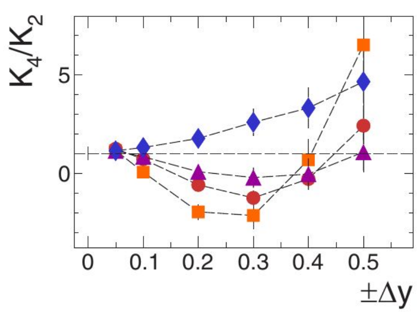
G. Pihan, MB, M. Kitazawa, T. Sami, M. Nahrgang, PRC 107 (2023)

Non-monotonic kurtosis as a signal for the critical point

- Monotonic increase in the variance.
- Non-monotonic Kurtosis only for the trajectories with critical point.
- This non-monotonic behavior of the kurtosis survives the rapid expansion for a diffusion length D = 1 fm.
- For increasing D the minimum moves to larger distances in rapidity.
- → Essential for the experiment to cover a wide range in rapidity to see the non-monotonicity.
- Interesting experimental data:
  (STAR AuAu, 0-5% most central, 0.4<pT<2GeV, |y|<ymax)



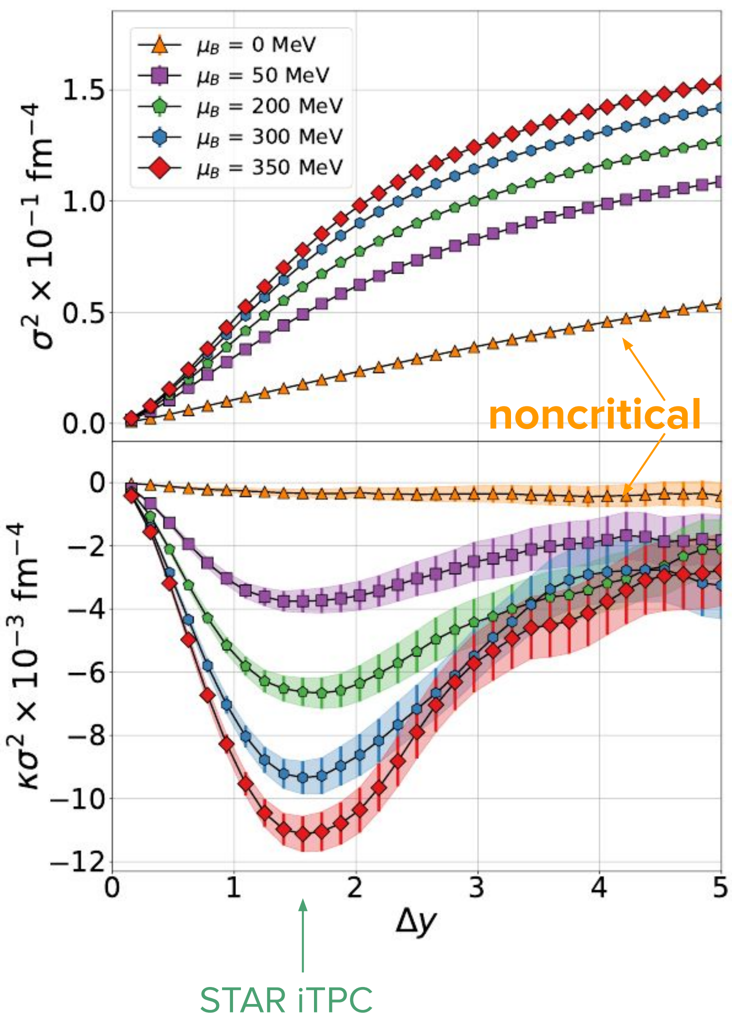
STAR Collab, PRC 104 (2021)



HADES Collab, PRC 102 (2020)

Word of caution: not yet an apple-to-to apple comparison possible!

rapidity dependence of fluctuation observables



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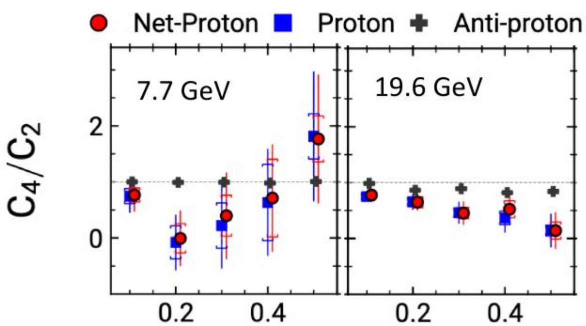
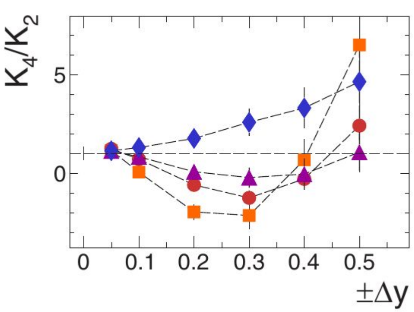


STAR Collab, PRC 104 (2021)



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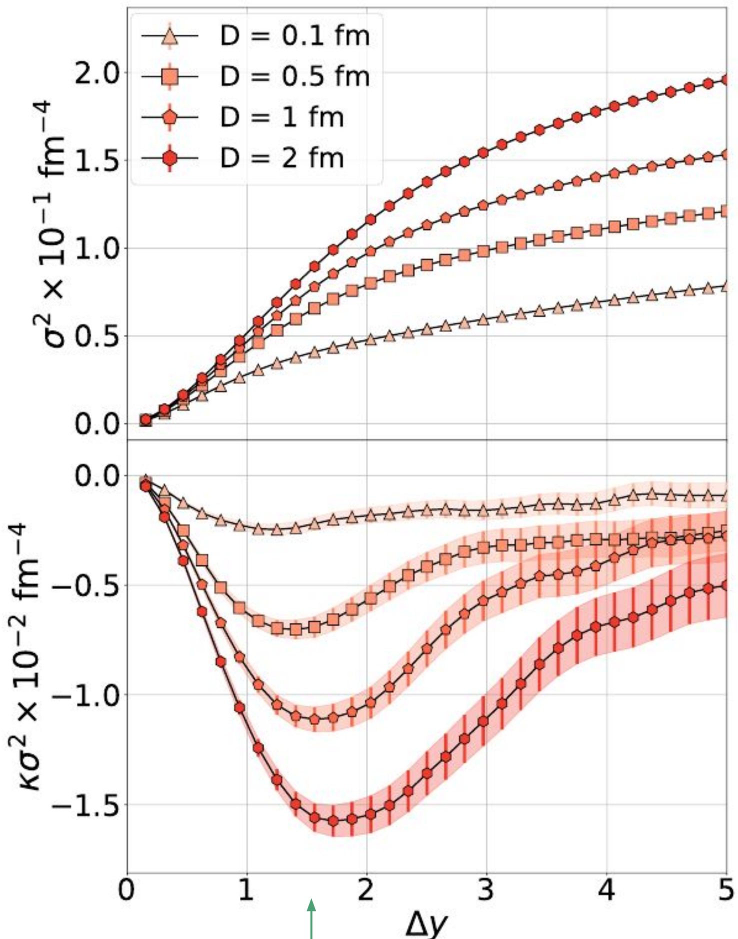
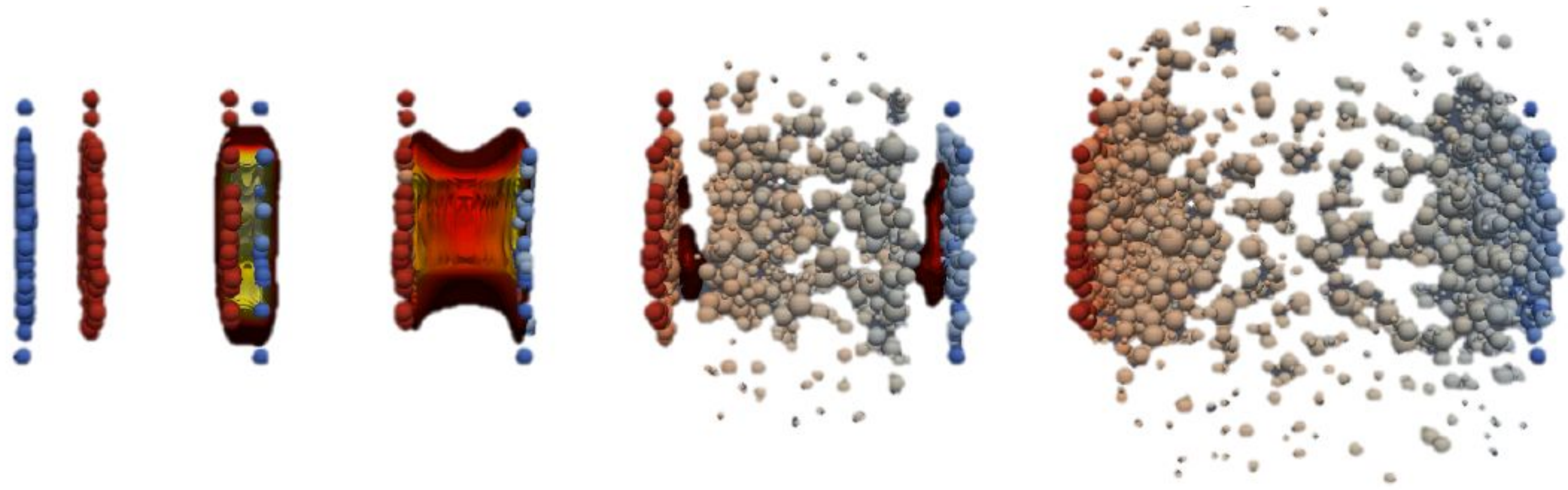


G. Pihan, MB, M. Kitazawa, T. Sami, M. Nahrgang, PRC 107 (2023)

STAR iTPC

Fluctuations all along the way



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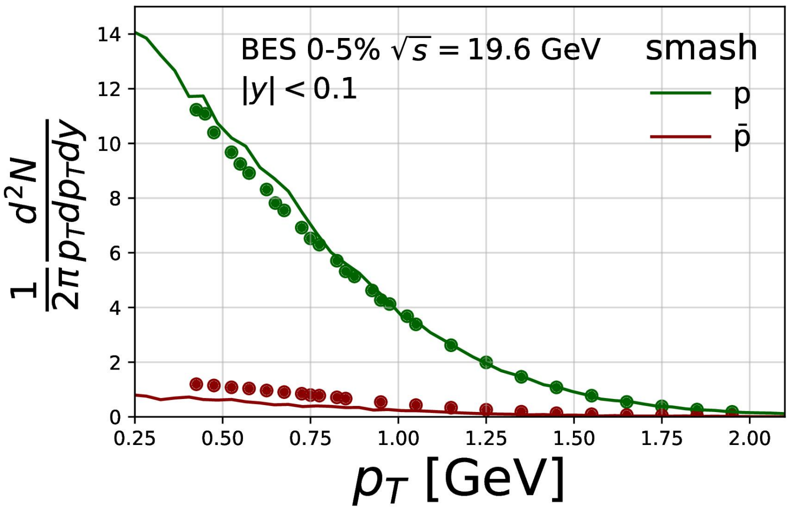
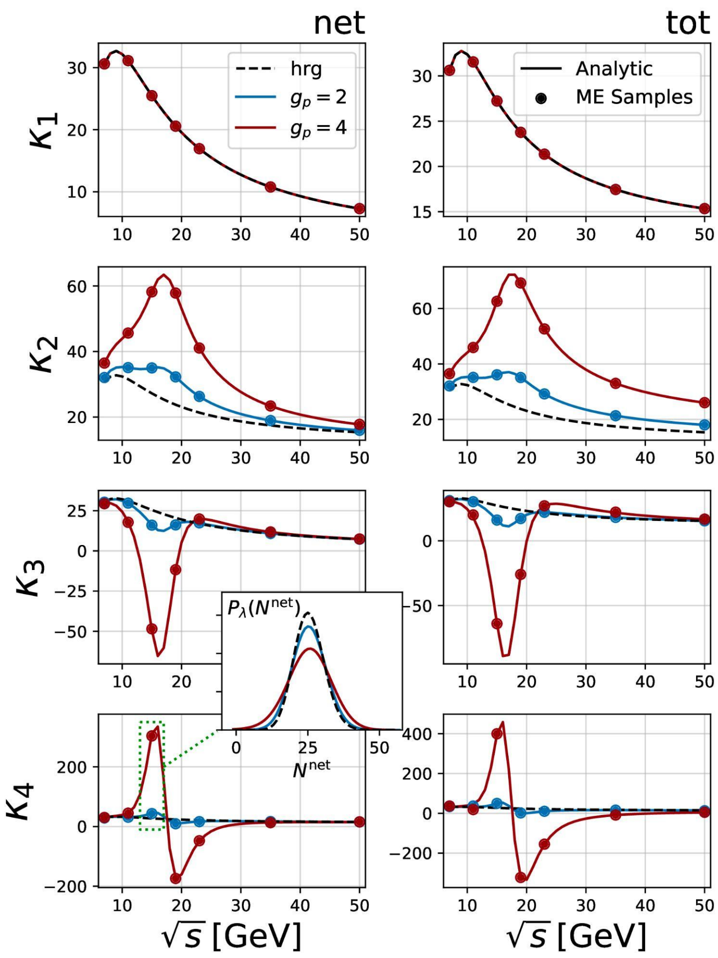
- Thermal fluctuations, including the formation and dynamics of critical fluctuations
- Fluctuations due to the hadronization process
- Fate of fluctuations in the hadronic phase
- Imperfect detection efficiency and finite acceptance

Fate of critical fluctuations in the hadronic phase

- Calculate up to 4th order cumulants of critical fluctuations from an 3d Ising model mapping to QCD and couple it to HRG cumulants (g=2,4).
- Reconstruct the particle distributions from the cumulants + maximum entropy constraint
- Assume simple geometry at particlization: uniform spatial distribution in a sphere (R=9fm) Momentum distribution $f_{i,k} = e^{-u \cdot k_i / T}$ velocity $\vec{u}(r) = \vec{e}_r u_0 r / R$ with u_0= 0.5



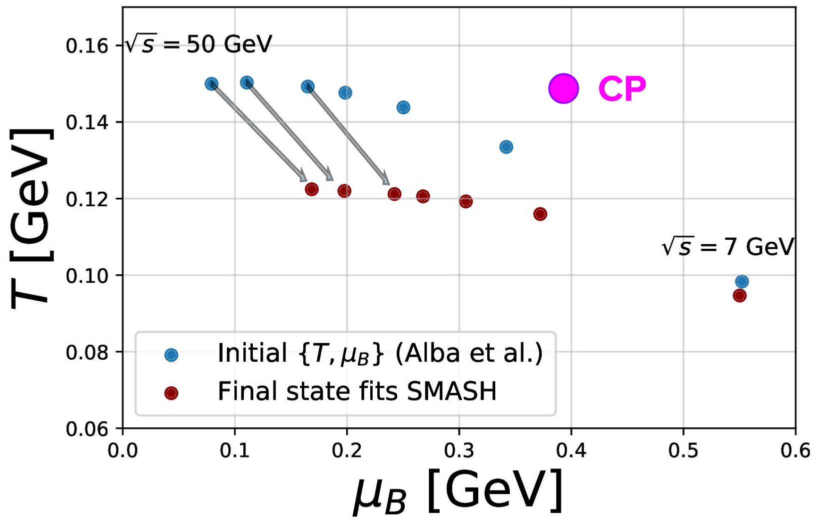


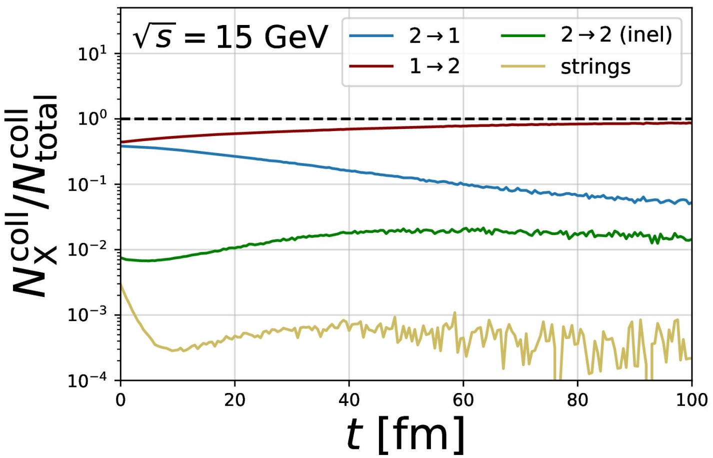
J. Hammelmann, MB, M. Nahrgang, H. Elfner, PRC 110 (2024)

Fate of critical fluctuations in the hadronic phase

- Apply smash (https://smash-transport.github.io) to the final hadronic interactions of the initialized particles.
- Resonance decay and regeneration are the dominant processes during the hadronic expansion.

Particle	Mass [GeV/c^2]	Degenercy
π	0.138	3
ρ	0.776	6
K	0.494	4
$K^\star(892)$	0.892	8
N	0.938	8
Δ	1.232	32
Λ	1.116	2
Σ	1.189	12





J. Hammelmann, MB, M. Nahrgang, H. Elfner, PRC 110 (2024)

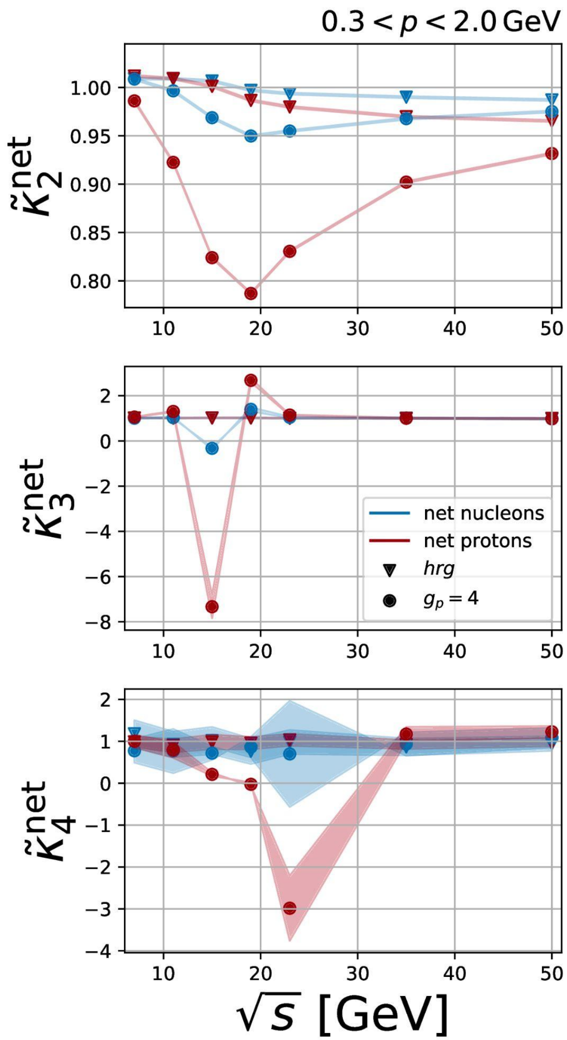
Impact of resonance dynamics vs decay

- Compare the dynamical effect of the resonance decay and regeneration to only resonance decay:

$$\tilde{\kappa}_n = \frac{\kappa_n^{\text{dynamical}}}{\kappa_n^{\text{decays}}}$$

- Net-proton cumulants are strongly impacted by the hadron dynamics compared to the net-nucleons -> importance of isospin randomization processes.

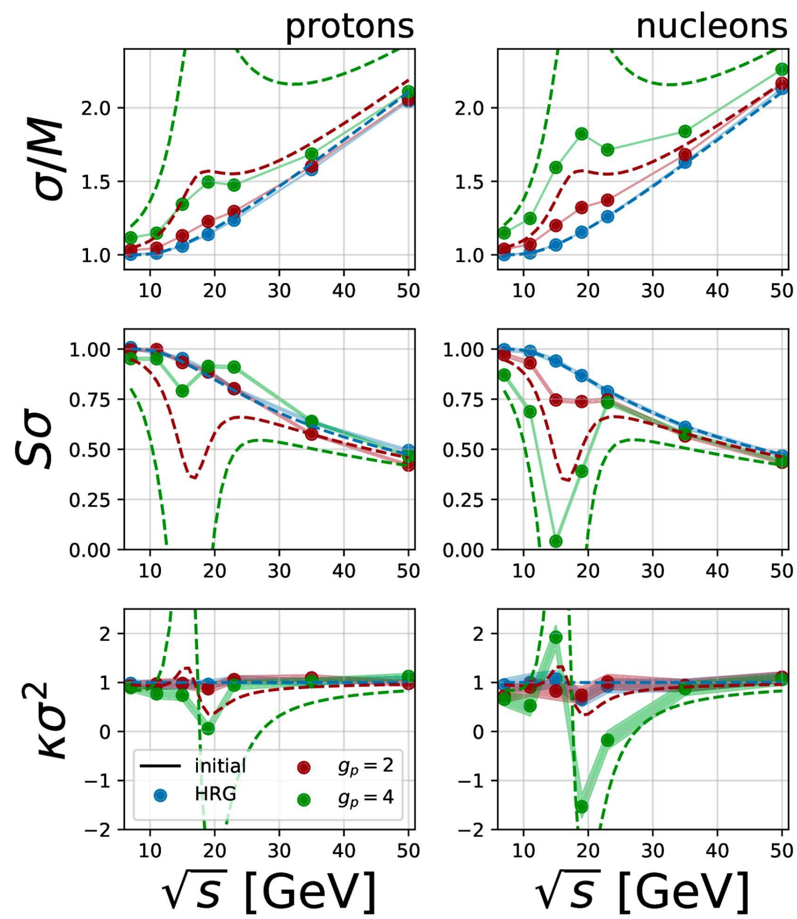
M. Kitazawa, M. Asakawa, Phys.Rev.C 85 (2012); MB, M. Nahrgang, S. Bass, T. Schaefer, Eur.Phys.J.C 77 (2017); M. Nahrgang, MB, P. Alba, R. Bellwied, C. Ratti Eur.Phys.J.C 75 (2015)



J. Hammelmann, MB, M. Nahrgang, H. Elfner, PRC 110 (2024)

Final proton and nucleon cumulant ratios

- For g = 2, no critical signal is seen in the net-proton variance and skewness, a very small signal in the kurtosis survives.
- For g = 4, the net-proton variance shows critical features -> not compatible with experiment.
- The nucleon critical signal is significantly more pronounced than for protons only.
- Signal depends strongly on the rapidity acceptance and can even change sign in the kurtosis.



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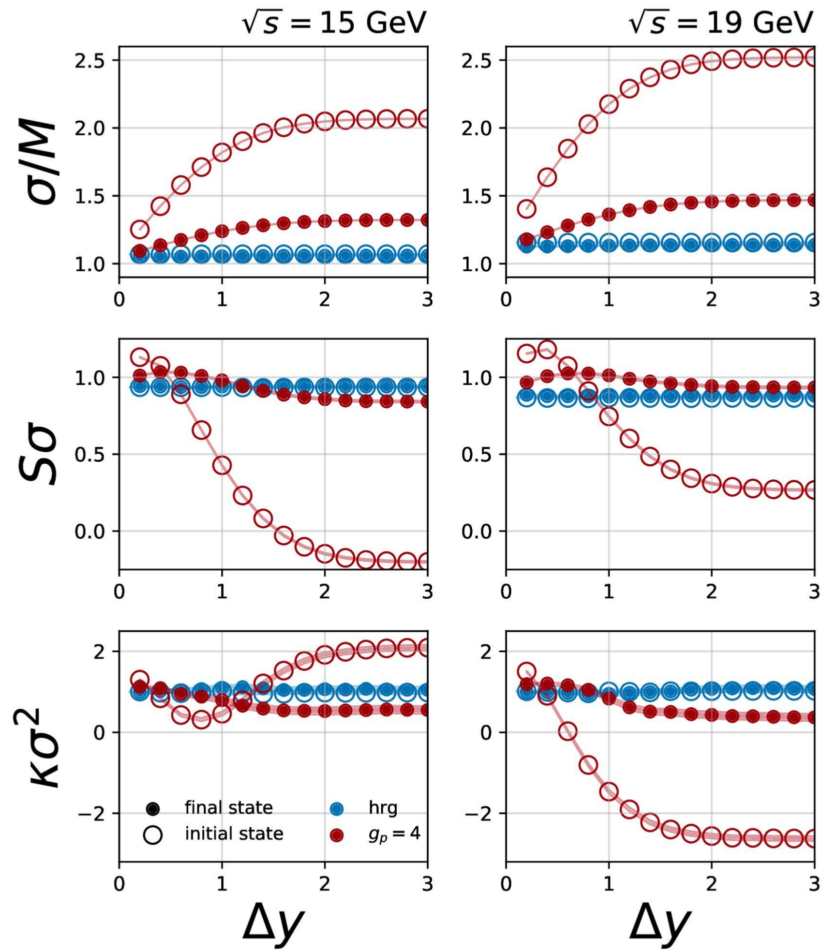
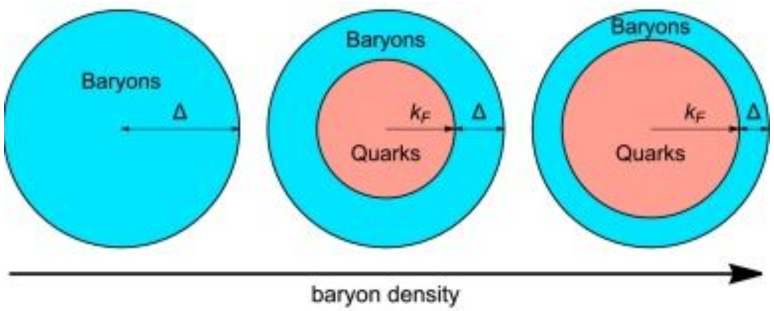


J. Hammelmann, MB, M. Nahrgang, H. Elfner, PRC 110 (2024)

Quarkyonic Matter in the QCD phase diagram

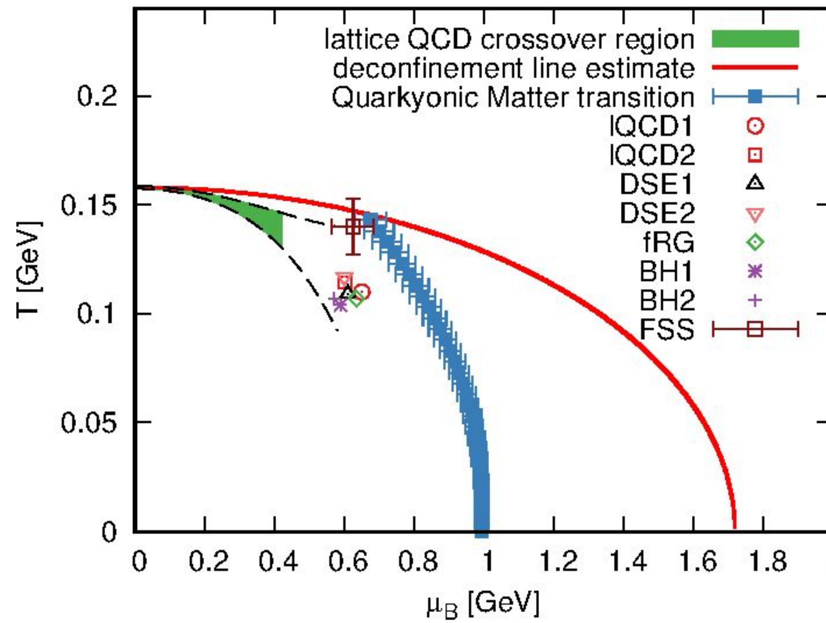
- With increasing density, $\mu_B \sim M_N$ to $\mu_B \sim M_N \sqrt{N_c}$, baryonic matter remains confined by becomes saturated. L. McLerran, R.D. Pisarski, NPA 796 (2007)
- Developpement of quark Fermi sea inside baryonic Fermi sea.



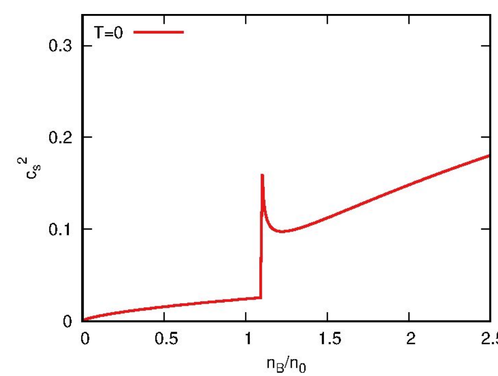
from V. Koch, V. Vovchenko, PLB 841 (2023)

- Baryons at Fermi surface are relativistic → stiffening of EoS as potential explanation for recent neutron star observations. L. McLerran, S. Reddy, PRL 122 (2019)

See talk by M. Nahrgang, Friday Sep. 12, 11h30



MB, Y. Fujimoto, L. McLerran, M. Nahrgang, PRC 111 (2025)



Conclusions

Confirming the existence of a QCD Critical Point experimentally has a crucial impact on the EoS at large densities.
Treating the dynamics of fluctuations near the Critical Point is important for quantitative statements about its existence based on heavy-ion collision data!

- Net-baryon fluctuations are strongly impacted by the expansion dynamics.
- Anticorrelations of baryons can signal the CP.
- Non-monotonic dependence of the kurtosis on the rapidity window can signal the CP.
- Results depend on diffusion coefficient and FO point.
- Resonance decay and regeneration strongly affect critical fluctuations.

It could be that a possible first-order phase transition is preluded by a transition to Quarkyonic Matter as the density increases. This would have consequences for fixed target or collider experiments at small beam energies as well as astrophysical observations.

APPENDIX

Approaches to fluid dynamical fluctuations

There are two main approaches of describing fluid dynamics with noise:

Hydro-kinetics

- Set of deterministic kinetic equations for n-point functions of fluid dynamical fields
- Renormalization (perturbatively) performed during the derivation
- Statistical average performed in the derivation of deterministic equations

A.Andreev, Sov. Phys. JETP 32 no. 5 (1971) and 48 no. 3 (1978); Y. Akamatsu et al., PRC 95 no. 1 (2017) and 97 no. 2 (2018); M. Stephanov et al., PRD 98 (2018); M. Martinez et al., PRC 99 no. 5 (2019); X. An et al., PRC 100 no. 2 (2019), PRL 127 (2021); L. Du et al., PRC 102 (2020); K. Rajagopal, NPA 1005 (2021)

Stochastic/fluctuating fluid dynamics

- Numerical implementation of the fluid dynamical equations with stochastic conservation law:

$$\partial_\mu T^{\mu\nu} = 0, \quad T^{\mu\nu} = T^{\mu\nu}_{\text{ideal}} + T^{\mu\nu}_{\text{viscous}} + S^{\mu\nu}_{\text{noise}},$$
$$\partial_\mu J^{\mu} = 0, \quad J^{\mu} = J^{\mu}_{\text{ideal}} + J^{\mu}_{\text{viscous}} + I^{\mu}_{\text{noise}}.$$

- Sample discretized noise event-by-event
- Observables are calculated from statistical averaging over events.
- Can easily be integrated in standard event generators of HIC!
- Many challenges....

Fluctuating Dissipative Fluid Dynamics

The correlators of the thermal noise terms in the energy momentum tensor and the conserved currents :

$$\langle S^{\mu\nu}(x_1) S^{\alpha\beta}(x_2)\rangle = 2T \begin{bmatrix} \eta\left(\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\nu\alpha}\right) \\ + \left(\zeta - \frac{2}{3}\eta\right)\Delta^{\mu\nu}\Delta^{\alpha\beta} \end{bmatrix} \delta^{(4)}(x_1 - x_2).$$

$$\langle I^{\mu}(x_1) I^{\nu}(x_2)\rangle = 2T\sigma\Delta^{\mu\nu}\delta^{(4)}(x_1 - x_2)\,.$$

Several issues arise from the discretization of the Dirac delta function in the noise

- Stochastic noise introduces a lattice spacing dependence.
- Correction terms due to renormalization become large for small lattice spacings.
- Large noise contributions can locally lead to negative energy densities.
- Large gradients introduced by the uncorrelated noise is a problem for PDE solvers.

EMMI Rapid Reaction Task Force: MB, A. Kalweit, M. Nahrgang et al. NPA 1003 (2020), 2001.08831

Fluctuating Dissipative Fluid Dynamics

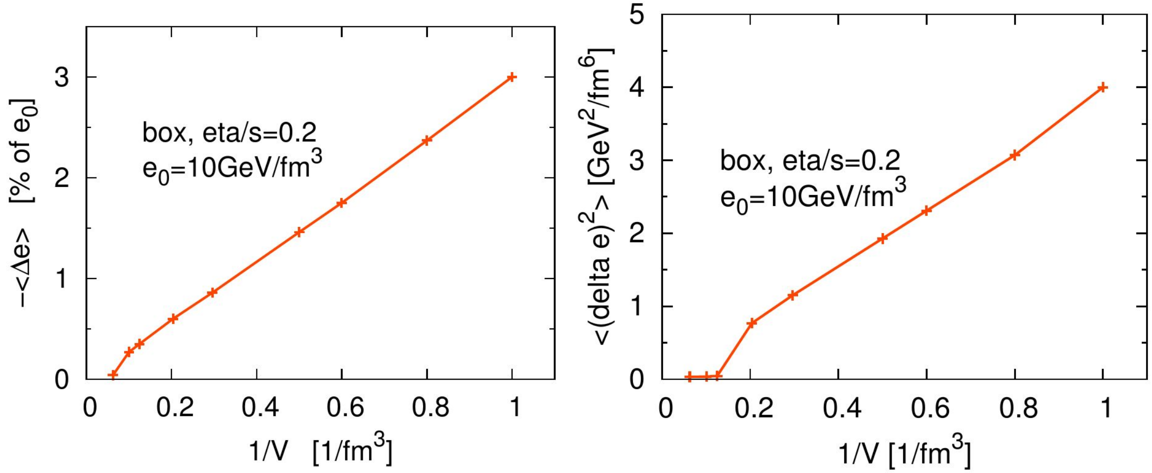
First implementations of FDFD have shown: need to limit the resolution scale; simulate noise down to a particular filter length scale, for which:

$$l_{\text{grid}} < l_{\text{filter}} \lesssim l_{\text{noise}} \ll l_{\text{hydro}}$$

Murase et al.: noise is smeared by Gaussians with widths of 1-1.5 fm (choice not discussed), large enhancement of flow observed. K. Murase et al, NPA 956 (2016);

Nahrgang et al.: noise is coarse-grained over distances of approx. 1fm, lattice spacing dependence of the energy density and its fluctuations observed.

M. Nahrgang et al, Acta Phys. Polon. 10 (2017);



Singh et al.: high-mode Fourier filter with a coarse graining scale of > 1fm, multiplicities and flow are little affected by the inclusion of fluctuations

M. Singh se et al, NPA 982 (2019);

Importance of fluctuations for transport coefficients

$$\eta \sim \int d^3x dt \langle T^{ij}(x,t) T^{ij}(0,0) \rangle$$

Included in fluid dynamics

NOT included in fluid dynamics

- symmetrized correlator:

$$G_S^{xyxy}(\omega, \mathbf{0}) = \int \mathrm{d}^3x\mathrm{d}t\, e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})} \left\langle \frac{1}{2}\{ T^{xy}(t, \mathbf{x}), T^{xy}(0, \mathbf{0})\} \right\rangle$$

- for the shear-shear contribution $\Rightarrow$

$$G_{R,\mathrm{shear-shear}}^{xyxy}(\omega, \mathbf{0}) = -\frac{7T}{90\pi^2}\Lambda^3 - i\omega \frac{7T}{60\pi^2}\frac{\Lambda}{\gamma_\eta} + (i+1)\omega^{3/2}\frac{7T}{90\pi^2}\frac{1}{\gamma_\eta^{3/2}}$$

cutoff-dependent fluctuation contribution to the pressure

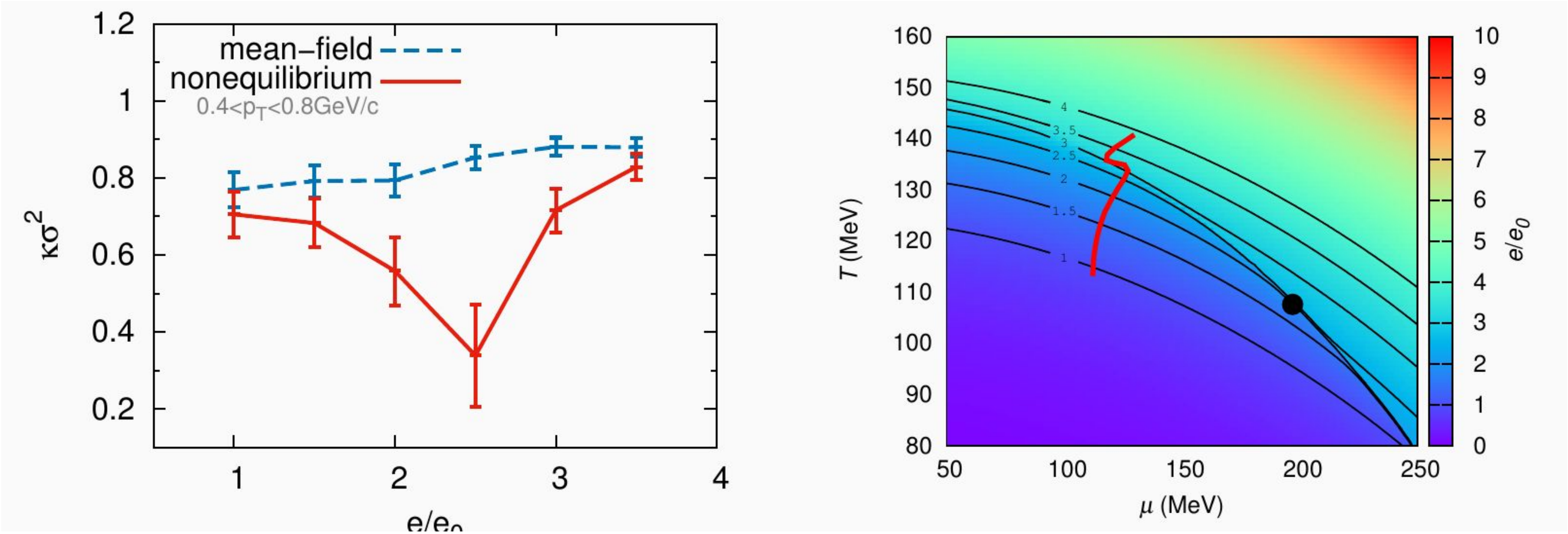
cutoff-dependent correction to η

frequency-dependent contribution to η and τ_π

P. Kovtun et al, PRD84 (2011); C.Chafin, T. Schäfer, PRA87 (2013); P. Romatschke et al, PRA87 (2013)

Net-proton fluctuations near the critical point

- UrQMD initial conditions rescaled to the EoS of the effective model.
- From densities to particles via Cooper-Frye particlization.
- At particlization: densities of the sigma field coupled to the FD densities.



C. Herold, M. Nahrgang, Y. Yan and C. Kobdaj, PRC 93 (2016) no.2

- No non-monotonic behavior in pure mean-field equilibrium calculations.
- Clear signal for criticality in net-proton fluctuations at transition energy density!
- Overall decreasing trend probably due to net-baryon number conservation

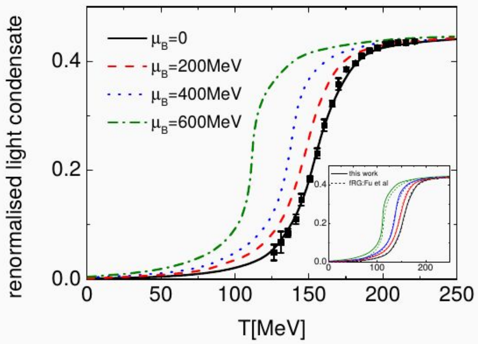
M. Nahrgang et al, EPJC72 (2012), 0903.2911

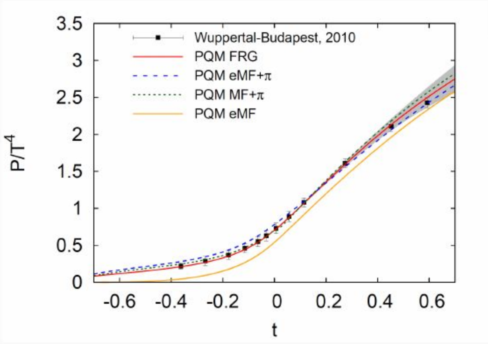
NchiFD + FRG ⟶ QCD assisted transport

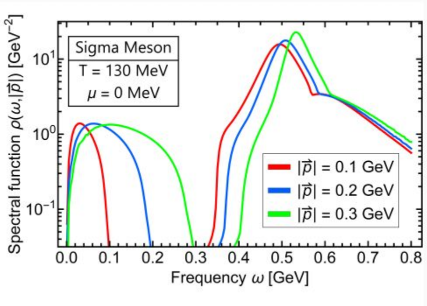
- Include effective potential beyond mean field, momentum dependent equilibrium sigma spectral function $\Rightarrow$ linear response regime of QCD.

First-principle approach to QCD from the Functional Renormalization Group (FRG)

Cyrol, Mitter, Pawlowski, Strodthoff PRD97 (2018)

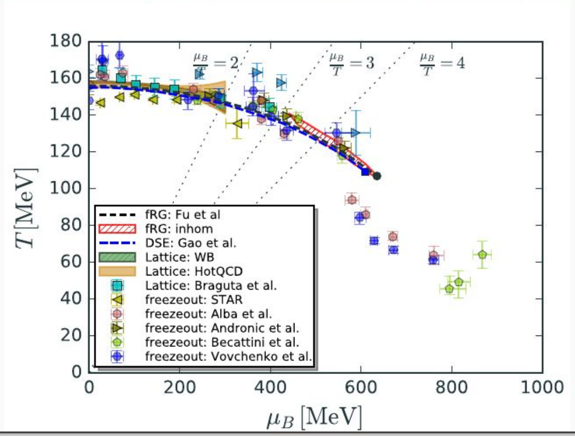






F. Gao, J. Pawlowski, 2010.13705; T. Herbst et al, PLB731 (2014); T. Herbst PRD88 (2013); F. Rennecke, J. Pawlowski, N. Wink

- Excellent description of phase structure at vanishing chemical potential.
- Phase structure qualitatively similar to the conjectured QCD phase diagram.
- Obtain spectral functions from analytical continuation.



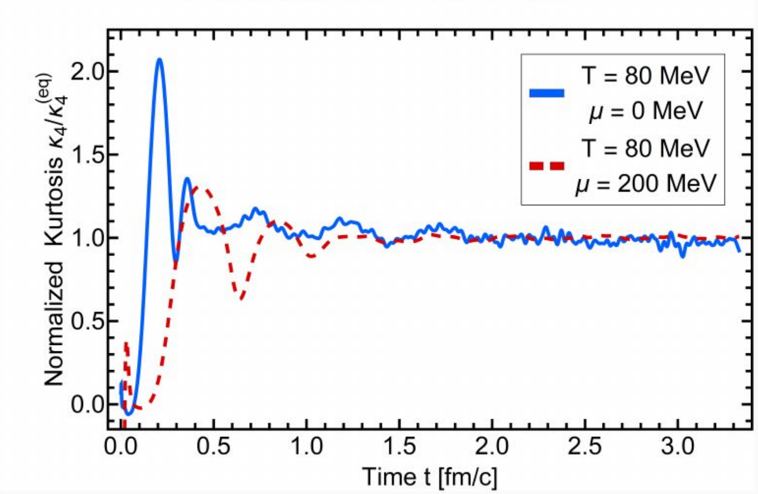
9 / 2

NchiFD + FRG ⟹ QCD assisted transport

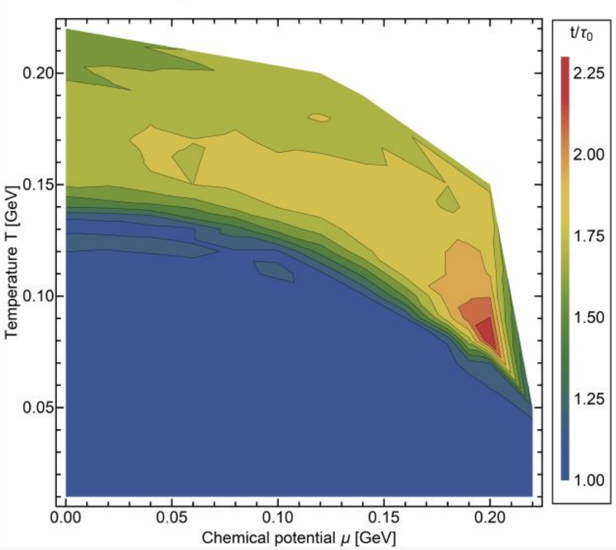
M. Bluhm et al., NPA982 (2019)

Transport equation: $\frac{\delta\Gamma}{\delta\sigma} = \xi$, where $\{\Re\Gamma^{(2)}_\sigma(\omega,\vec{p}), \Im\Gamma^{(2)}_\sigma(\omega,\vec{p}), U\} \in \Gamma$

Normalized Kurtosis:



Equilibration time:



- Critical end point and the phase structure are clearly identifiable.
- Critical slowing down in the vicinity of the critical point, but no dramatic enhancement of τ_{relax} in a dynamic setup!

Diffusive dynamics of the net-baryon density

$\partial_\mu N_B^\mu = 0$ net-baryon charge conservation

neglect coupling to energy and momentum density, consider space-time independent fluid with spatially homogeneous T at fixed μ_B:

$\rightarrow$ diffusive dynamics follows the minimized free energy $\mathcal{F}$

$$\partial_t n_B(t,x) = \kappa \nabla^2 \left(\frac{\delta \mathcal{F}[n_B]}{\delta n_B} \right) + \nabla \mathbf{J}(\mathbf{t},\mathbf{x})$$

stochastic current included to study intrinsic fluctuations

$$J(t,x) = \sqrt{2T\kappa}\,\zeta(t,x)$$

$\rightarrow$ $\zeta(t,x)$ is Gaussian and uncorrelated (white noise)

$$\langle \zeta(x,t)\zeta(0,0) \rangle = \delta(x)\delta(t)$$

$\Rightarrow$ respects the fluctuation-dissipation theorem!

$\rightarrow$ correlated noise possible!

Critical energy density from 3d Ising Model

Ginzburg-Landau

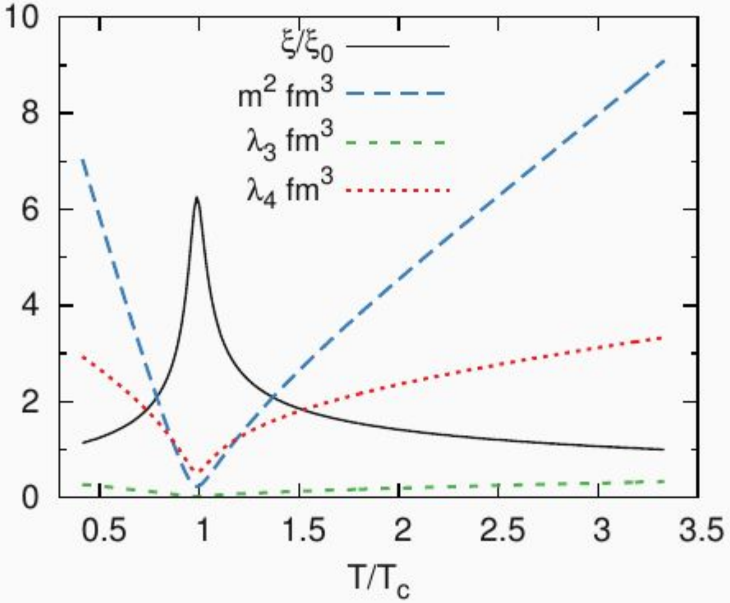
$$\mathcal{F}[n_B] = T\int d^3r \left(\frac{m^2}{2n_c^2}\Delta n_B^2 + \frac{K}{2n_c^2}(\nabla \Delta n_B)^2 + \frac{\lambda_3}{3n_c^3}\Delta n_B^3 + \frac{\lambda_4}{4n_c^4}\Delta n_B^4 + \frac{\lambda_6}{6n_c^6}\Delta n_B^6 \right)$$

The couplings depend on temperature via the correlation length $\xi(T)$:

Gauss + surface

$$m^2 = 1/(\xi_0\xi^2)$$
$$K = \tilde{K}/\xi_0$$
$$\lambda_3 = n_c\,\tilde{\lambda}_3\,(\xi/\xi_0)^{-3/2}$$
$$\lambda_4 = n_c\,\tilde{\lambda}_4\,(\xi/\xi_0)^{-1}$$
$$\lambda_6 = n_c\,\tilde{\lambda}_6$$

M. Tsypin PRL 73 (1994); PRB 55 (1997)



parameter choice: $\Delta n_B = n_B - n_c$

$\xi_0 \sim 0.5$ fm, $T_c = 0.15$ GeV, $n_c = 1/3$ fm^{-3}

$K = 1/\xi_0$ (surface tension)

$\tilde{\lambda}_3, \tilde{\lambda}_4, \tilde{\lambda}_6$ (universal, but mapping to QCD)

in this Fig.: $\tilde{\lambda}_3 = 1$, $\tilde{\lambda}_4 = 10$

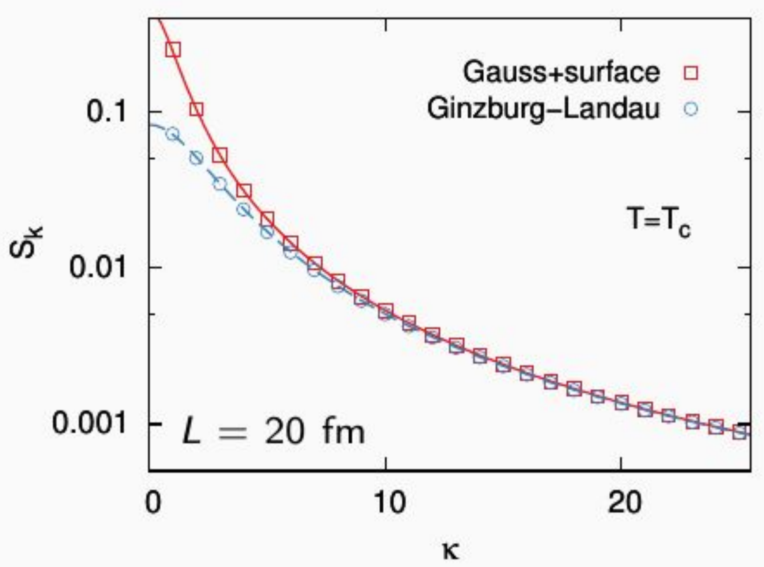
M. Nahrgang and MB, PRD 99 (2019) and PRD 102 (2020)

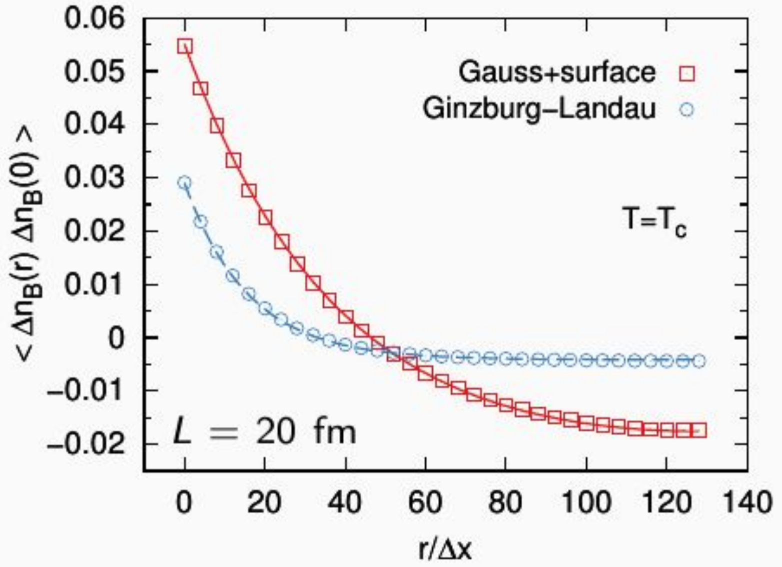
Studied in a static and cooling box of QGP

Validated in the equilibrium limit for the Gauss + surface model:

- Structure factor and equal-time correlation function are well reproduced
- Approaches continuum as resolution is increased
- Baryon conservation effects under control

Important step for all fluctuating codes!





M. Nahrgang and MB, PRD 99 (2019) and PRD 102 (2020)

Scaling of equilibrium cumulants

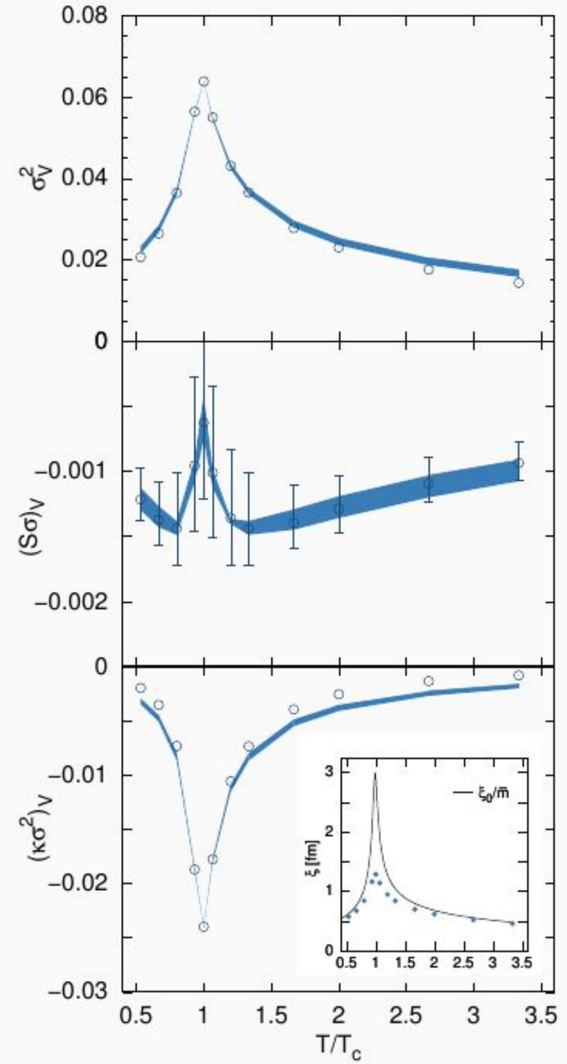
- Expected scaling in an infinite system ($\xi \ll V$): M. Stephanov PRL 102 (2009)

  $\sigma_V^2 \propto \xi^2, \quad (S\sigma)_V \propto \xi^{2.5}, \quad (\kappa\sigma^2)_V \propto \xi^5$

- Here, a finite system with exact baryon conservation ($\xi \lesssim V$)! Can be systematically studied in $\xi/V \Rightarrow$ affects equilibrium scaling!

- E.g. for the skewness terms $\propto \lambda_3\lambda_4$ and $\propto \lambda_3\lambda_6$ contribute with opposite sign.

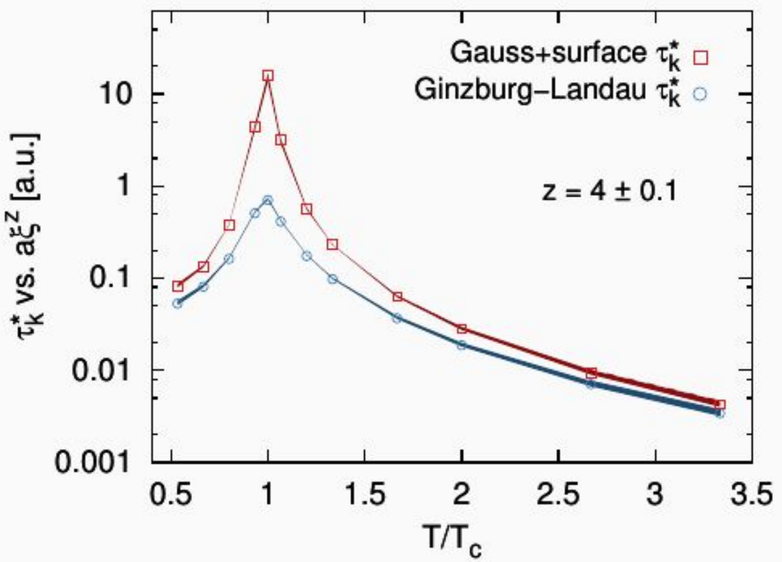
$$\sigma_V^2 \propto \xi^{1.3\pm0.05}$$
$$(S\sigma)_V \propto -\#\xi^{1.47\pm0.05} + \#\xi^{2.4\pm0.05}$$
$$(\kappa\sigma^2)_V \propto \xi^{2.5\pm0.1}$$



M. Nahrgang and MB, PRD 99 (2019) and PRD 102 (2020)

Dynamical critical scaling

- Dynamical structure factor for Gaussian model in continuum:
  $S(k,t) = S(k)\exp(-t/\tau_k)$ with $\tau_k^{-1} = \frac{Dm^2}{n_c}\left(1 + \frac{K}{m^2}k^2\right)k^2$
- Analyze ξ-dependence of relaxation time for modes with $k^* = 1/\xi$:



for both models: $\tau_k^* = a\,\xi^z$ with

$$z = 4 \pm 0.1$$

$$a = \frac{n_c \xi_0}{D(1+\tilde{K})}$$

$\Rightarrow$ Simulations reproduce scaling of model B!

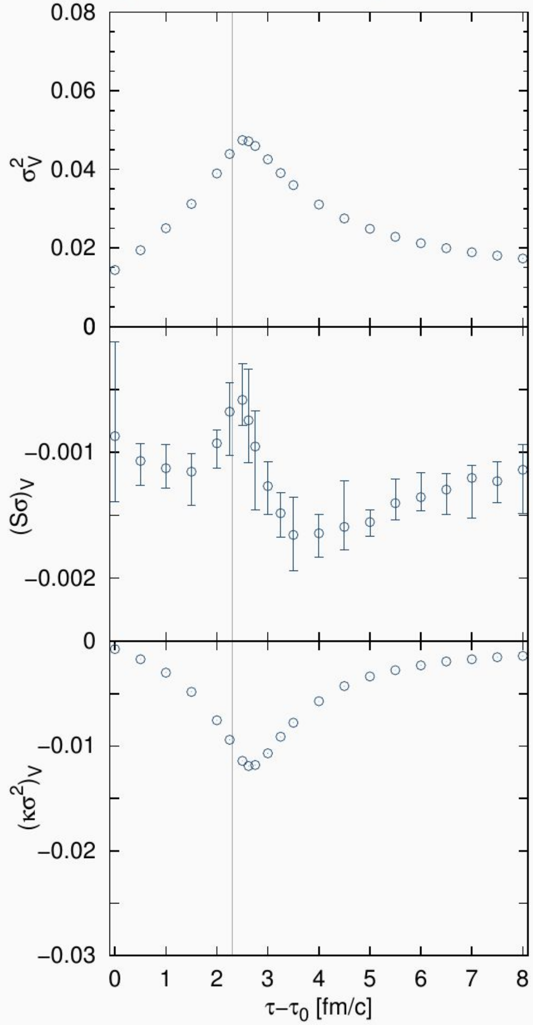
For the full dynamics of a HIC, couple to fluctuations in $T^{\mu\nu}$ ⟶ model H

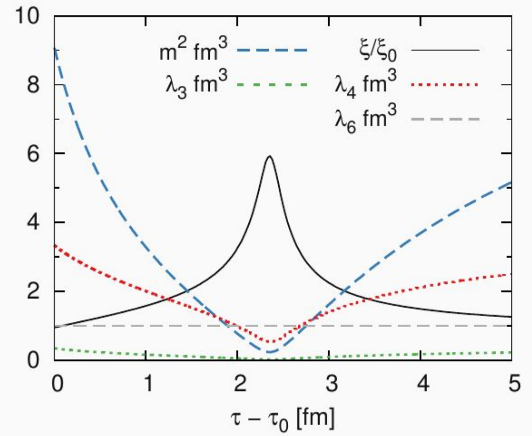
According to Hohenberg, Halperin, Rev. Mod. Phys. 49 (1977)

M. Nahrgang and MB, PRD 99 (2019) and PRD 102 (2020)

Time evolution of critical fluctuations

For a Bjorken-like temperature evolution:





- shift of extrema for variance/kurtosis (retardation effects) to later times corresponding to $T(\tau) < T_c$
- |extremal values| in dyn simulations < equilibrium values (nonequilibrium effects):

$$(\sigma_V^2)_{\rm dyn}^{\rm max} \approx 0.75\,(\sigma_V^2)_{\rm eq}^{\rm max}$$
$$((\kappa\sigma^2)_V)_{\rm dyn}^{\rm min} \approx 0.5\,(\kappa\sigma_V^2)_{\rm eq}^{\rm min}$$

- expected behavior with varying D and c_s^2 (expansion rate)

M. Nahrgang and MB, PRD 99 (2019) and PRD 102 (2020)

Renormalizing critical dynamics in model A of Hohenberg-Halperin in 3 dimensions

Cassol-Seewald et al. 0711.1866
Farakos et al. 9412091, 9404201
Gleiser, Ramos 9311278

Stochastic relaxation equation of chiral order parameter

$$\frac{\partial^2\varphi}{\partial t^2} - \nabla^2\varphi + \eta\frac{\partial\varphi}{\partial t} + \frac{\partial V_{\text{eff}}}{\partial\varphi} = \xi$$

Ginzburg-Landau effective potential, ϵ encodes phase transition

$$V_{\text{eff}}(\varphi) = \frac{1}{2}\epsilon\varphi^2 + \frac{1}{4}\lambda\varphi^4$$

White thermal noise

$$\langle\xi(\vec{x},t)\rangle = 0 \quad \text{and} \quad \langle\xi(\vec{x},t)\xi(\vec{x}',t')\rangle = 2\eta T\, \delta(\vec{x}-\vec{x}')\delta(t-t')$$

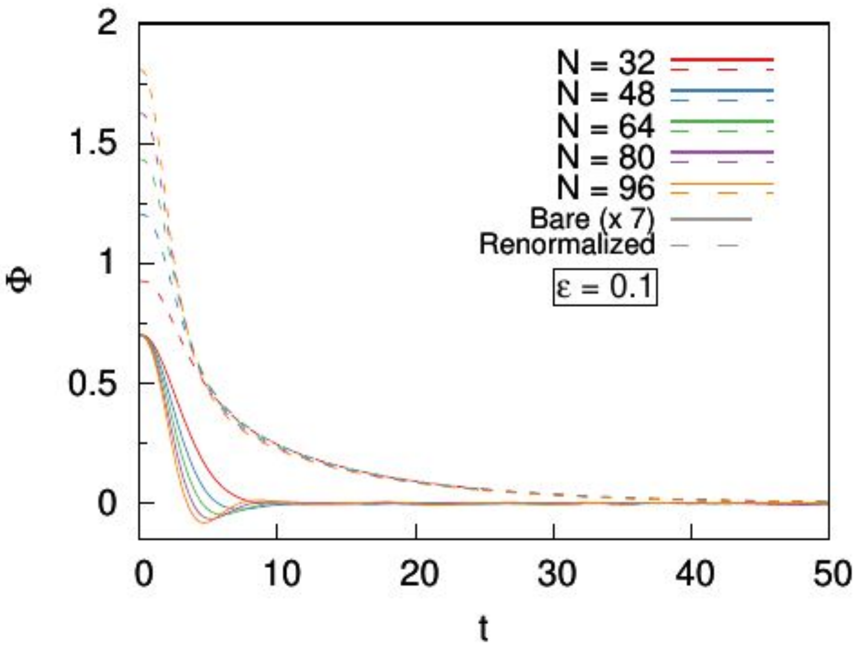
Counterterm for mass renormalization

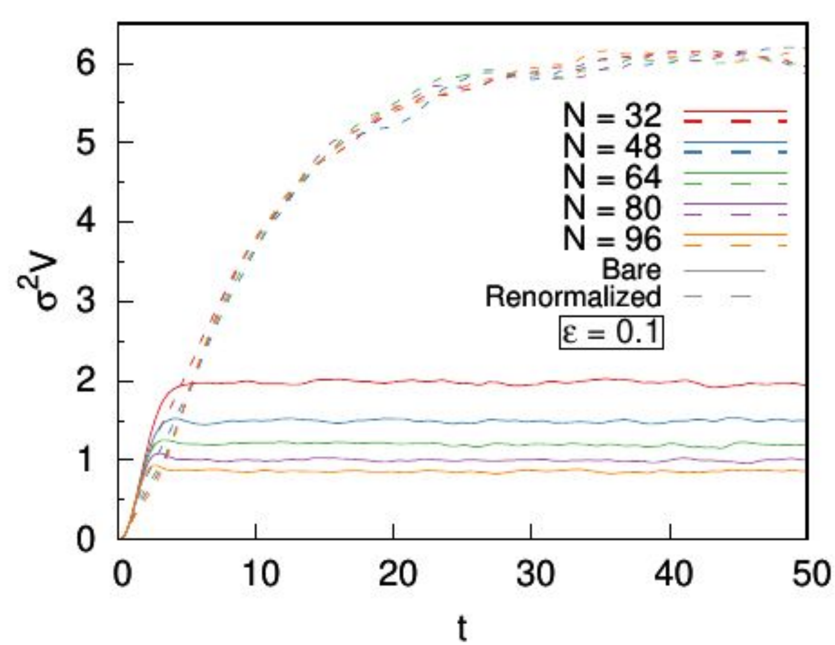
$$V_{\text{CT}} = \left\{ -\frac{3\lambda\Sigma}{4\pi}\frac{T}{\mathrm{d}x} + \frac{3}{8}\left(\frac{\lambda T}{\pi}\right)^2\left[\ln\left(\frac{6}{M\mathrm{d}x}\right) + \zeta\right]\right\}\frac{\varphi^2}{2}$$

Renormalizing critical dynamics in model A

Time evolution near a critical point of

mean & variance





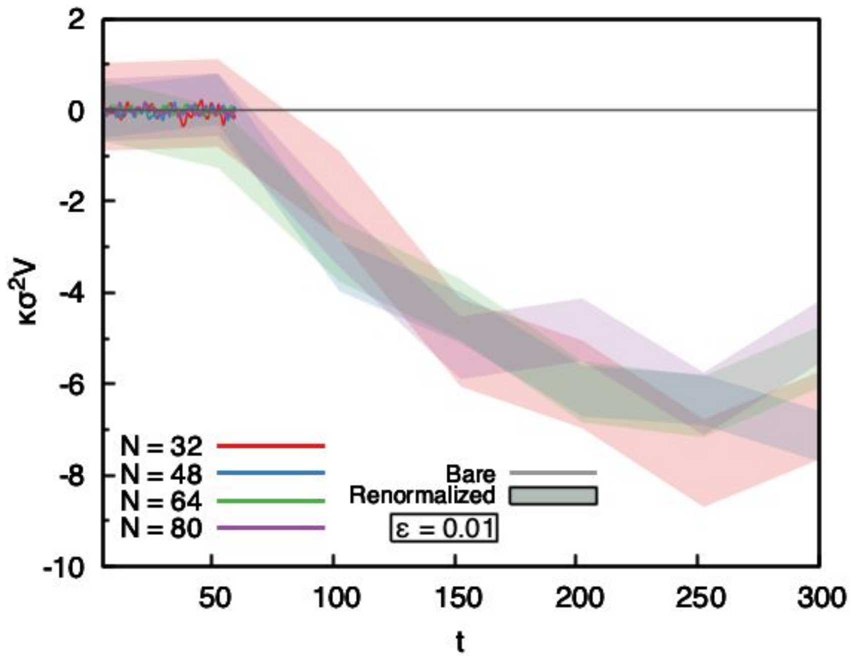
Restoration of lattice spacing independence after including the counterterm

N. Attieh, M. Bluhm, N. Touroux, M. Kitazawa, T. Sami, MN, in preparation

Renormalizing critical dynamics in model A

Time evolution near a critical point of

kurtosis



A nonzero kurtosis is observed after inclusion of the counterterm. Lattice spacing dependence or independence cannot be resolved within the available statistics.

N. Attieh, M. Bluhm, N. Touroux, M. Kitazawa, T. Sami, MN, in preparation

The role of numerics and computational resources

Time for two heavy nuclei to collide and produce particles:

~ 10^{-23} seconds

Time for a simulation of two colliding heavy nuclei and particle production:

~ 1 hour

Example: even with a Gaussian Process Emulator the Bayesian analysis of the temperature dependence of bulk and shear viscosity costs 100 Mio CPU hours.

This assumes $O(10^4)$ events per point, fluctuation studies easily require $O(10^8)$ events per point...