Staggered bosons and critical spin chains

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Based on 2303.12837, + Work in progress with P.T. Lloyd

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Teaching old bosons a new trick

When considering Hamiltonian formulations of physical questions
with bosonic variables (degrees of freedom),
we need both position and momenta.

In the Hamiltonian formalism they appear on the same footing.

Can we harness this idea to make new interesting models?
Get away from the distinction of p vs x somehow.

Staggered fermions

- Fermion degrees of freedom are distributed over a lattice region (even vs odd sites). Other half of Kogut-Susskind.
- Helps with doublers.
- Also, Majorana fermions (half a fermion) are interesting for topological reasons: Kitaev.

Goals

- Search for interesting lattice realizations of gapless field theories (towards minimal number of variables)

- Enjoy some symmetry protection that is not available in conventional field theories.

- One example: half bosons in 1D (and some generalizations)

Overview

- Hamiltonian formulation of the chiral boson.
- Half boson on a lattice: staggered bosons.
- Topology and zero modes.
- Interacting models and critical spin chains.
- Fractons.

Chiral boson is “half a boson”

$$\partial_t \phi = c \partial_x \phi$$

Only a left mover: does not give rise to modular invariant partition function (a.k.a. a nice Euclidean path integral).

A full boson with a nice Euclidean partition function has both a left and a right mover.

The left mover is nevertheless a proper field theory (integer QHE).
Laughlin, Wen, Stone,…

Hamiltonian formulation

We write a Poisson bracket structure between fundamental degrees of freedom

$$\{\phi(x), \phi(x')\} = \partial_x \delta(x - x')$$

Time derivative is not the canonical conjugate!

We only need ϕ (one bosonic variable, rather than two)

$$H = \int c \frac{\phi^2}{2}$$

There is no relevant deformation (polynomial) that gaps this system.

(Easy proof, anomaly matching)

What are these commutation relations?

Poison brackets become commutators in quantum theory.

c=1 (chiral) current algebra in position space

$$\phi \simeq J(x^+)$$

The right hand side is the anomaly (contact term: total derivative)

Turn it into a lattice

- Work idea of half boson on a lattice.
- Anomaly matching of 2d gravitational anomaly prevents a theory of a lattice that produces just a left mover $(c_L - c_R)_{UV} = 0$
- What do we get?
- Gapless vs. gapped question.

How to make half a boson

$$x, p \to q$$

We still want non-trivial Poisson brackets.

Idea is that the boson degrees of freedom become slightly delocalized, so that a notion of x, p reside at different sites (staggered degrees of freedom).

In practice

We do a discretized version of the derivative of the delta function.

$$\{q_i, q_j\} = \delta_{i,j-1} - \delta_{i,j+1}$$

There is a Sign choice: this sign choice is called left moving, if we change signs, we call it right moving.

Poisson bracket matrix

$$\omega^{IJ} = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ -1 & 0 & 1 & \ddots \\ 0 & -1 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

Constant and antisymmetric: defines a classical phase space.

Hamiltonian 1.0

Copy/paste the chiral boson Hamiltonian

$$H = \frac{1}{2}\sum q_i^2$$

We get a discretized version of chiral equation of motion.

$$\dot{q}_i = q_{i+1} - q_{i-1}$$

$$\dot{q} \sim 2\partial_x q$$

Mode expansion: Fourier in position

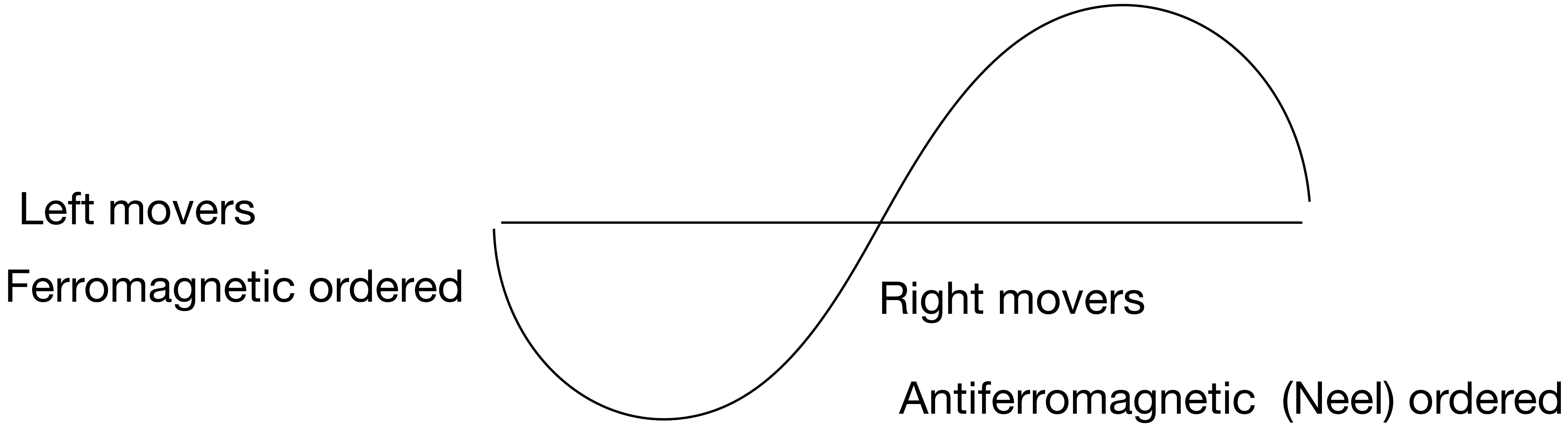
$$\omega(k) = -2\sin(k)$$

Nielsen-Ninomiya argument predicts doublers (anomaly matching as well).
The system has to have a right mover!

Straightforward to quantize: raising/lowering depends on sign of $\omega(k)$

The mode at k is conjugate to the mode at -k.

INFRARED $\omega(k) = 0$



Important: $\omega(k)$ is a single valued function of k.
Deformations don't alter the fact that
there are crossings of zero.

Neel ordering

$$q_j \to \tilde{q}_j = (-1)^j q_j$$

Turns the left moving half boson
into a right moving half boson:
it changes signs in the Poisson bracket.

Non-trivial Parity invariance

$$q_j \to \tilde{q}_{-j} = (-1)^j q_{-j}$$

At the level of Fourier modes

$$a_k^\dagger \to a_{\pi-k}^\dagger$$

Symmetry protection

Left and right movers can not mix if translation invariance is preserved: they are at different values of k (the modes do not hybridize)

Massless bosons protected, even in the presence of perturbative interactions.

It is a critical theory

$$k = 2\pi n/L$$

$$\tilde{\omega} = \omega(L/4\pi) \sim n$$

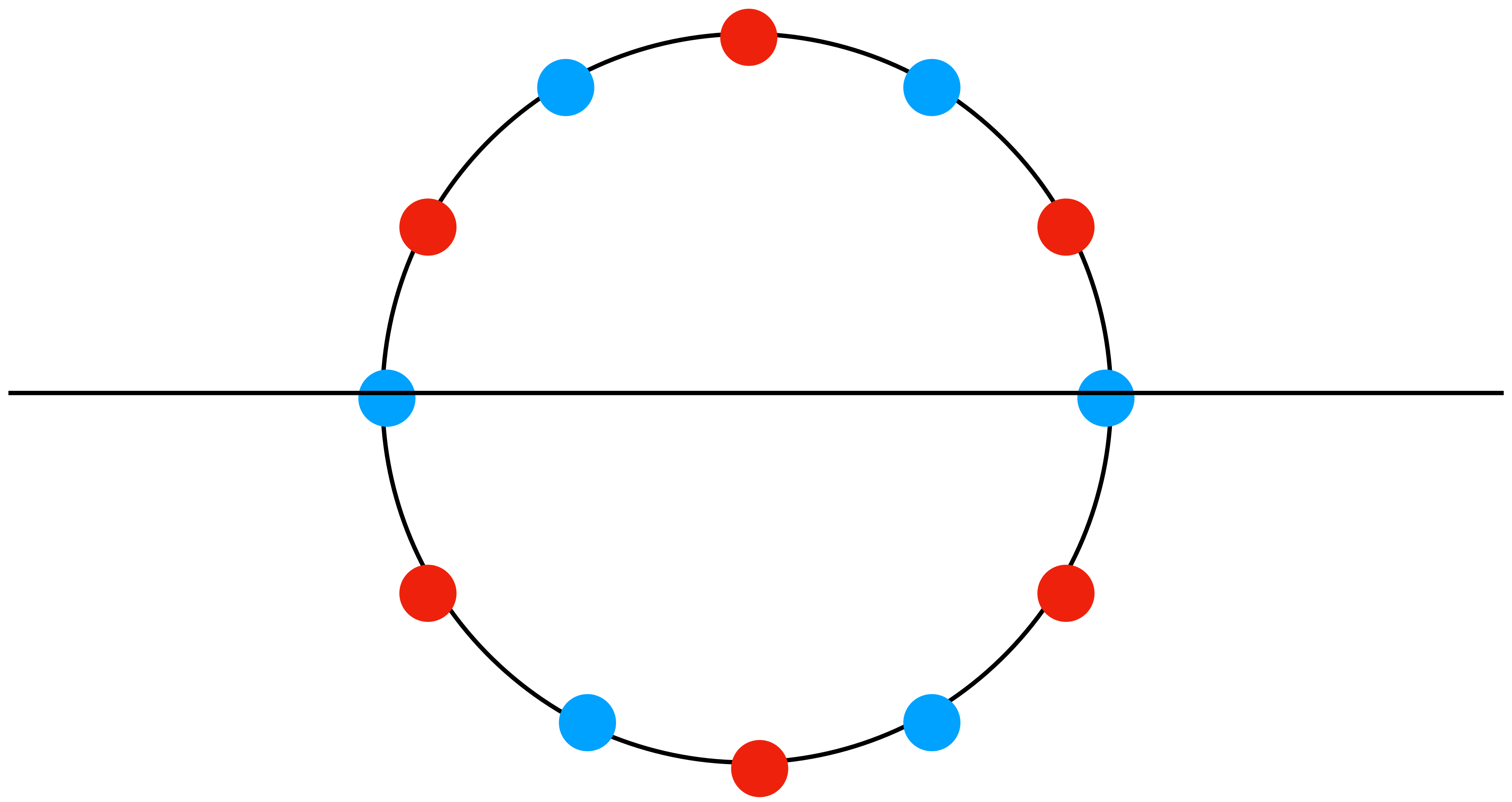
Only one positive frequency mode per n (near k=0).
Negative n is negative frequency (lowering operator).

Similar statement for $\pi - k \sim n2\pi/L$

Zero modes

Finite periodic lattice:

Even number of sites: even number of zero modes
(2,0) depending on if periodic (blue)
or anti periodic (red).

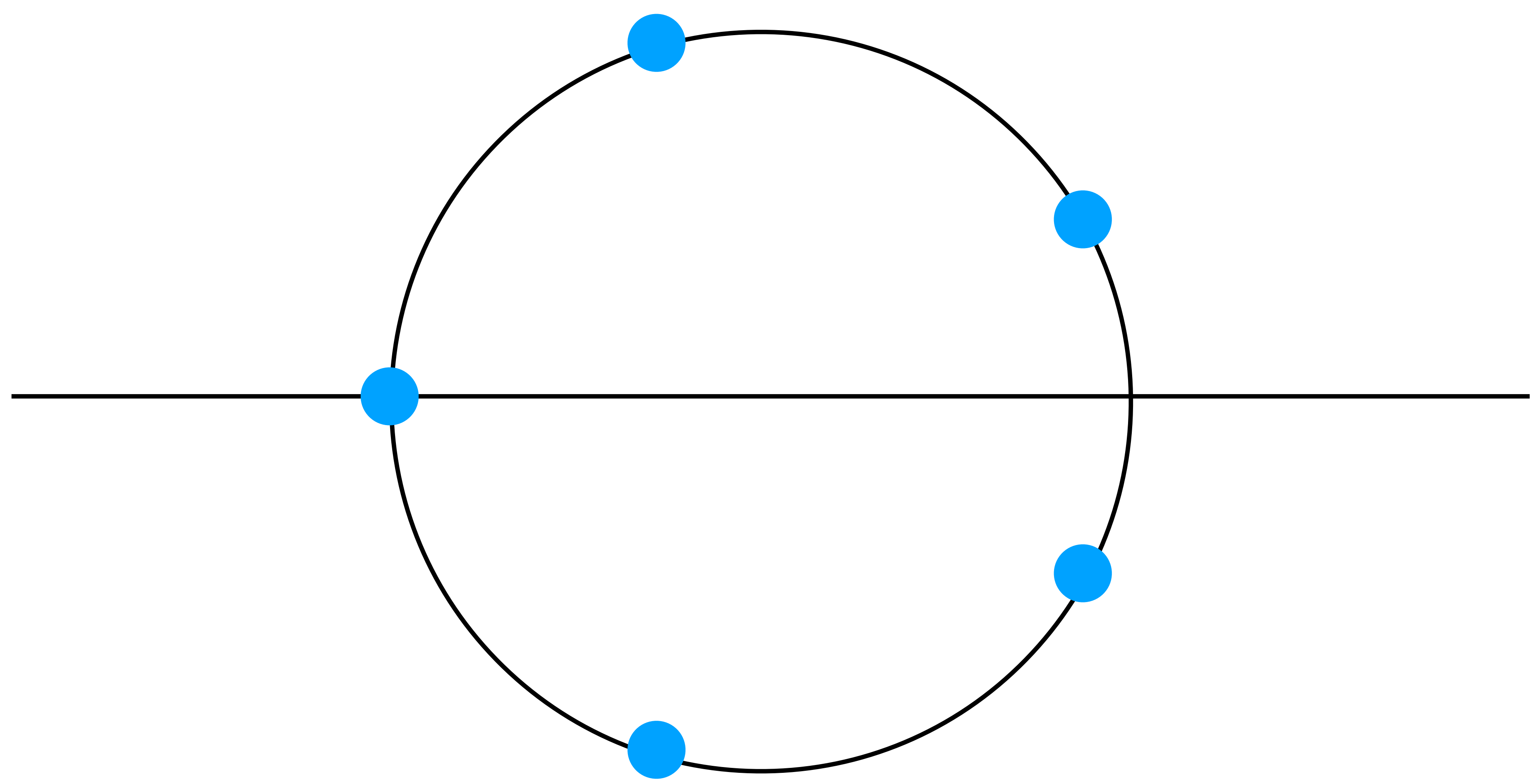


These states are parity invariant

Odd number of lattice sites

Odd lattice sites: one zero mode.
For right movers, n is half integer.

Parity is broken



REASON: a non degenerate Poisson bracket requires an even number of variables.

Open interval

- One zero mode if odd number of sites (NN boundary conditions)

- No zero modes if even number of sites (DN boundary conditions)

These are very similar to the counting of zero modes for Majorana fermions. Reason is similar: need to pair two Majorana fermions to get a “frequency”: mass matrix for Majorana fermions is an antisymmetric matrix.

Making half bosons from full bosons

Projections from a full boson into half bosons:

$$x_i, p_i \to q_i = p_i + x_{i+1}$$

$$w_i = p_{i+1} + x_i$$

Two half bosons: one "left mover" and one "right mover".
They share the zero modes.

$$2N \to 2N - 2$$

Some modes are missing
in the projection: conjugate variables
to zero modes

Another projection (doubled lattice)

$$N(x,p) \to 2N(q)$$

$$q_{2i} = p_i, \quad q_{2i+1} = x_{i+1} - x_i$$

More carefully

$$2N \to 2N - 1$$

Miss conjugate variable to zero mode.
And we also miss one zero mode?

Zero mode for odd q missing.

If we add it by hand, we have that

$$x_{j+1} - x_j \rightarrow const$$

The extra zero mode in the x variables is interpreted as classical winding in the x variables

Staggered bosons automatically come with (continuous) winding configurations.

Classical version of T-duality:
can't distinguish $\dot{\phi} \leftrightarrow \nabla \phi$

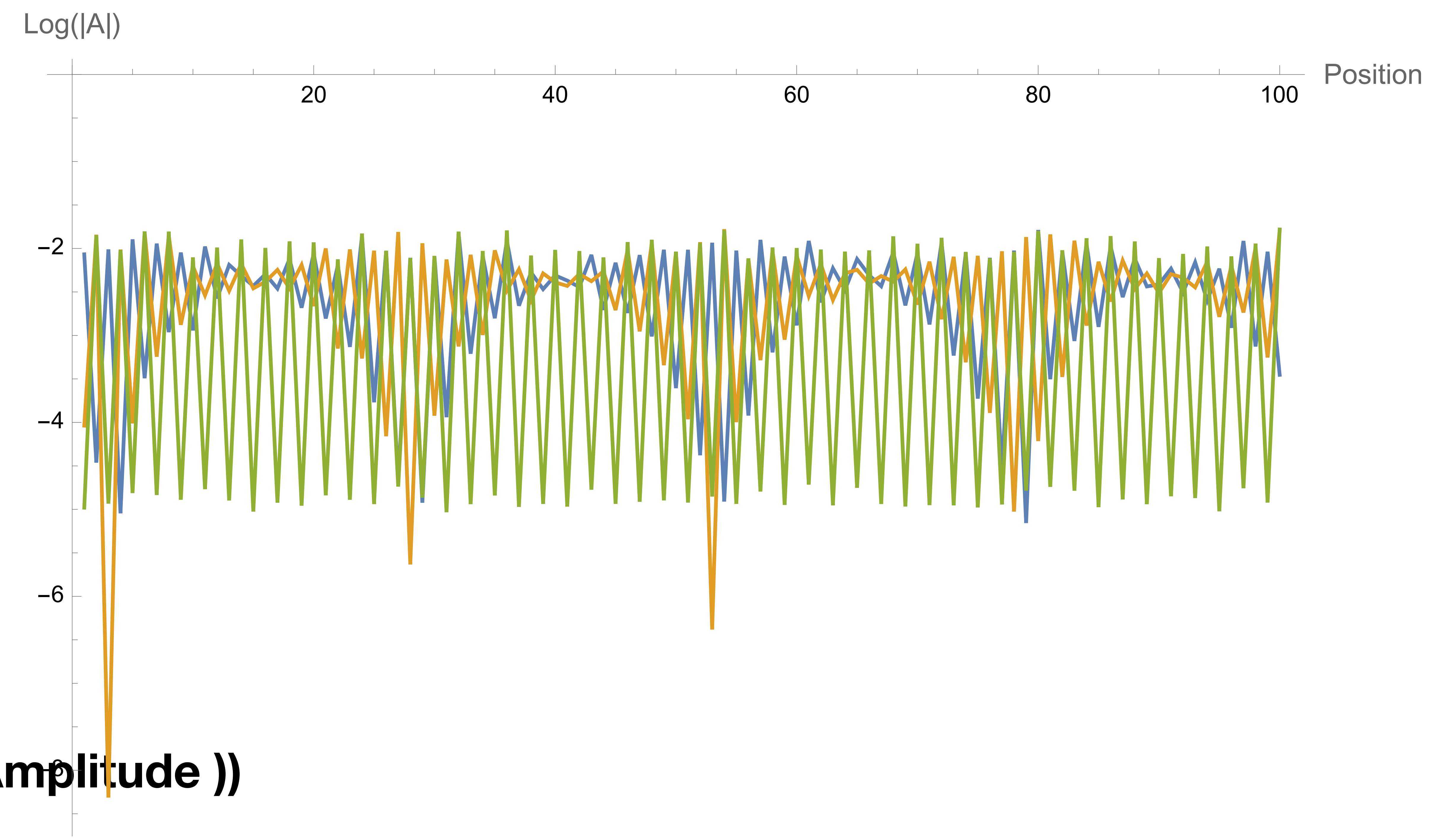
Theory is still critical if we add noise

$$H = \frac{1}{2} \sum \eta(i) q_i^2$$

$$\eta(i) \sim 0.7 - 1.0$$

Low frequency modes are delocalized

Position



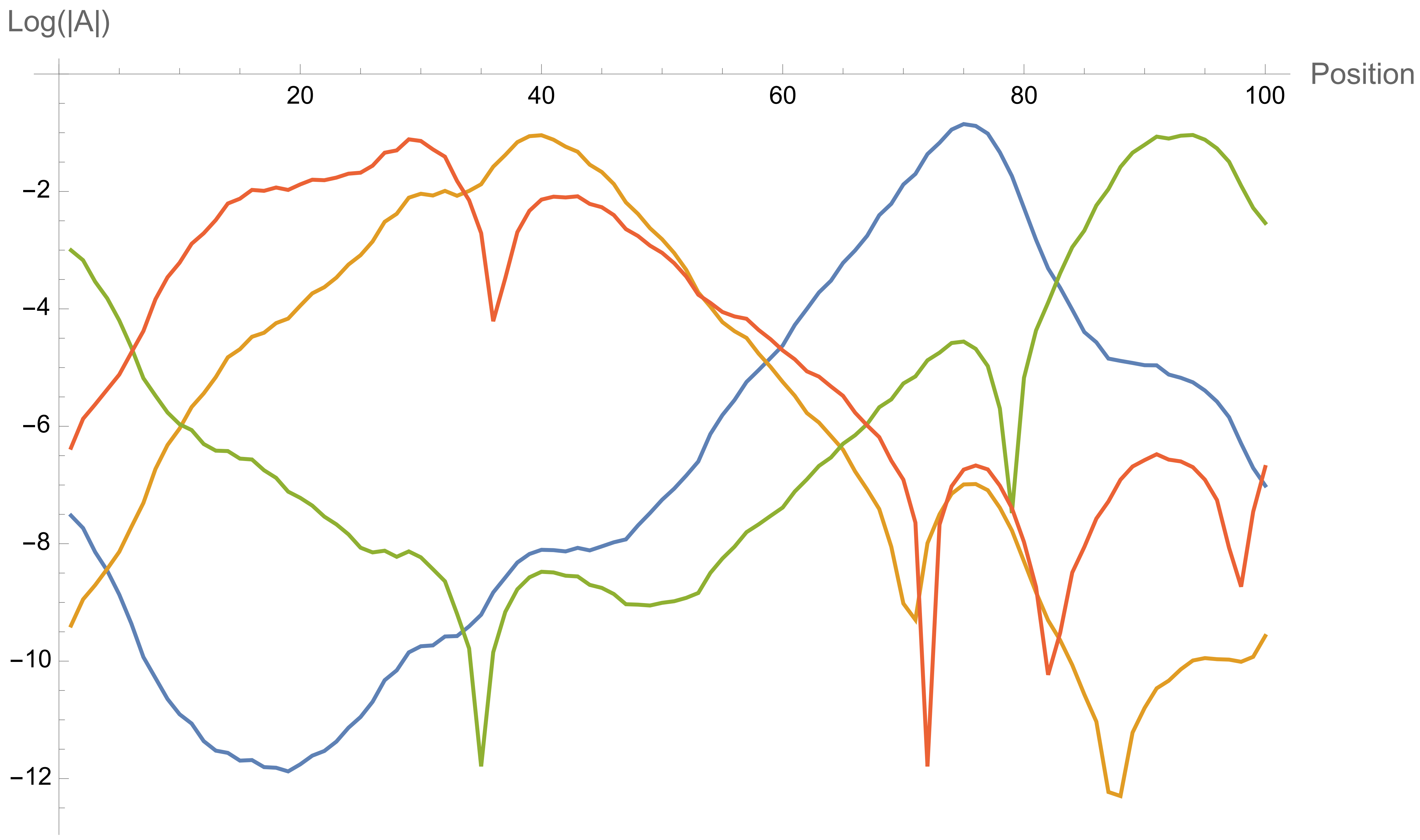
Log (Abs(Amplitude))

Modes near zero frequency of spectrum

No Anderson localization of low lying modes

High frequency modes localized

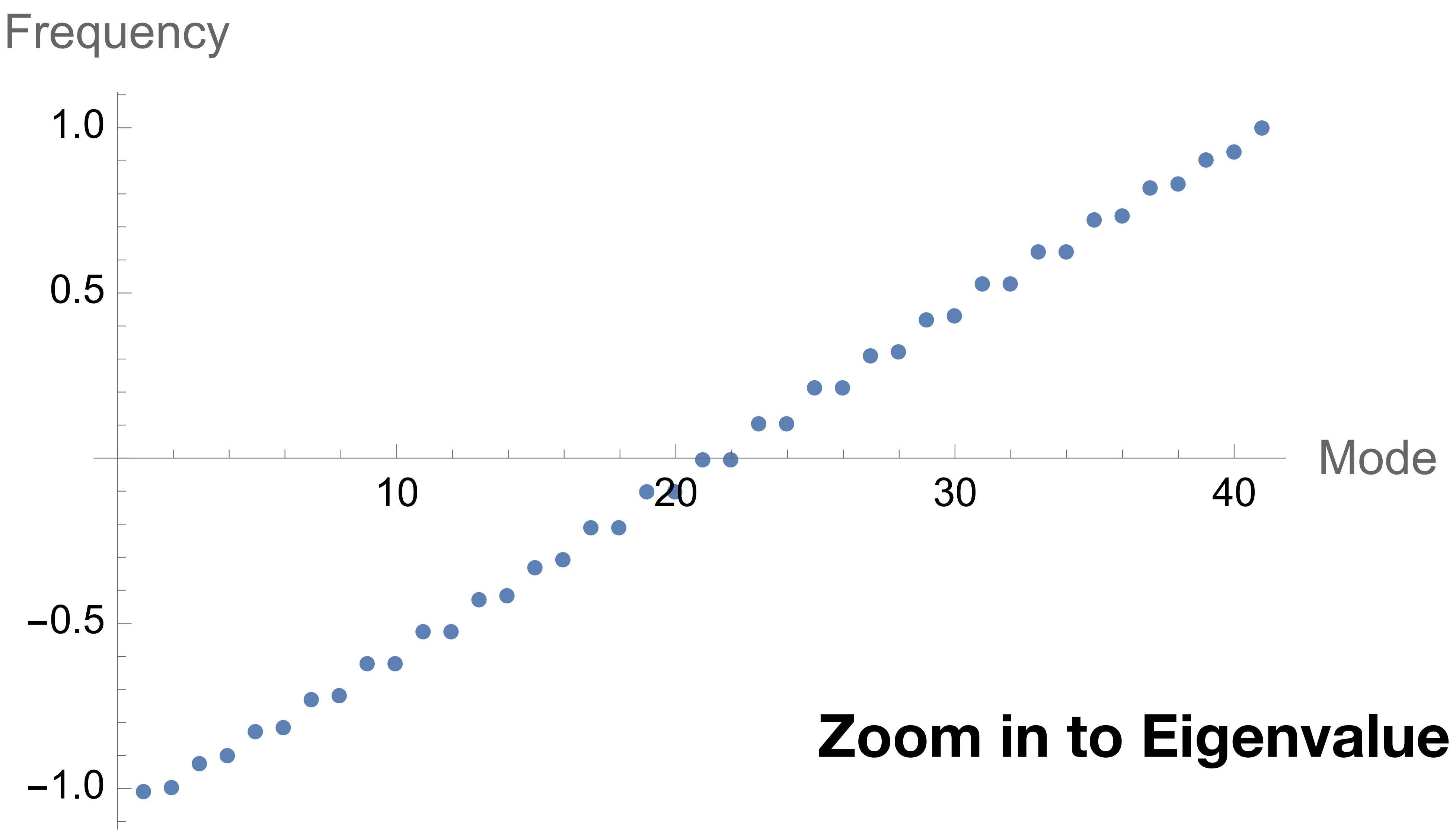
Position



Log (Abs(Amplitude))

Low frequency (with noise)

w



Zoom in to Eigenvalue # (ordered by w)

Still critical (evenly spaced),
with double degeneracy from left and right movers

Rough Reason

$$\phi \sim \partial\chi$$

The half boson variable is like the current variable in the continuum limit.

$$H \sim \int \eta(x)\partial\chi^2$$

It is like having a (noisy at discretization level) curved coordinate in the IR: we can get rid of it by a change of variables (for position on the lattice x).

Interacting models and Finite Hilbert spaces

Gauging translations

Translations of q are symmetries of Poisson bracket. Finite translations can be gauged in quantum theory.

$$K_j = \exp(i\alpha q_j)$$

With Baker-Campbell-Hausdorf

$$K_j K_{j+1} = \gamma K_{j+1} K_j$$

$$\gamma \simeq \exp(i\alpha^2)$$

If γ is a root of unity

$$K_j^n$$

Commutes with everything (central):
K becomes a “finite matrix” of roots of unity.

Can be mapped to clock-shift matrices

$$K_{2i} \sim \exp(i\alpha p_i), K_{2i+1} \sim \exp(i\alpha(x_{i+1} - x_i))$$

Basically: magnetic translation algebra.

Because of gauging, can choose K^n=1 (central element)

$$K_{2i} \simeq Q_i \quad K_{2i+1} \sim P_i \otimes P_{i+1}^{-1}$$

Spin chain Hamiltonians

$$\hat{H} = -\sum K_i + K_i^{-1} = -\sum_j \left[Q_j + Q_j^{-1} + P_j \otimes P_{j+1}^{-1} + P_j^{-1} \otimes P_{j+1} \right]$$

Critical spin chains!

Each Q,P has a Hilbert space of dimension n attached to it.
Q,P are generalizations of Pauli matrices.

$$K_j K_{j+1} = \gamma K_{j+1} K_j$$

$$\gamma = -1 \rightarrow QP = -PQ$$

They anticommute and reduce to 1 qubit per Q,P pair

Simplest cases

Critical Ising in a magnetic field ($\gamma = -1$).

Three state Potts at criticality ($\gamma^3 = 1$)

Spin chains at criticality with c=1 ($\gamma^n = 1$)
For n > 3

n=4 is two copies of Ising

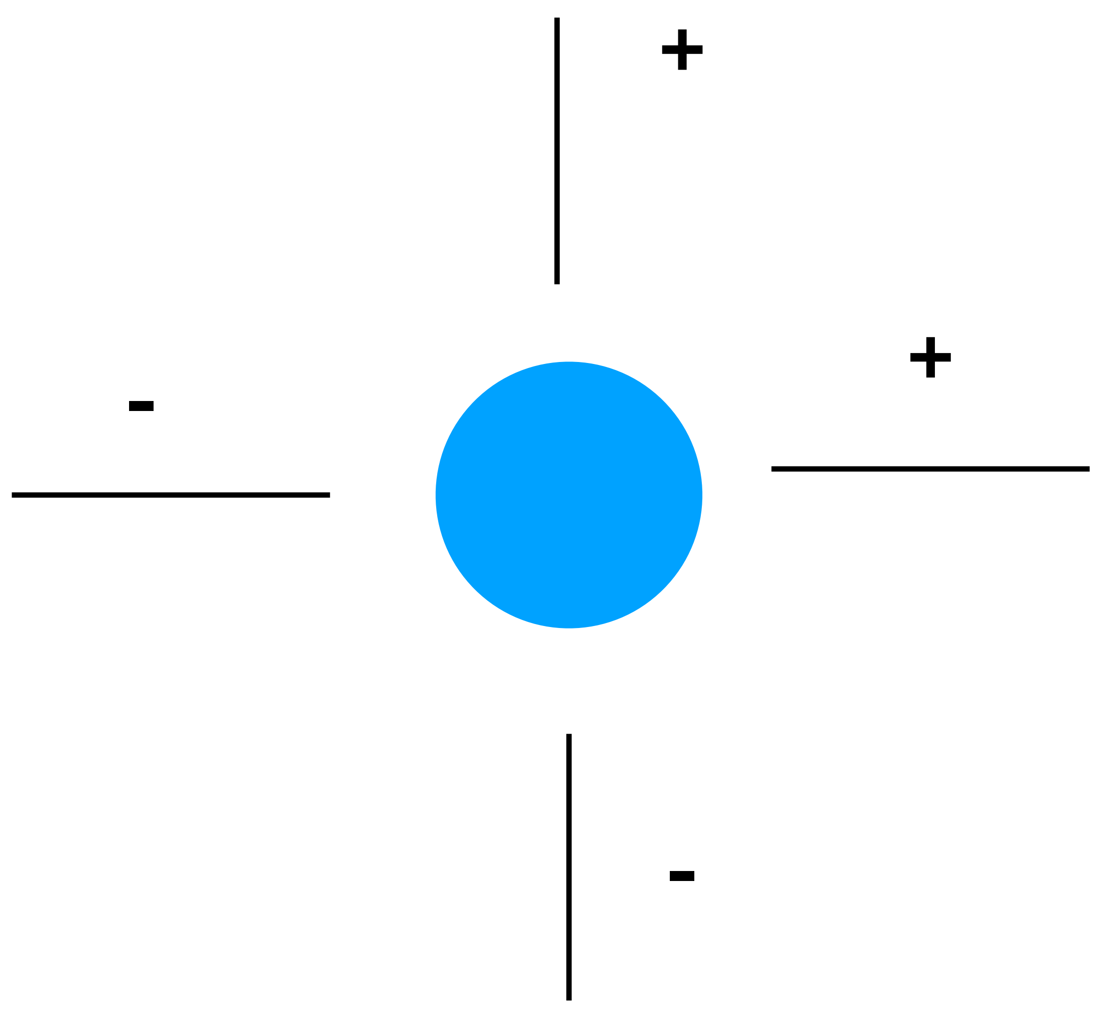
n>4 is BKT: there is a U(1) current algebra that survives to the IR.

Doing numerics on it with P.T. Lloyd

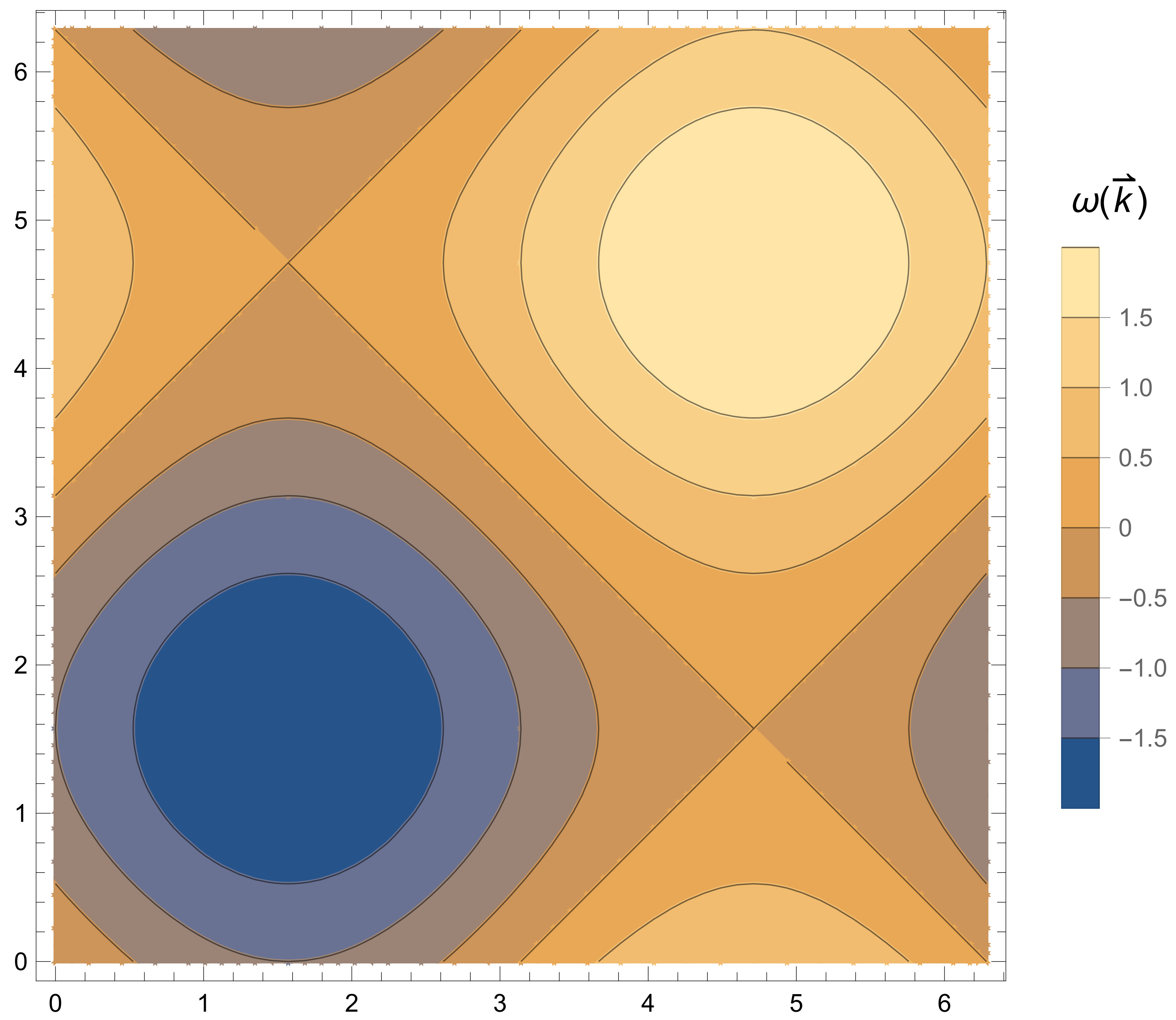
Higher dimensions

Add an extra dimension, keep half bosons on lattice sites and pick translation invariant non-vanishing bracket with all nearest neighbors.

$$\{q_{i,j}, q_{\ell,m}\} = (\delta_{i,\ell-1} - \delta_{i,\ell+1})\delta_{j,m} + \delta_{i,\ell}(\delta_{j,m-1} - \delta_{j,m+1})$$



We get the following dispersion relation



$$\omega(\vec{k}) = -2\sin(k_x) - 2\sin(k_y)$$

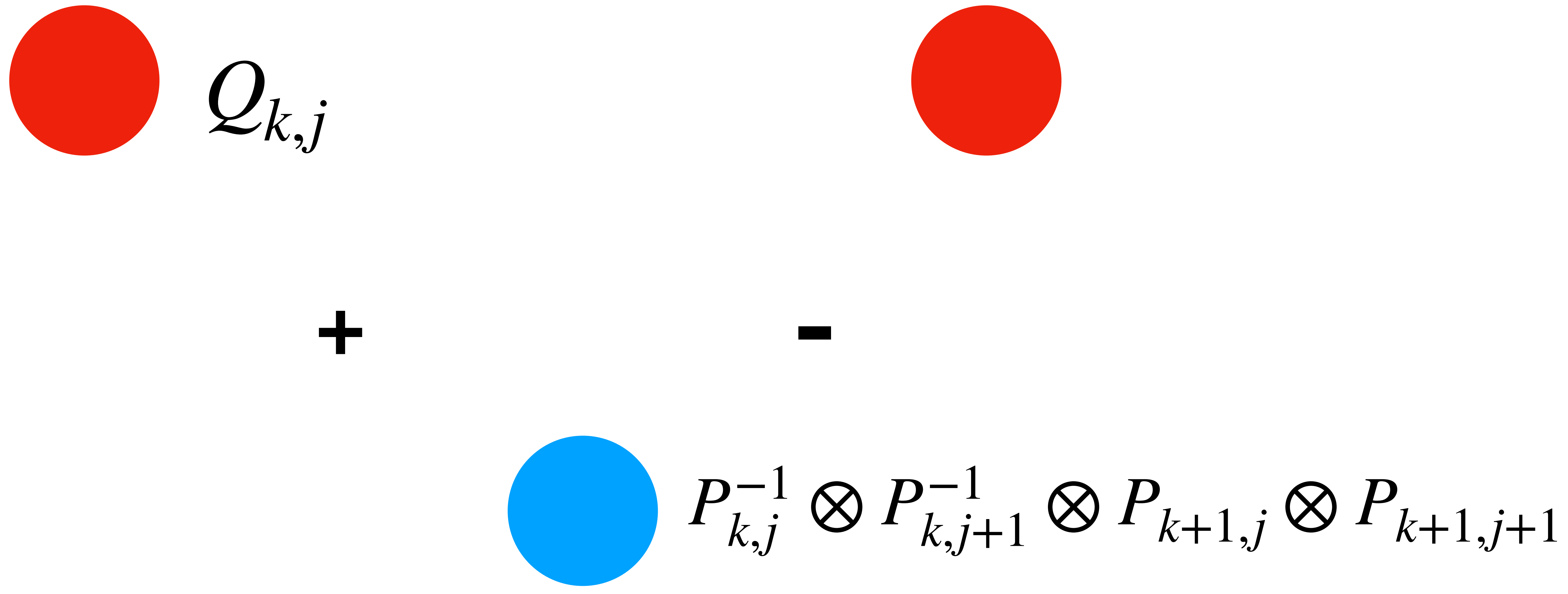
Lines of zero modes that cross.

Protected by "single validness of $\omega(k)$"

These are gapless fracton models
(extra symmetries on lattice)

Suggests we rotate lattice by 45 degrees.

Can also be mapped to P,Q matrices upon periodic gauging.
Split even and odd lattice sites
(face centered 2 D lattice)



This cannot be interpreted as hopping:
“strongly coupled” in hopping intuition:
does look like a plaquette.

Recap

- Half-bosons in 1D: gapless and symmetry protected if translation invariance is preserved.

- Result is robust against “disorder”: gaplessness persists.

- On periodic identification of half bosons one automatically gets critical spin chains (at the exact BKT transition point)

- Leads to interesting fracton (free+interacting) 2+1 D theories.