Lee-Yang singularities, series expansions and the critical point

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Chirality and Criticality: Novel Phenomena in Heavy Ion Collisions

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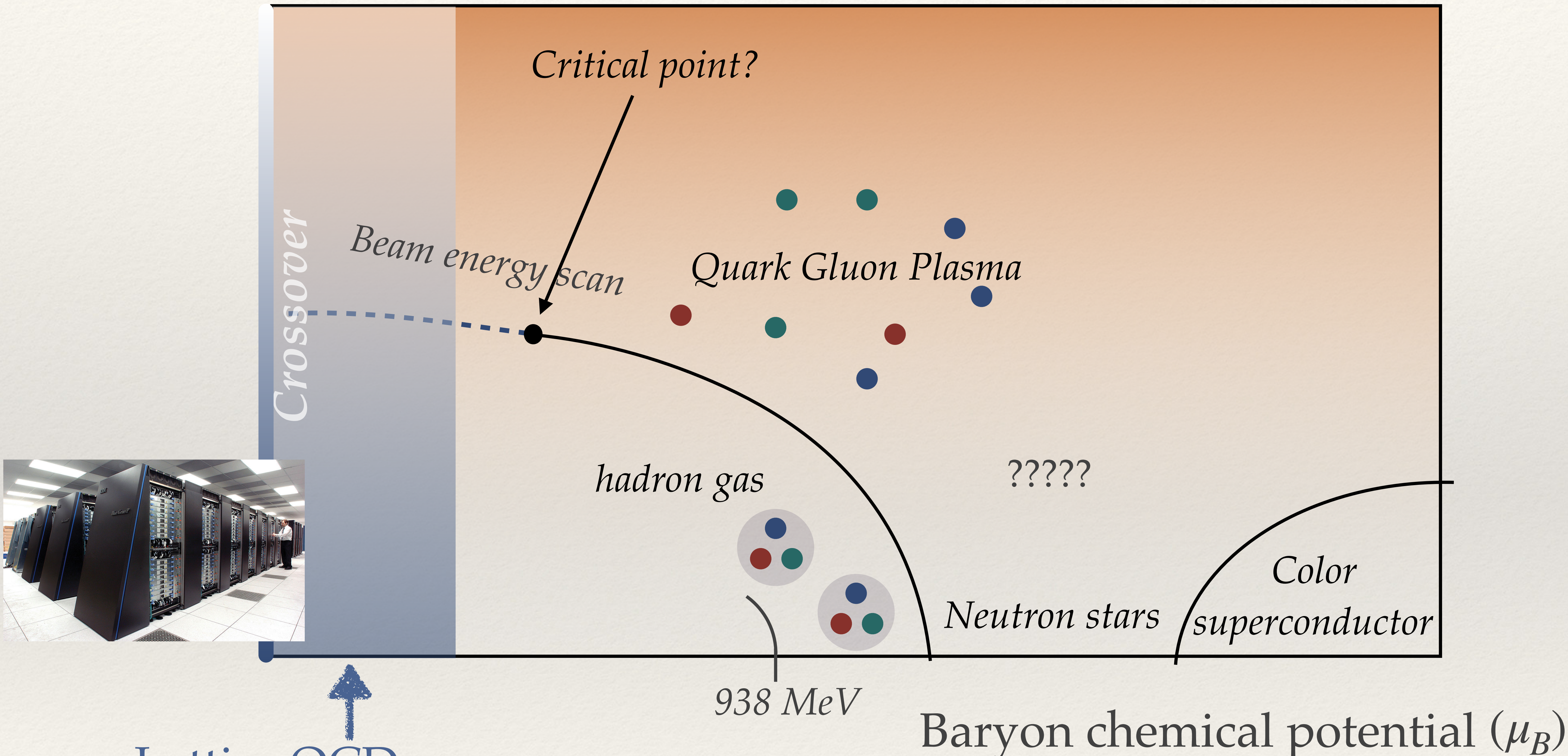
Based on:

GB *PRL* 127 (2021) 17, 171603

GB, G. Dunne (UConn), Z. Yin (UNC→ Stanford) *PRD* 105 (2022) 10, 105002

Motivations



Lattice QCD

Taylor series around $\mu_B = 0$

Analytical cont. from $i\mu_B$

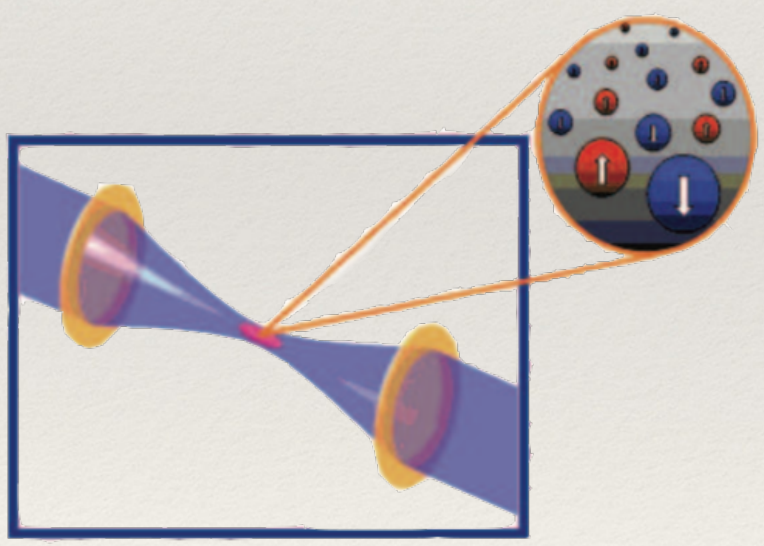
Motivations

Strongly correlated fermions, ultra-cold atoms

- At high T/low density, *Virial expansion*

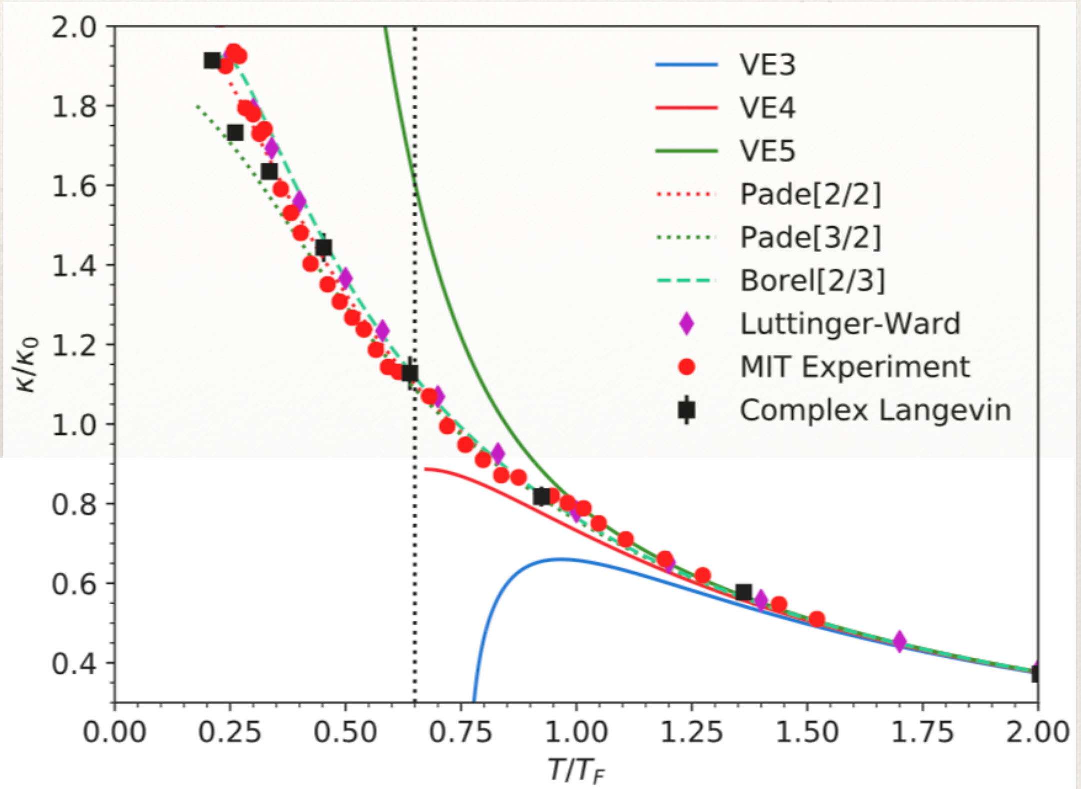
$$\Omega = -T \log \mathcal{Z} \sim -T \sum_{n=0}^{N} b_n e^{n\mu/T}$$

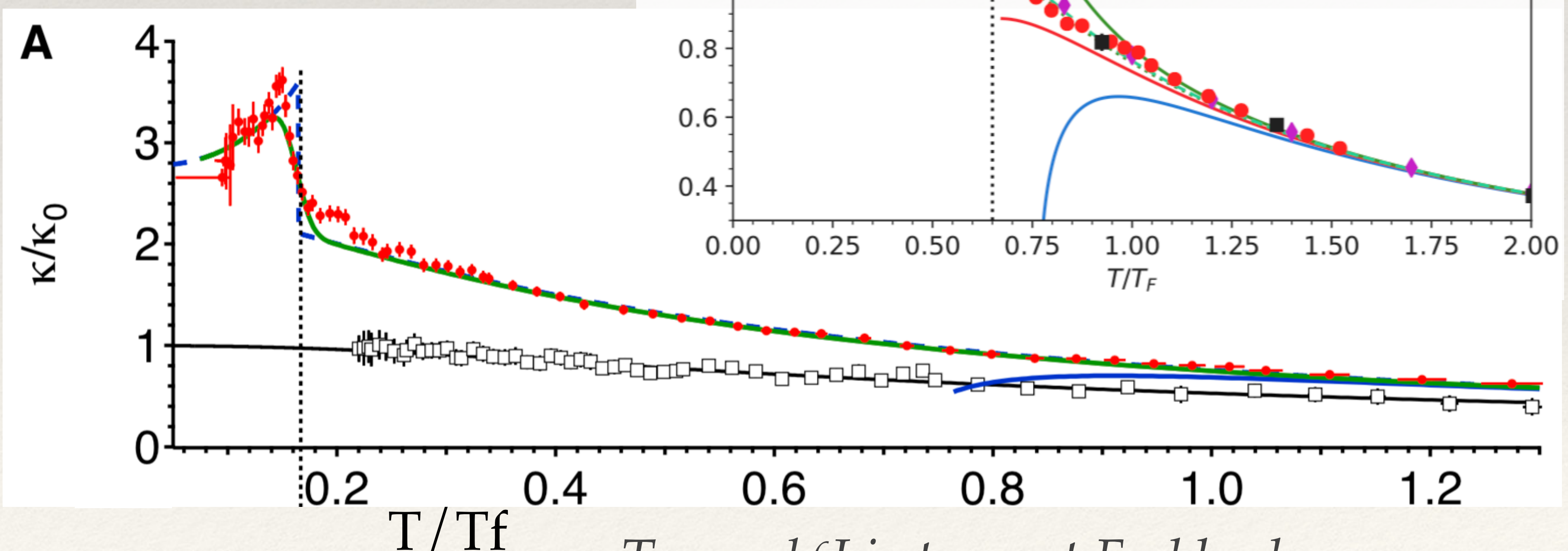
- Low T behavior ?
  Superfluid transition?, spin imbalance?….



[fig. Liu, Phys. Rep. **524**, 2, pg.37-83]

[Hou, Drut;
PRA **102**, 033319]





Trapped ^{6}Li atoms at Feshbach resonance

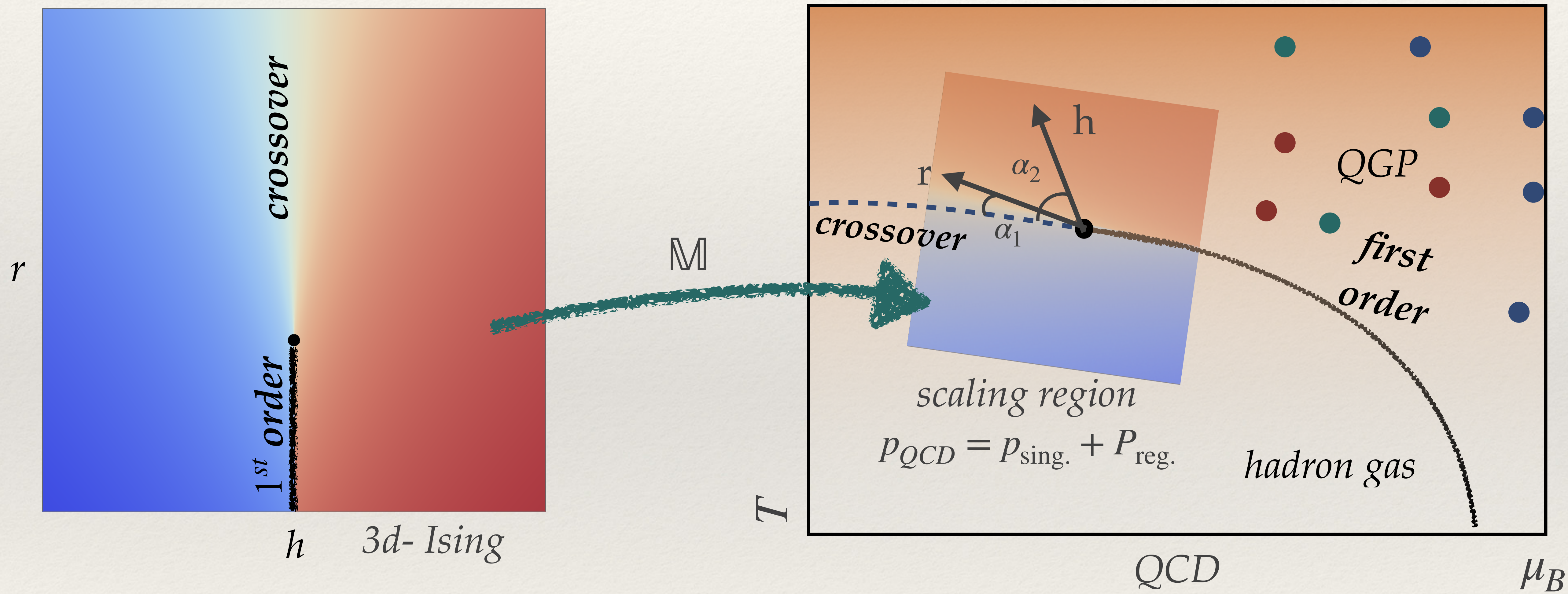
[MIT exp., Science **335**, 6068, pg. 563-567]

Motivations

COLLABORATION

[*Nucl.Phys.A* 1017 (2022) 122343]

$$\begin{pmatrix} r \\ h \end{pmatrix} = \mathbb{M} \begin{pmatrix} T - T_c \\ \mu - \mu_C \end{pmatrix} = \begin{pmatrix} r_T & r_\mu \\ h_T & h_\mu \end{pmatrix} \begin{pmatrix} T - T_c \\ \mu - \mu_C \end{pmatrix}$$



Given the e.o.s. as truncated Taylor series around μ=0, what can we say about *the critical e.o.s* ?

More generally how much of the e.o.s. we can reconstruct?

Motivations

Given a finitely many terms in the series expansion the equation of state, obtained away from a critical point, what can we say about *the critical phenomena* ?

How much of the critical e.o.s can we reconstruct?

Mathematically: reconstructing a function near a singularity from a truncated local expansion at a regular point.

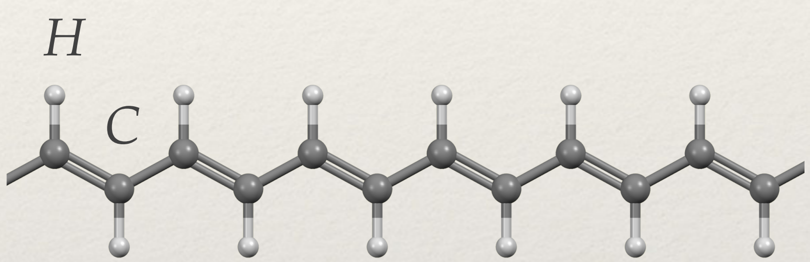
$$f(x) \sim c_0 + c_1 x + c_2 x^2 + \ldots c_n x^n$$

Gross-Neveu Model

$$S = \int d^2x \sum_{a=1}^{N_F} \left(i\bar{\psi}_a(\partial\!\!\!/ - m_q)\bar{\psi}_a + \frac{g^2}{2}(\bar{\psi}_a\psi_a)^2 \right)$$

[Gross, Neveu, '74]

- Solvable in large N_f limit (mean field is exact at $N_f\!=\!\infty$)
- Asymptotically free, dimensional transmutation
- Chiral symmetry breaking ($\mathbb{Z}_2 : \psi \to \gamma^5\psi$)
- Toy model for QCD
- Condensed matter: model for *trans-polyacetylene*



$$\frac{\pi}{Ng^2} = \log\frac{\Lambda}{m}$$

Dimensional transmutation

$$\gamma \equiv \frac{\pi}{Ng^2}\frac{m_q}{m} = \log\frac{m[m_q]}{m[0]}$$

Explicit χSB parameter

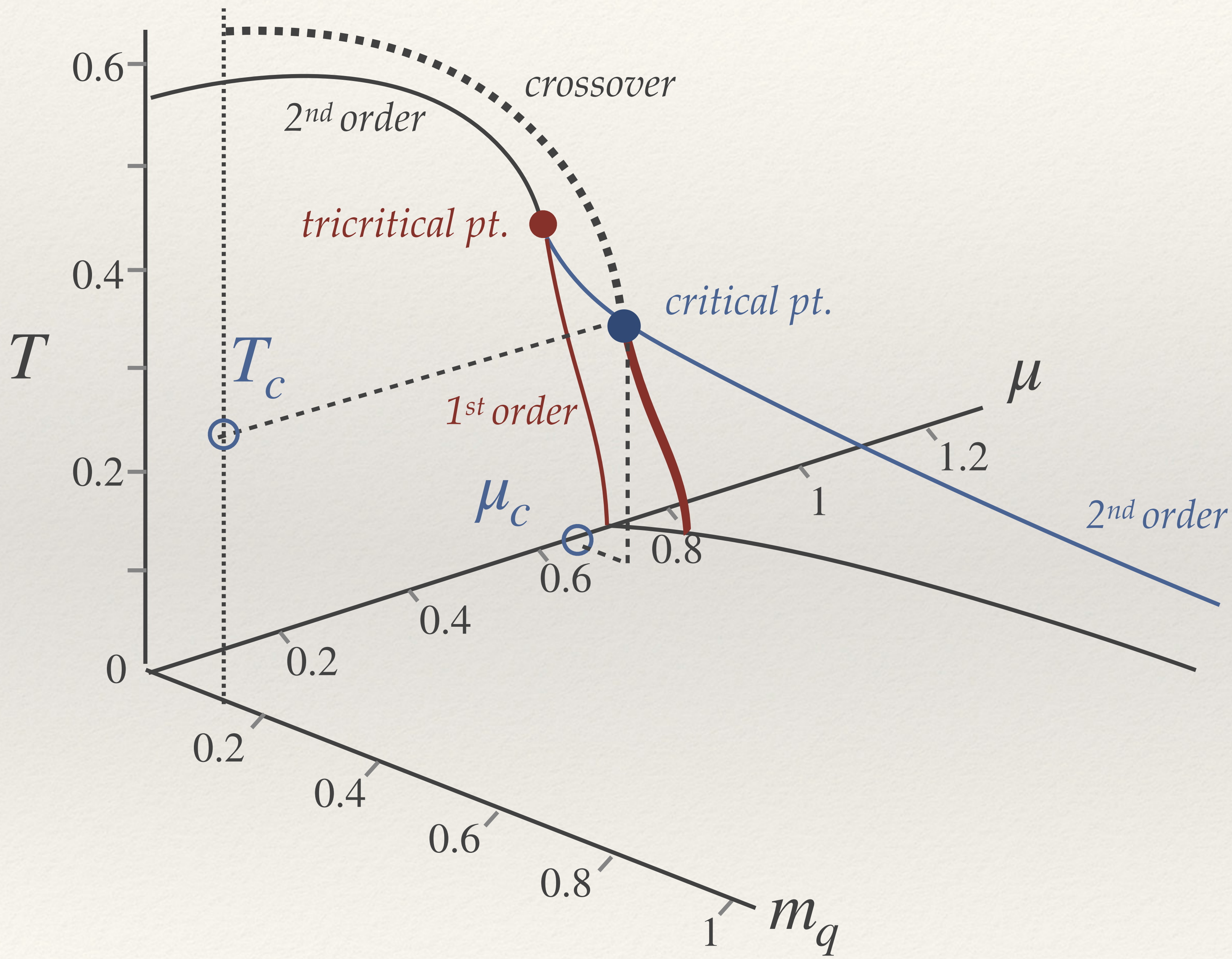
Thermodynamics:

$$\Omega[T,\mu] = \min_{\phi}\left(\frac{\phi^2}{2\pi}\left(\log\phi - \frac{1}{2} + \gamma\right) - \frac{\gamma}{\pi}\phi - T\int\frac{dk}{2\pi}\log\left[\left(1 + e^{-\left(\sqrt{k^2+\phi^2}-\mu\right)/T}\right)\left(1 + e^{-\left(\sqrt{k^2+\phi^2}+\mu\right)/T}\right)\right]\right)$$

Gross-Neveu Model

"homogeneous" phase diagram (toy example for QCD)*

[Barducci et al. '95]



- Assume $m_q \neq 0$
- Near the critical p.t
  $\Rightarrow \mathbb{Z}_2$ Ising e.o.s

  mean field exponents

  $\beta = 1/2, \;\delta = 3, \;\sigma_{LY} = 1/2$
- Focus on the crossover

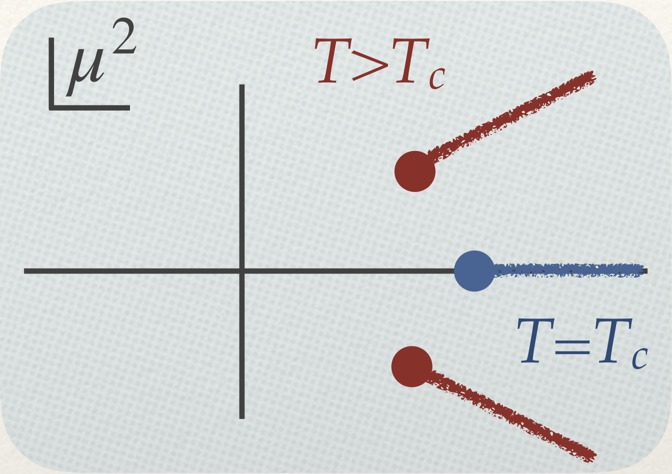
  $$T \gtrsim T_c$$
- T, μ normalized to m

*for the phase diagram including crystalline phases see [Schnetz ,Thies, Ulrichs '05 ; GB, Dunne, Thies '08]

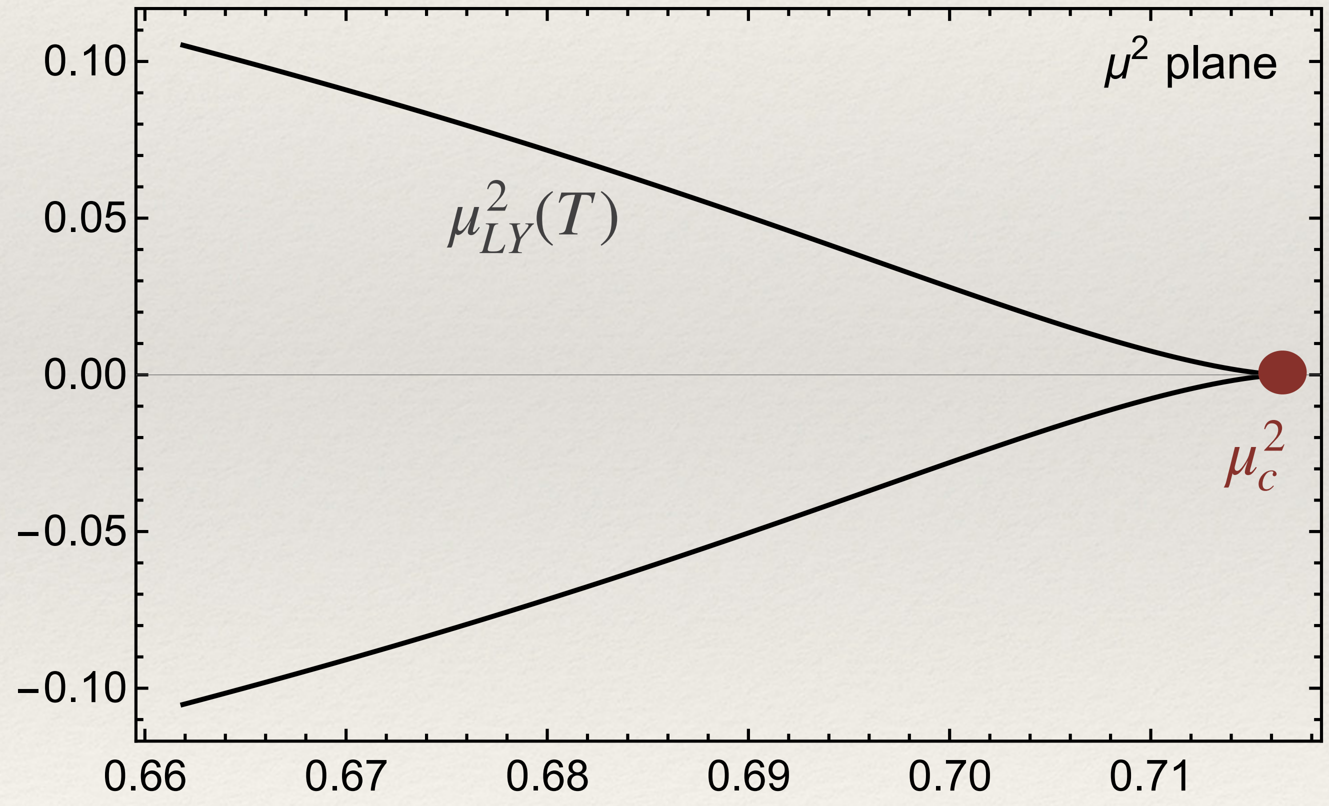
Lee-Yang edge singularities

- The equation of state has complex singularities
- Zeroes of partition function $\mathcal{Z}(\zeta)$ $(\zeta = e^{\mu/T}$: fugacity)
- Coalesce into branch cuts in thermodynamic limit
- Pinch the real axis at a second order transition

[Lee-Yang, 52′]



[See talk by Skokov]



In the context of QCD see eg. [Halasz, Jackson Verbaarshot ’97, Ejiri ’05, Stephanov ’06, Mukherjee, Skokov ’20,…]

Lee Yang edge singularity

- The scaling e.o.s, $f_s(w)$, has singularities at $w = \pm i w_{LY}$ $(w := hr^{-\beta\delta})$

$$\mu_{LY}(T) \approx \mu_c - \frac{h_T}{h_\mu}(T - T_c) \pm i w_{LY} \frac{\left(\det \mathbb{M}\right)^{\beta\delta}}{h_\mu^{\beta\delta+1}}(T - T_c)^{\beta\delta}$$

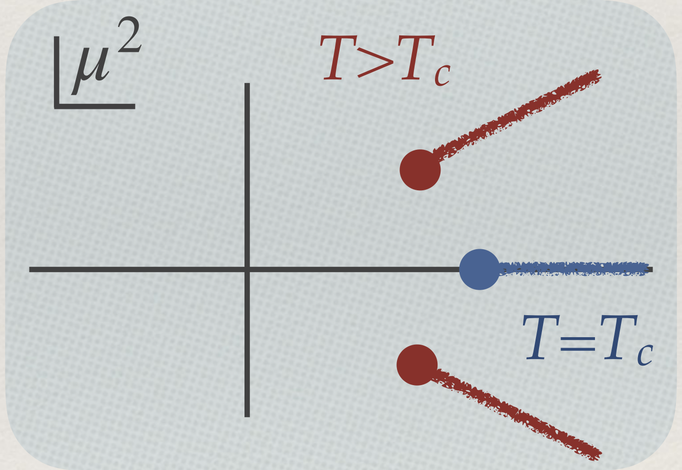
$(\tan \alpha_1)^{-1}$

slope of the crossover line

$\det \mathbb{M} \propto (\tan \alpha_2 - \tan \alpha_1)$

relative angle between r, h axes

see [Pradeep, Stephanov '19]



- The e.o.s. near the LY singularity: $M(w) \sim (w \pm i w_{LY})^{\sigma_{LY}}$, $(M$: magnetization)

$\sigma_{LY,d=3} \approx 0.1$, $\sigma_{LY,d=6} = 1/2$ (mean field)

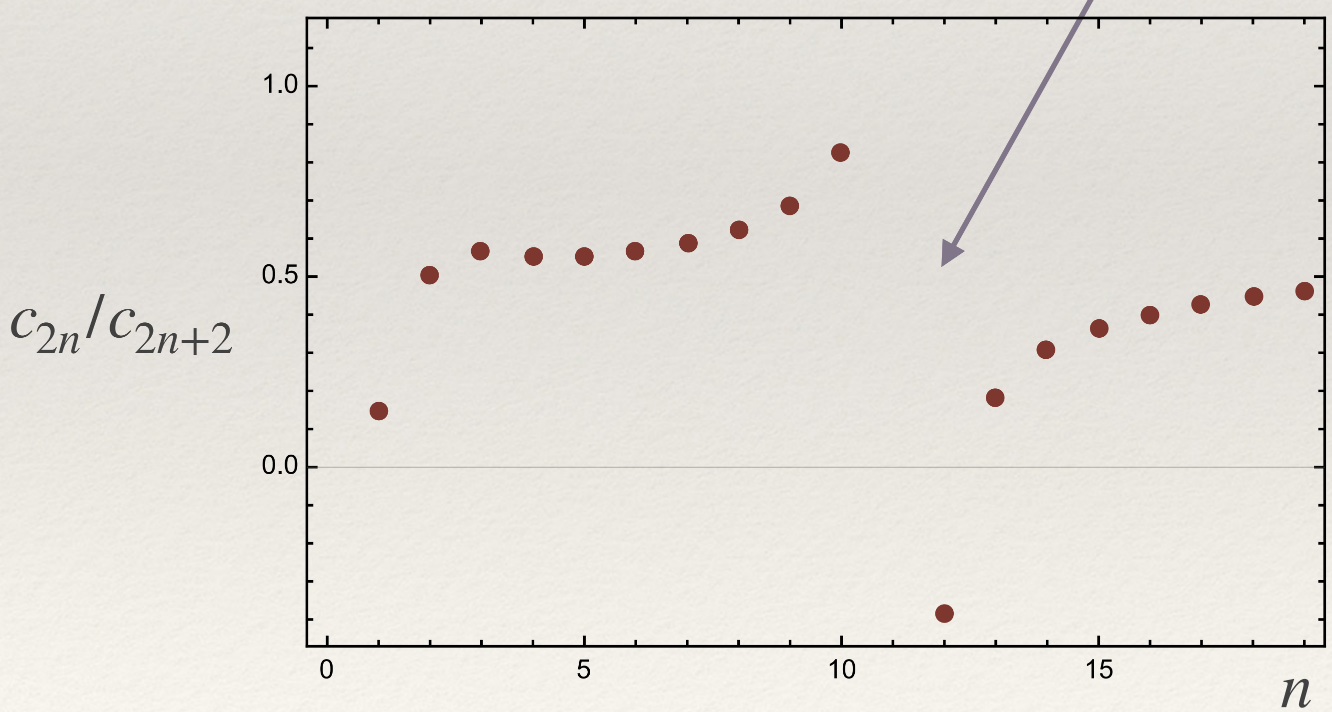
[Fisher, '74; An, Stephanov, Mesterházy '16; Connelly, Johnson, Mukherjee, Skokov '20]

When life gives you Taylor series...

Large order behavior e.g. $\chi(T,\mu) = \sum_n c_{2n}(T)\mu^{2n} \to c_{2n}/c_{2n+2} \sim |\mu_{LY}^2|, \quad n \to \infty$

- Singularities are complex conjugate pairs: *interference* effect in large order behavior
- Difficult to extract the radius of convergence

$$c_{2n} \sim \frac{\Gamma(\sigma+n)}{\Gamma(1+n)\,|\mu_{LY}^2|^{n+\sigma_{LY}}} \cos(\theta(n+\sigma_{LY}) - \pi\sigma), \quad (\theta := \arg \mu_{LY}^2)$$



We can do better!

When life gives you Taylor series...

Large order behavior e.g. $\chi(T,\mu) = \sum_n c_{2n}(T)\mu^{2n} \to c_{2n}/c_{2n+2} \sim |\mu_{LY}^2|, \quad n \to \infty$

We can do better!

- I - Extract the location of $\mu_{LY}(T)$
- II - Reconstruct the equation of state beyond the radius of convergence
- III - Reconstruct the equation of state beyond the leading Riemann sheet (globally!)

When life gives you Taylor series...

Taylor series:

$$\chi(\mu, T) = \sum_{n=0}^{N} c_{2n}(T)\mu^{2n}$$

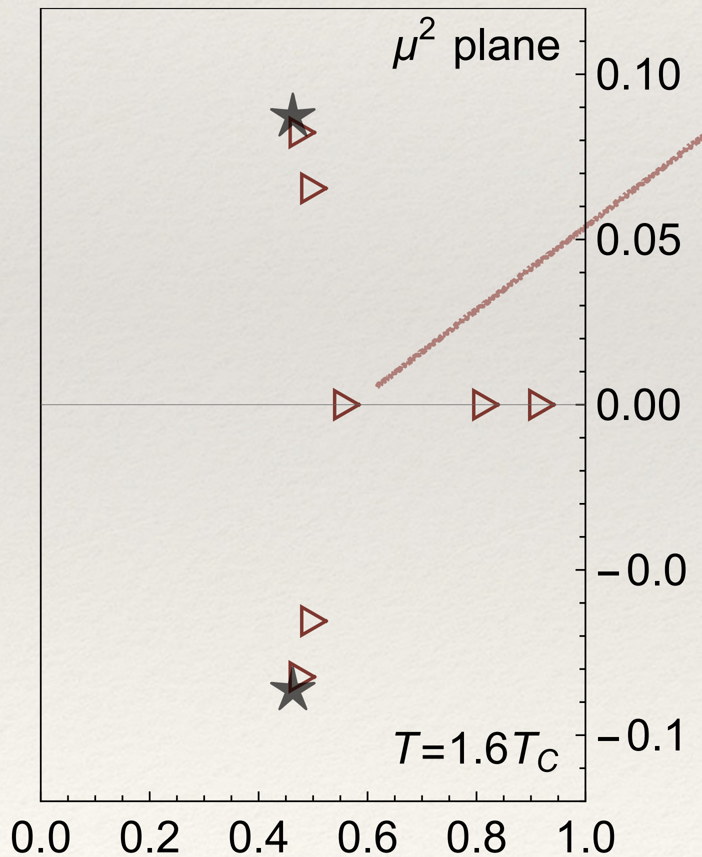
Padé approximant (diagonal)

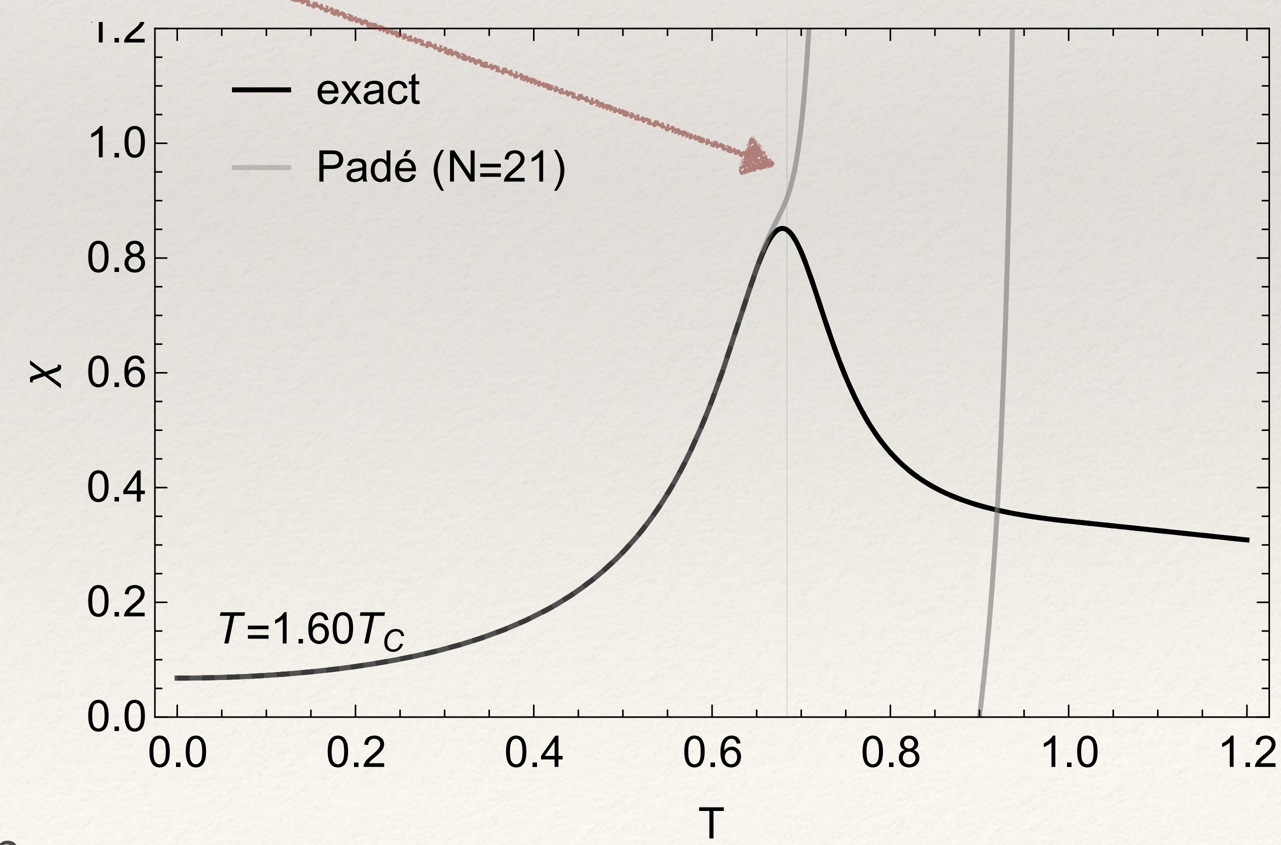
$$\mathrm{P}\chi(\mu, T) = \frac{p_0(T) + p_1(T)\mu^2 + \ldots + p_{N/2}(T)\mu^N}{q_0(T) + q_1(T)\mu^2 + \ldots + q_{N/2}(T)\mu^N}$$

Unphysical poles

Padé cannot reconstruct the susceptibility beyond the radius of convergence!

$(\mu^2 \gtrsim |\mu_{LY}^2|)$



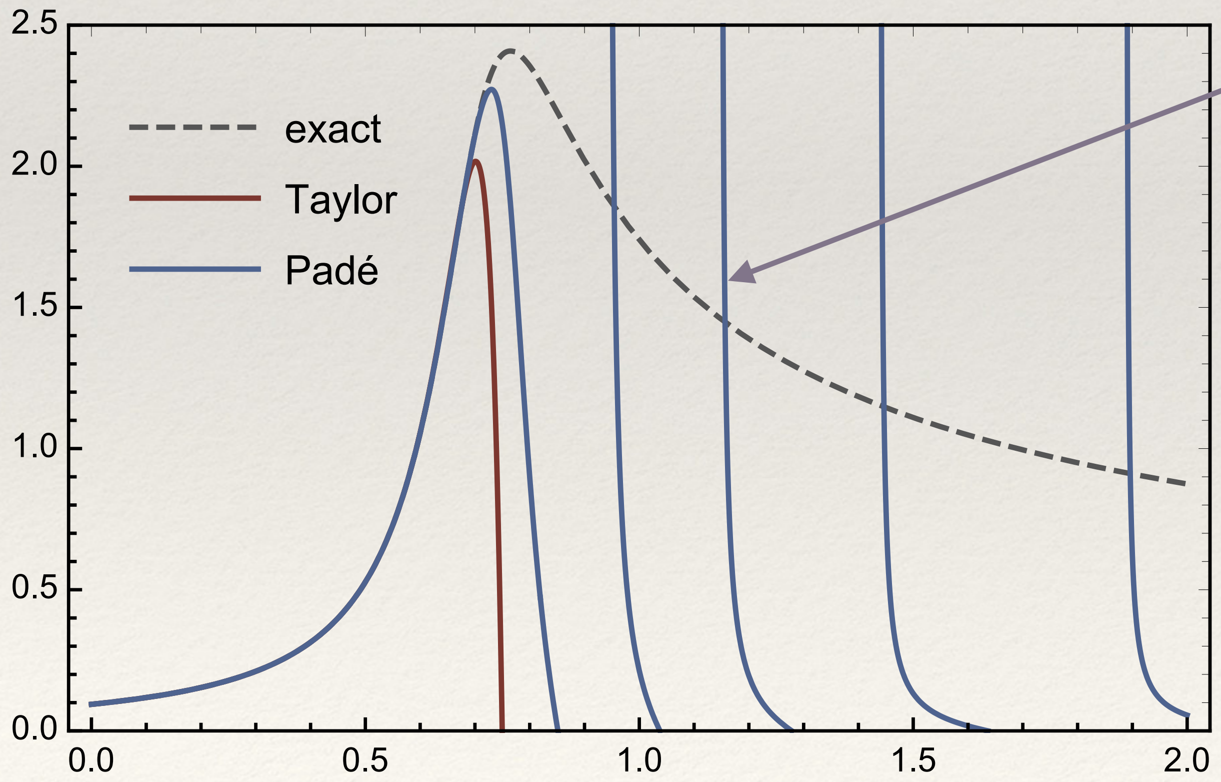


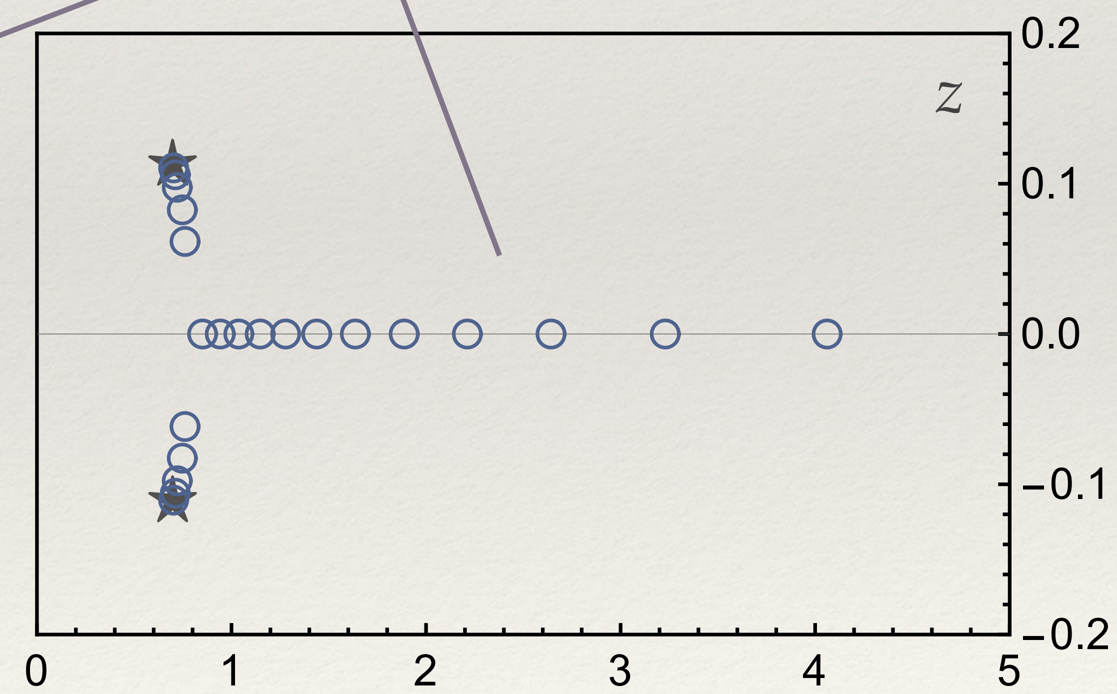
When life gives you Taylor series...

Spurious poles are unavoidable in Padé when there are conjugate pair of singularities…

$$\text{e.g.} \quad f(z) = \frac{1}{2}\left(\frac{1}{\sqrt{z - z_c}} + \frac{1}{\sqrt{z - z_c^*}}\right)$$

Spurious poles

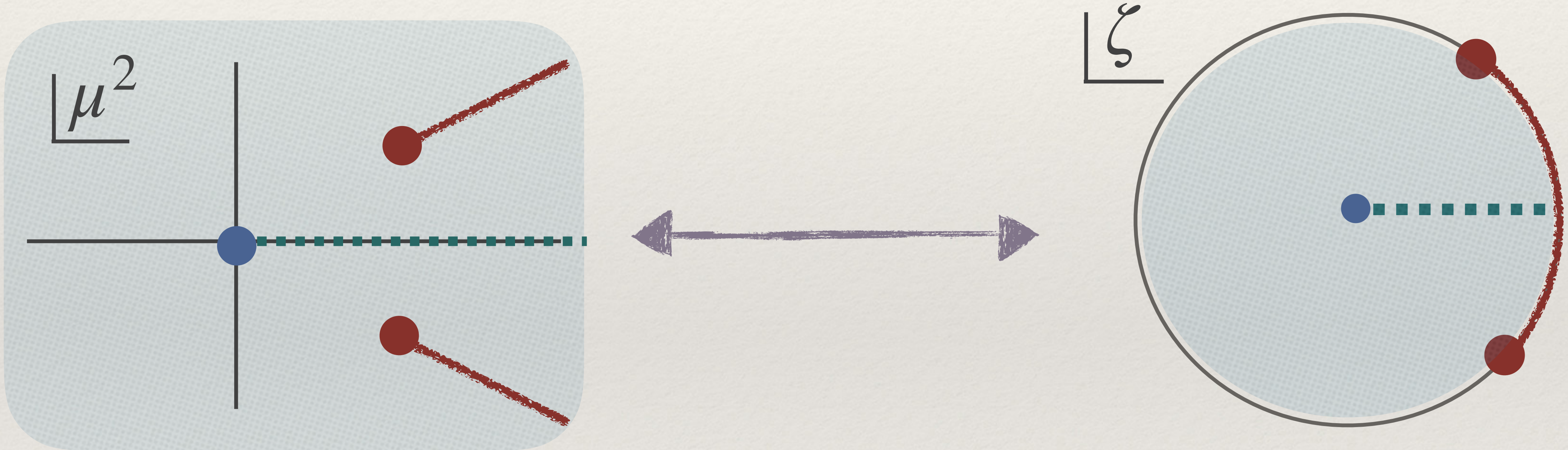




Conformal Maps

We can still do better!

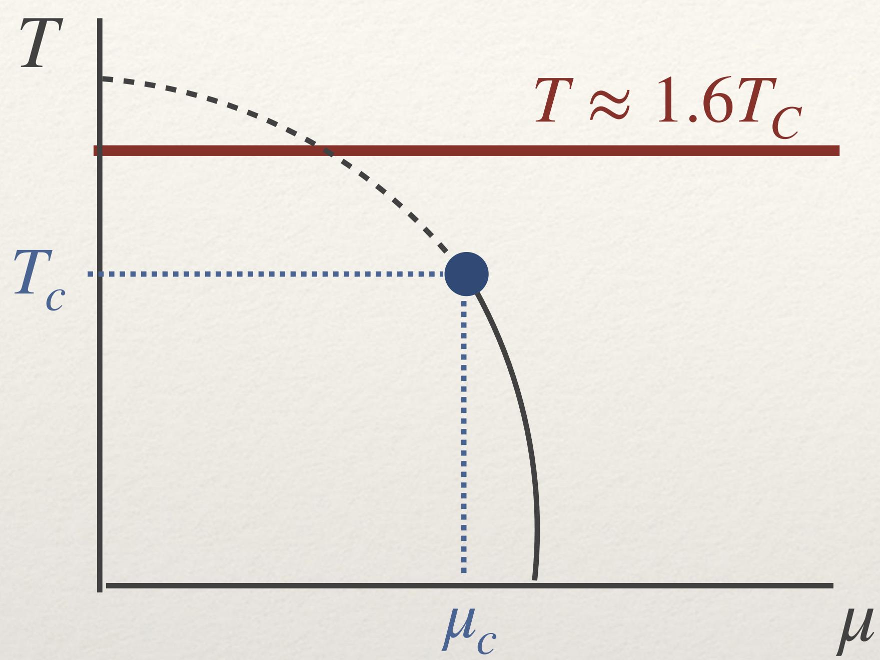
Conformal map $\phi(\zeta) = \left(\frac{\theta}{\pi}\right)^{\theta/\pi}\left(1-\frac{\theta}{\pi}\right)^{1-\theta/\pi}\frac{4\mu_{LY}^2\zeta}{(1+\zeta)^2}\left(\frac{1+\zeta}{1-\zeta}\right)^{2\theta/\pi}$

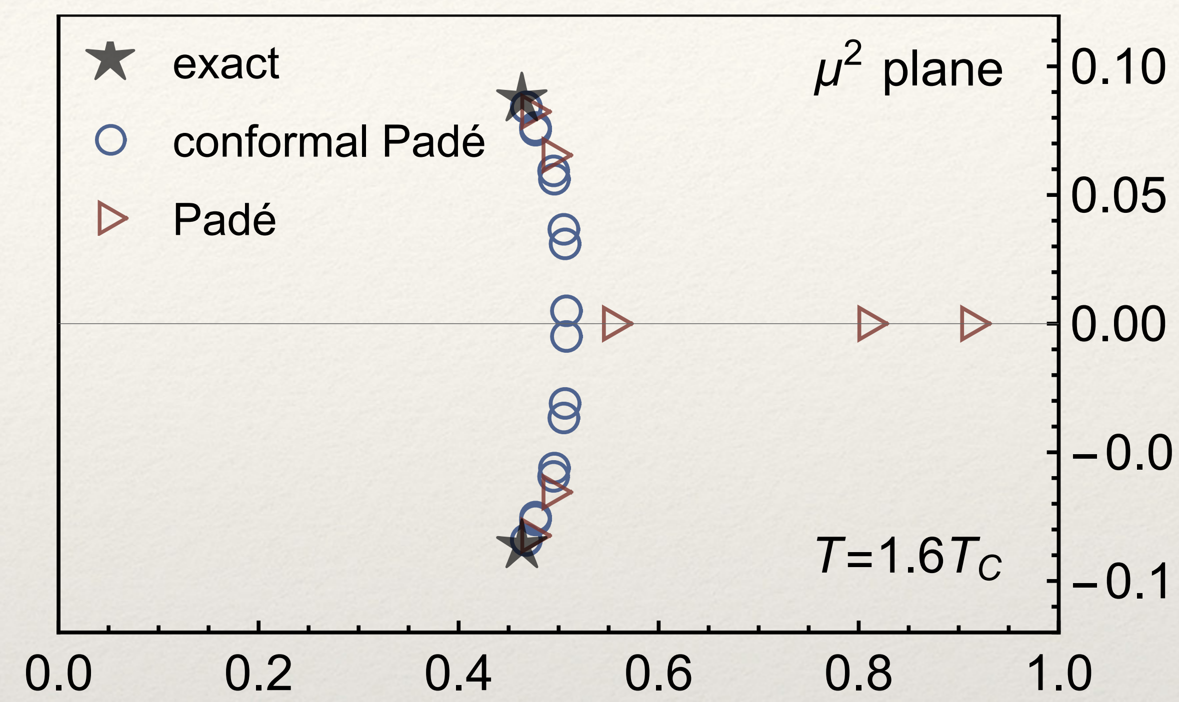


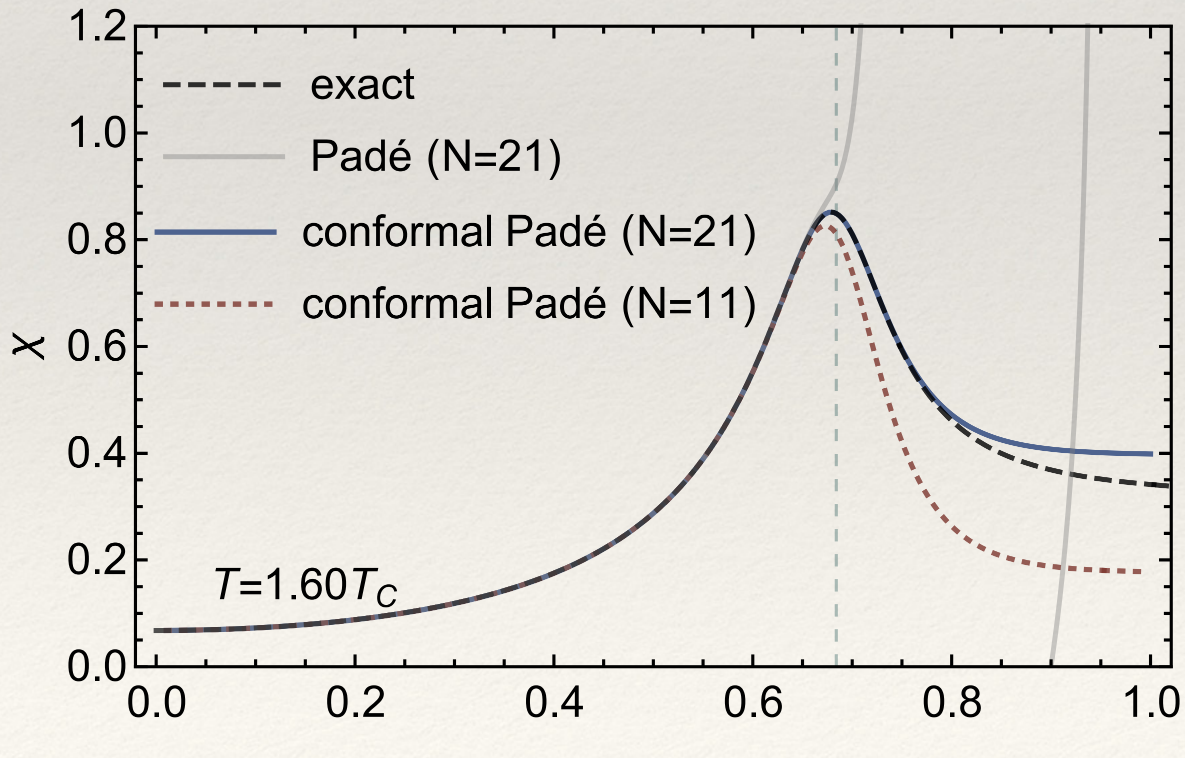
Do Padé resummation in the ζ plane (unit circle)

"conformal Padé" $$\mathrm{P}\chi(T,\phi(\zeta)) = \left.\frac{\tilde{p}_0(T)+\tilde{p}_1(T)\zeta+\ldots+\tilde{p}_{N/2}(T)\zeta^N}{\tilde{q}_0(T)+\tilde{q}_1(T)\zeta+\ldots+\tilde{q}_{N/2}(T)\zeta^N}\right|_{\zeta=\phi^{-1}(\mu^2)}$$

Results: susceptibility

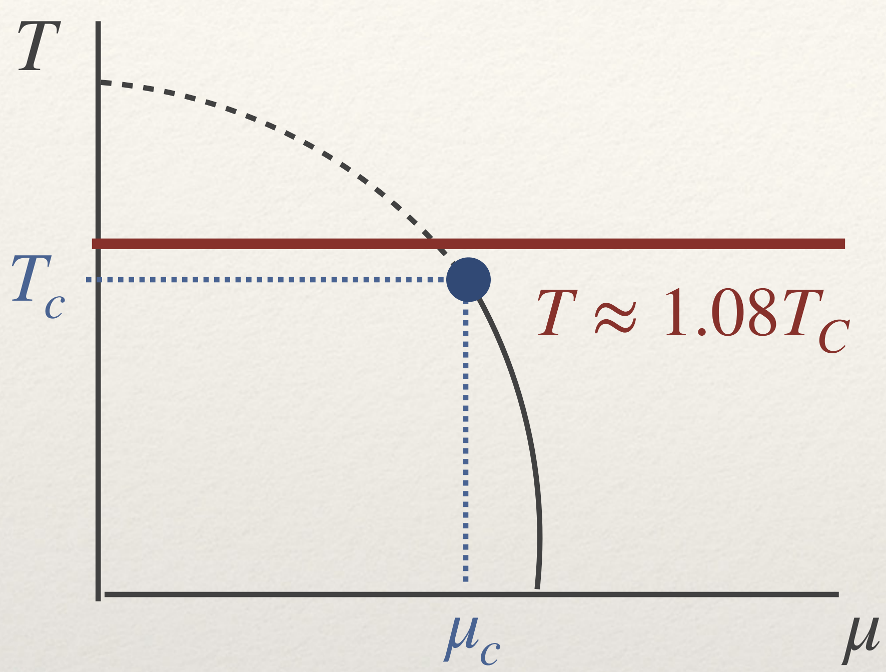


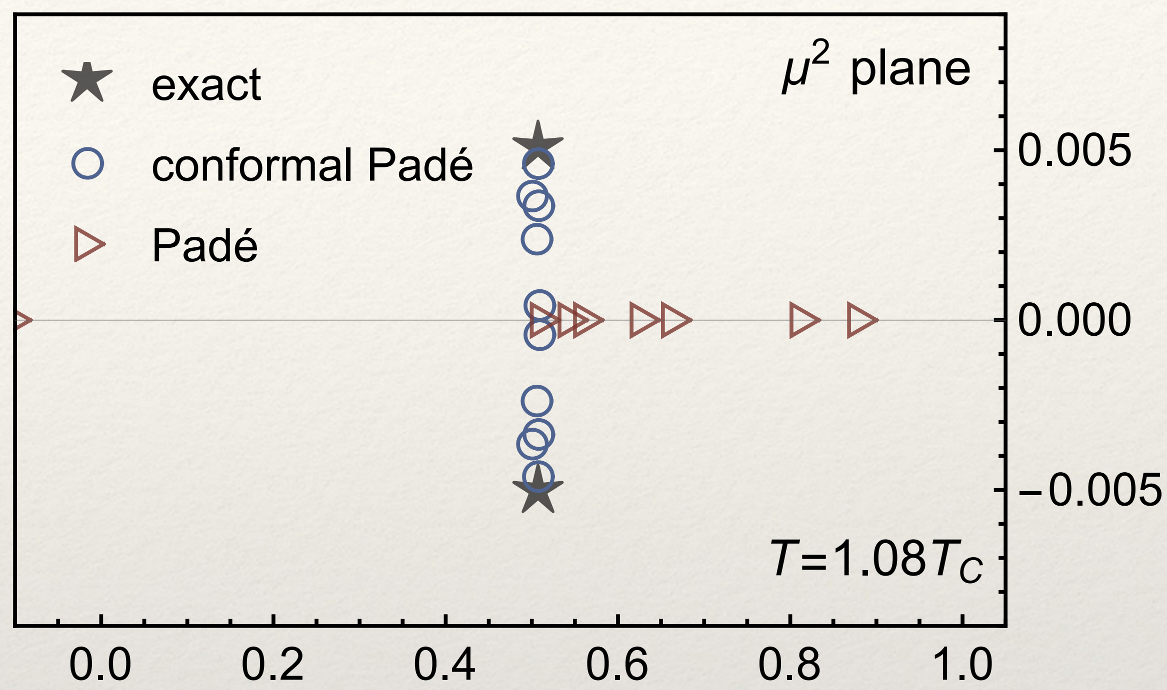


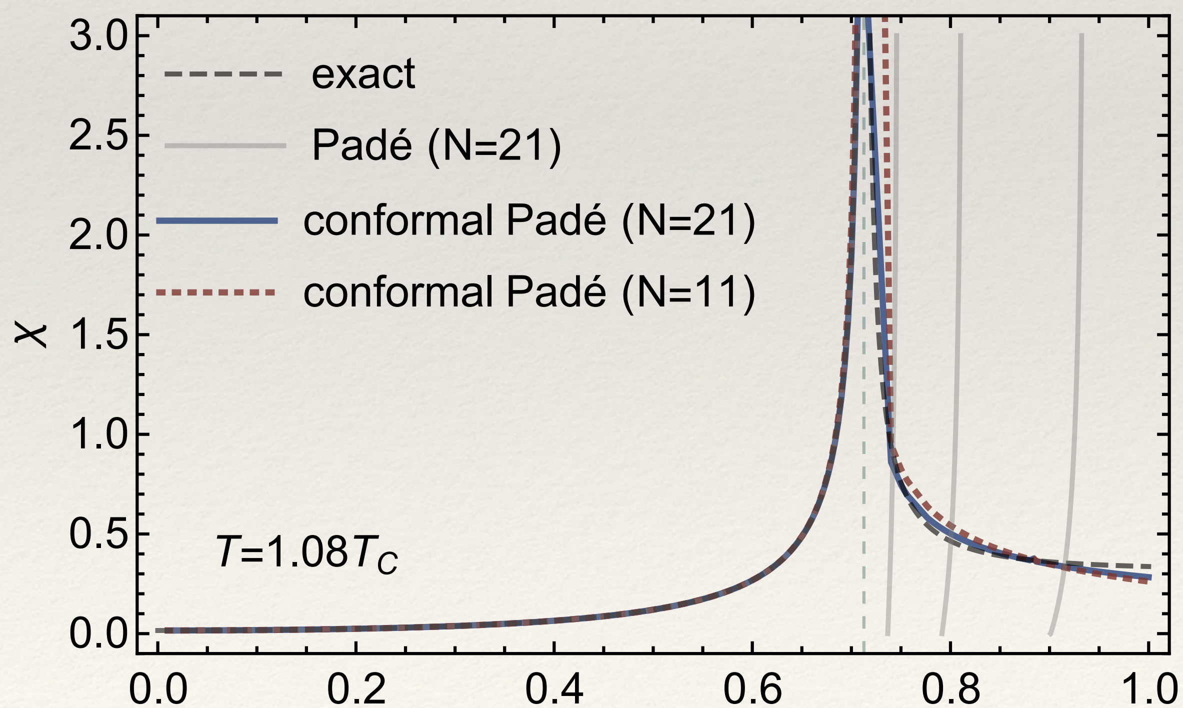


conformal Padé does not introduce unphysical poles on the real axis!

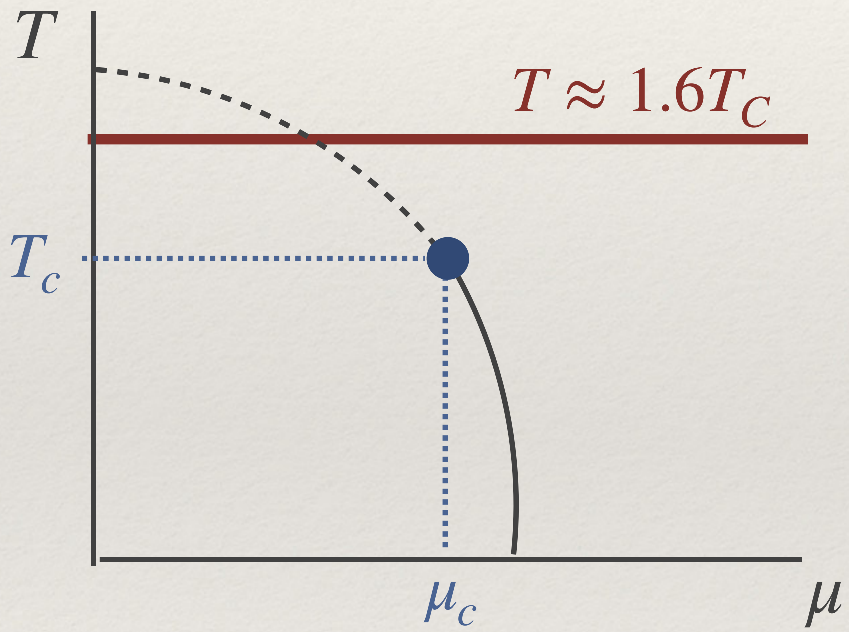
Results: susceptibility

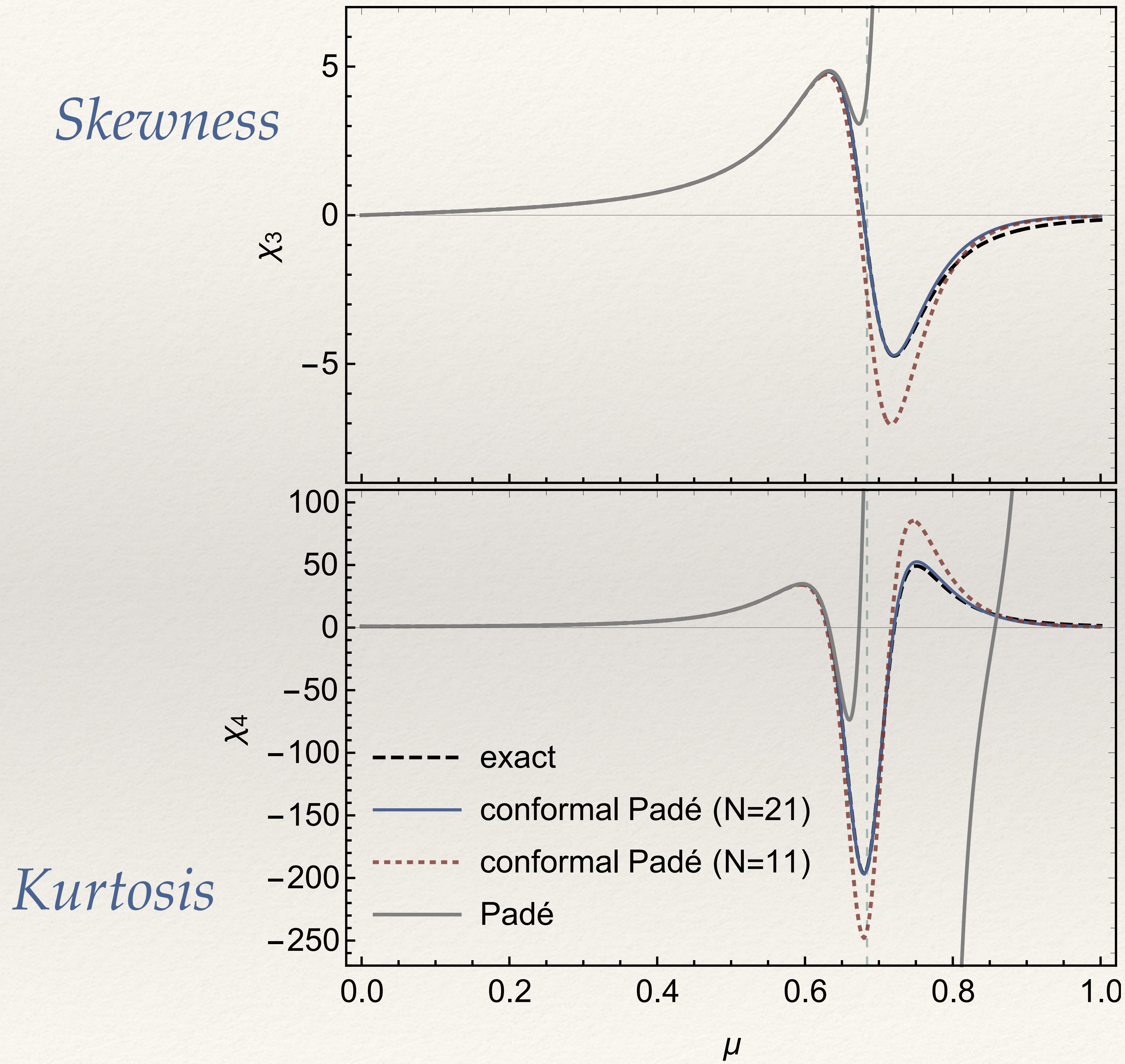






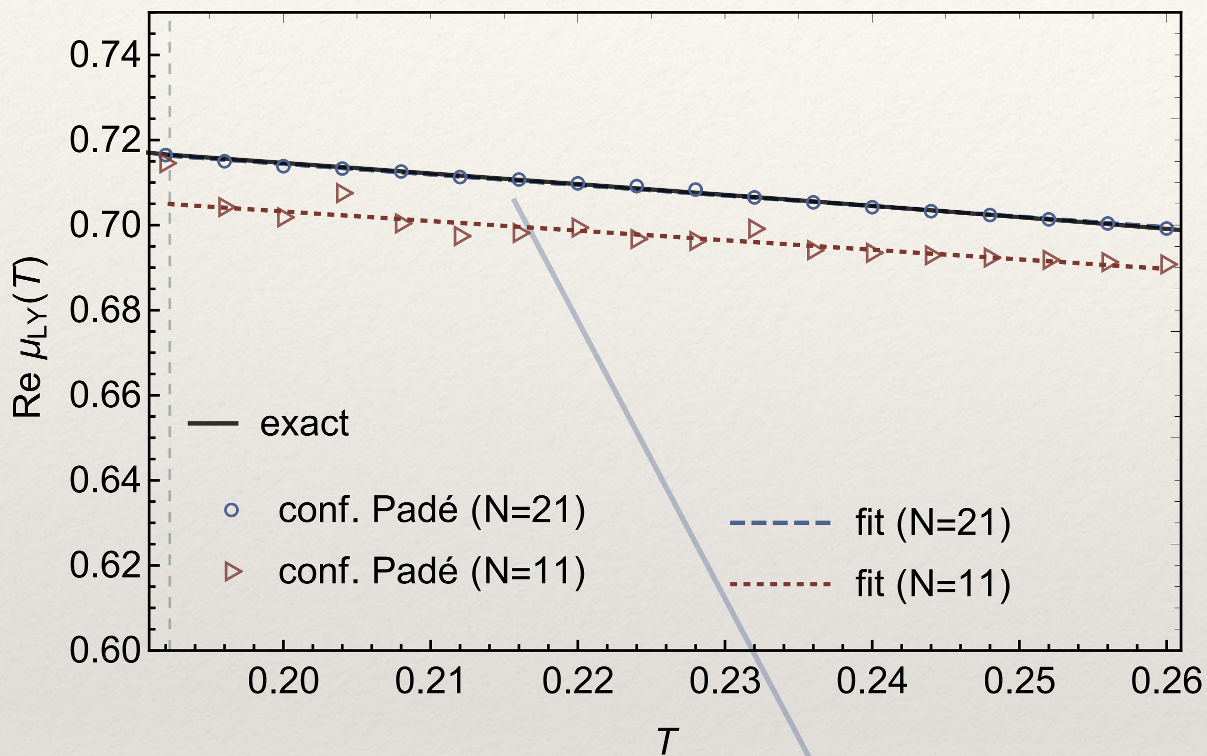
Results: skewness, kurtosis

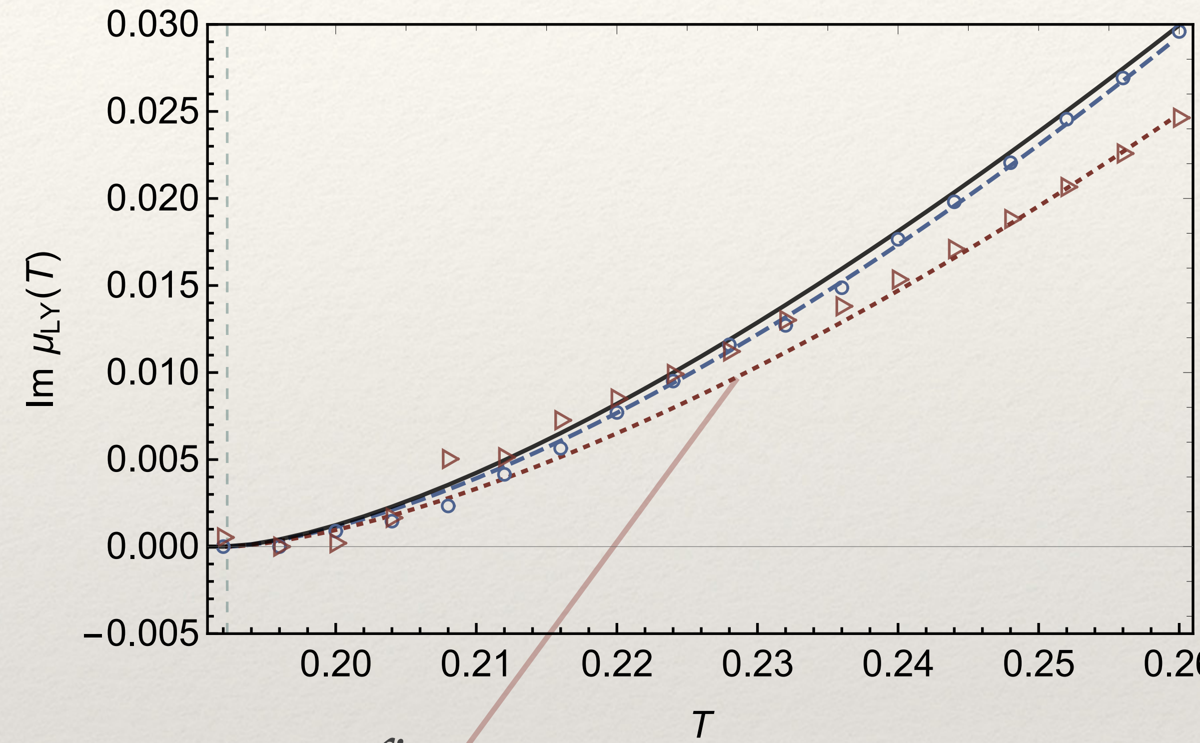




Results: Lee-Yang trajectory

- Find $\mu_{LY}^2(T)$ from conformal- Padé for different temperatures





fit *fit*

$$\mu_{LY} \approx \mu_c - \frac{h_T}{h_\mu}(T - T_c) + i w_{LY} \frac{r_\mu^{3/2}}{h_\mu}\left(\frac{r_T}{r_\mu} - \frac{h_T}{h_\mu}\right)^{3/2} (T - T_c)^{3/2} \qquad w_{LY} = \frac{2}{3\sqrt{3}}$$

- Extract $\mu_c, T_c,$ crossover slope, $\frac{h_T}{h_\mu}$, and $\frac{r_\mu^{3/2}}{h_\mu}\left(\frac{r_T}{r_\mu} - \frac{h_T}{h_\mu}\right)^{3/2}$

Results

$$\mu_{LY}(T) \approx \mu_c - \frac{h_T}{h_\mu}(T - T_c) + i w_{LY} \frac{r_\mu^{3/2}}{h_\mu}\left(\frac{r_T}{r_\mu} - \frac{h_T}{h_\mu}\right)^{3/2} (T - T_c)^{3/2}$$

$$w_{LY} = \frac{2}{3\sqrt{3}}$$

	T_c	μ_c	h_T/h_μ	c
exact	0.192	0.717	0.249	4.684
conf. Padé (N=21)	0.195	0.716	0.248	4.323
conf. Padé (N=11)	0.185	0.707	0.225	3.666

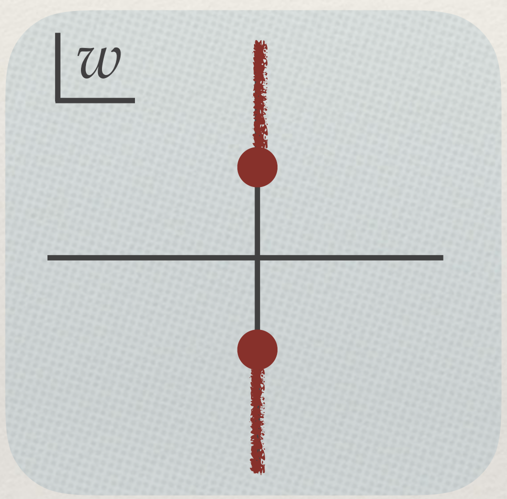
Crossing the branch cuts…

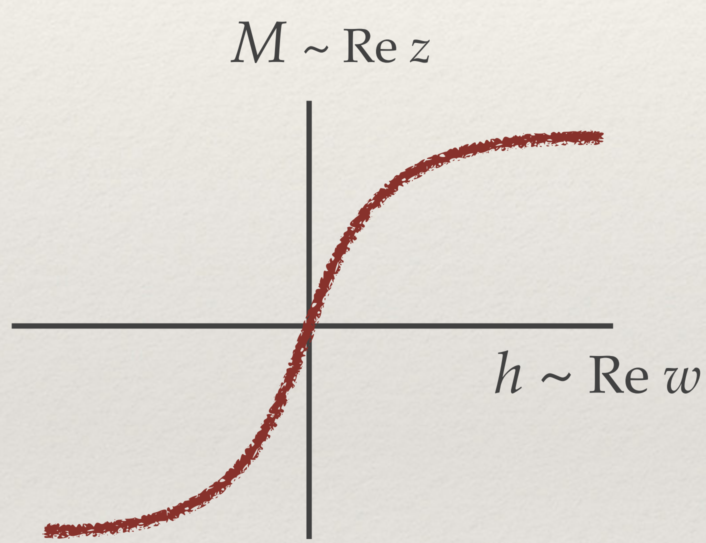
$w = hr^{-\beta\delta}$
$z = Mr^{-\beta}$

Ising model: $w = F(z)$

$F(z) = z + z^3$ (mean field)

High Temperature (T>T$_c$)





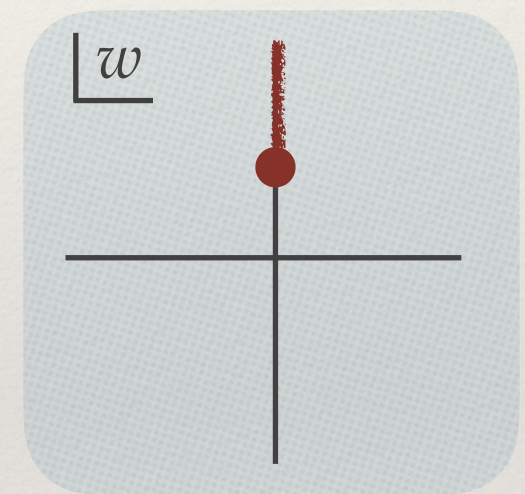
high T sheet
r>0

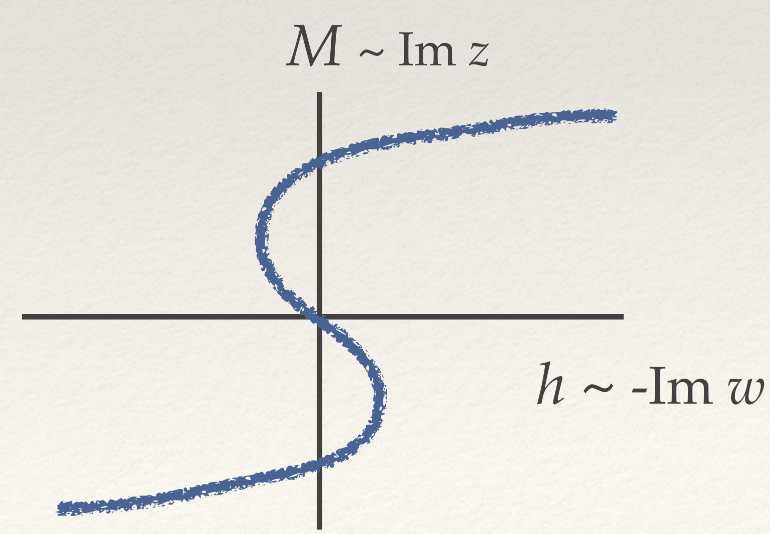
$z(w) = w - w^3 + 3w^5 - 12w^7 + \ldots$

high T expansion

Low Temperature (T<T$_c$)

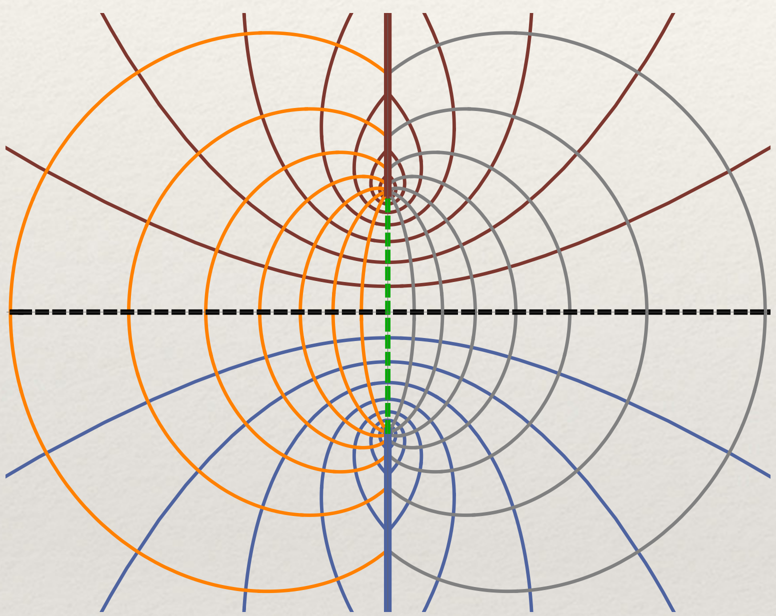
low T sheet
r<0, h>0





Uniformization

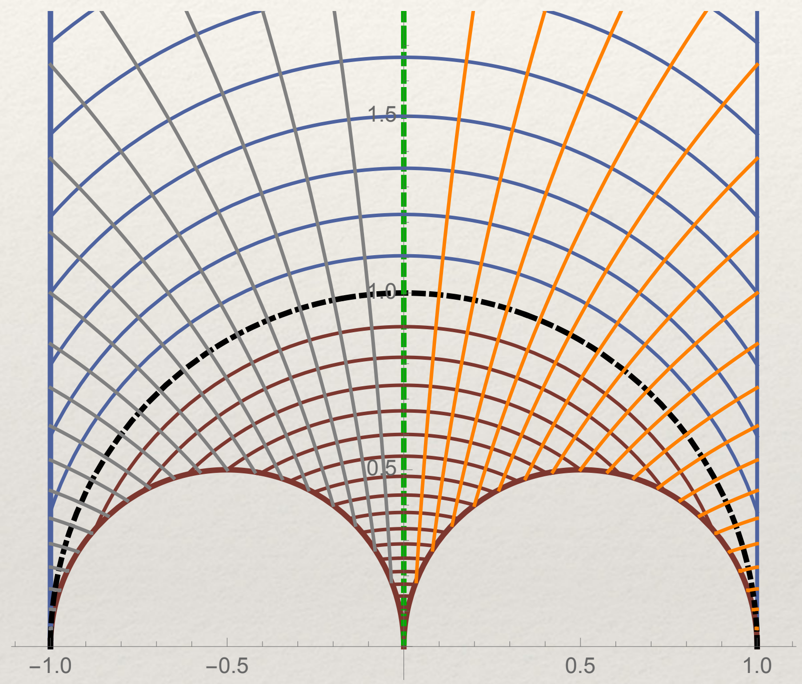
w plane



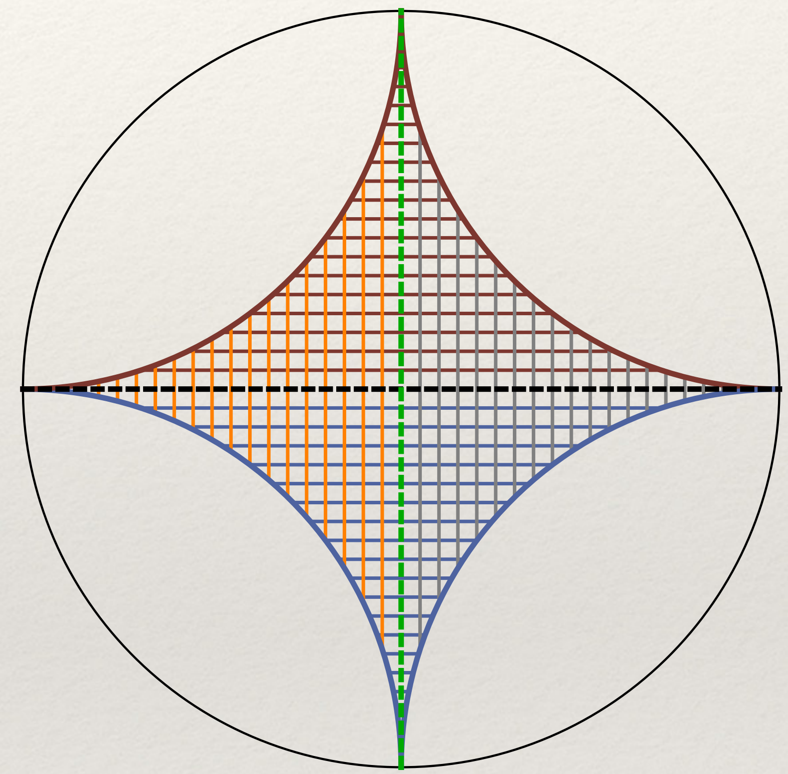
high T sheet

r>0

τ plane



ζ plane



$$w \to w(\tau) = i(-1 + 2\lambda(\tau))$$

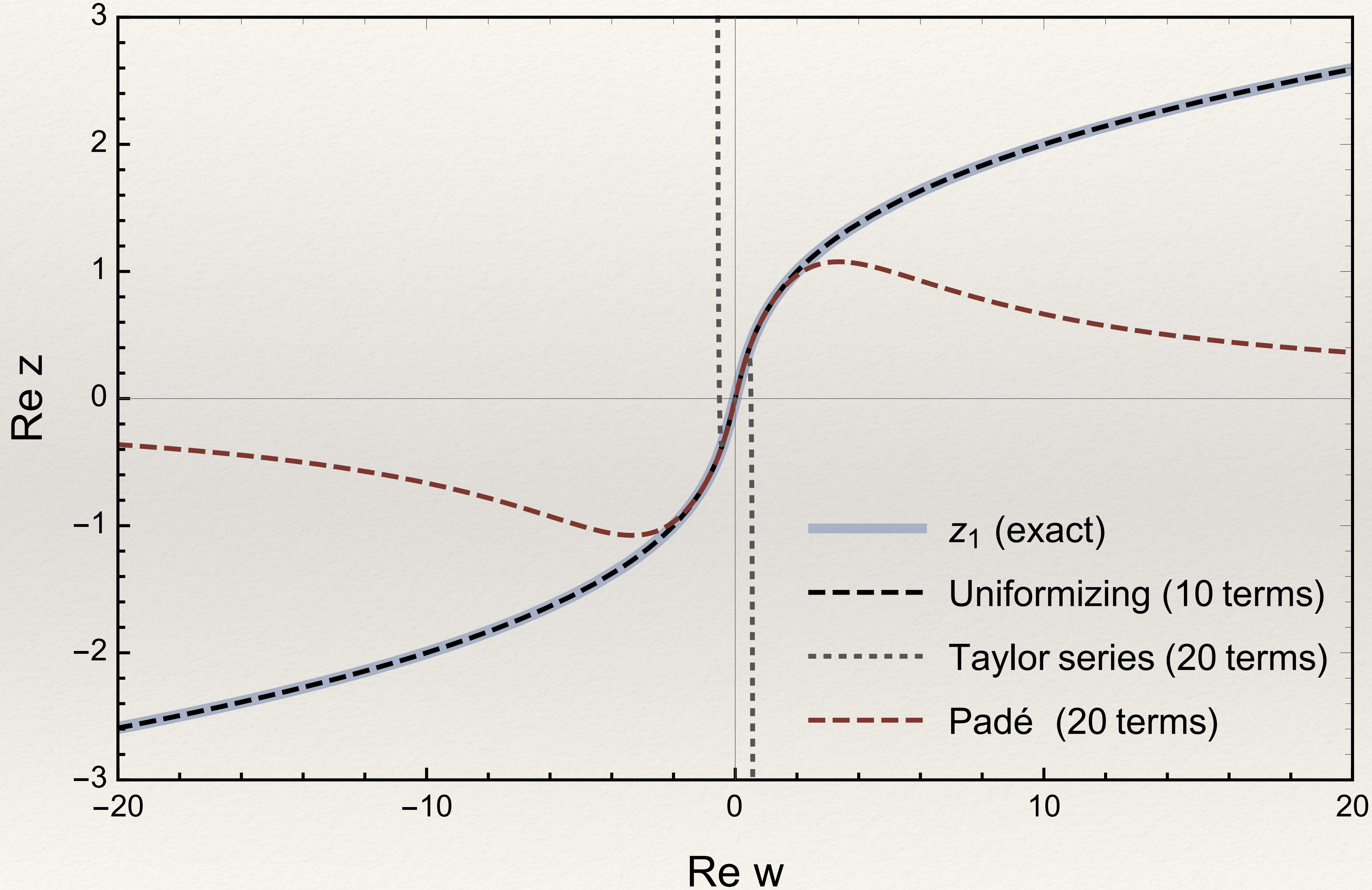
$$\tau(\zeta) = i\left(\frac{1 + i\zeta}{1 - i\zeta}\right)$$

$$\lambda(\tau) = \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}$$ (elliptic modular function)

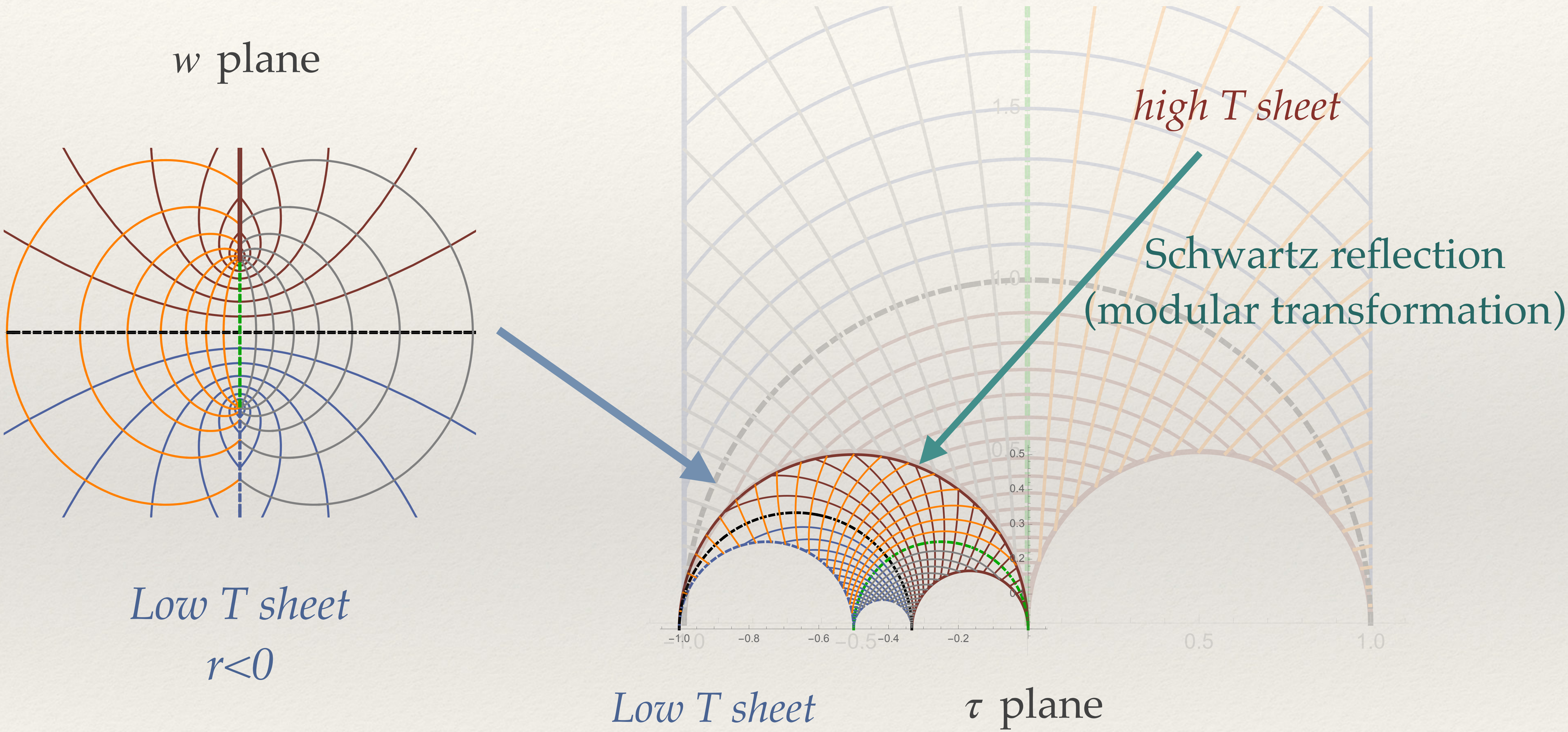
$$\theta_2(\tau) = \sum_{n=1}^{\infty} e^{2\pi i\tau(n+1/2)^2}, \quad \theta_3(\tau) = \sum_{n=1}^{\infty} e^{2\pi i\tau n^2}$$

Uniformization

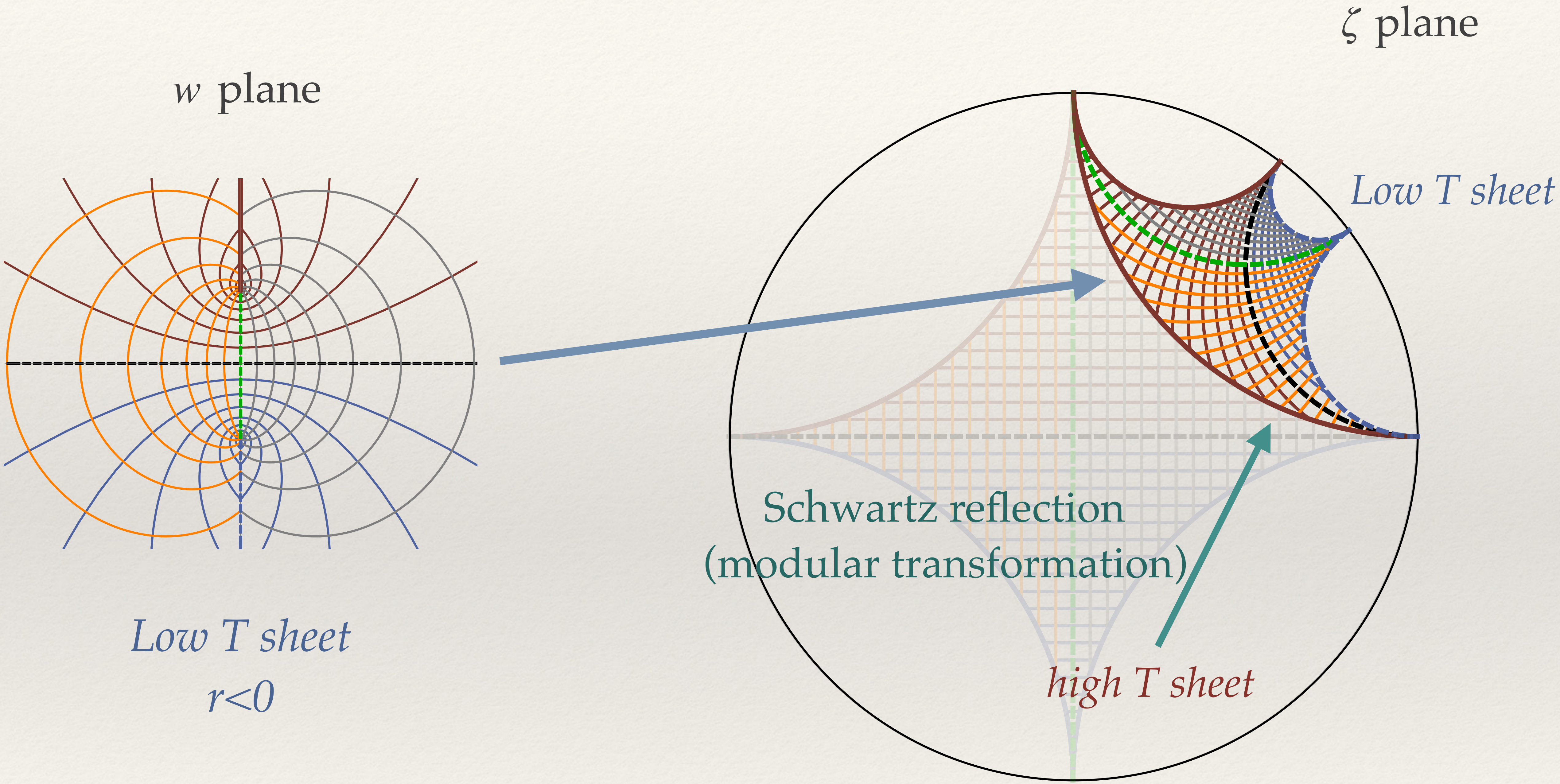
High T



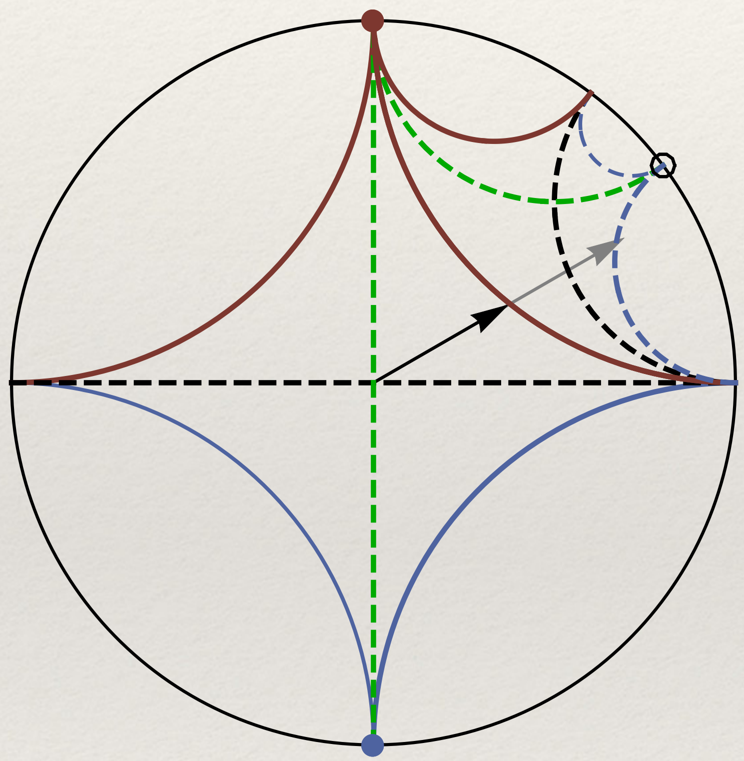
Uniformization

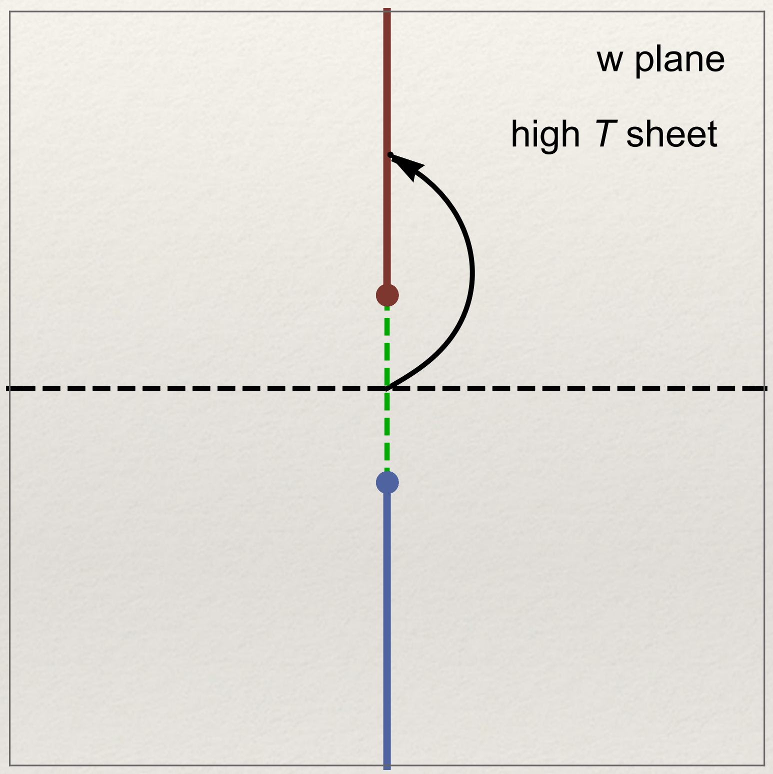


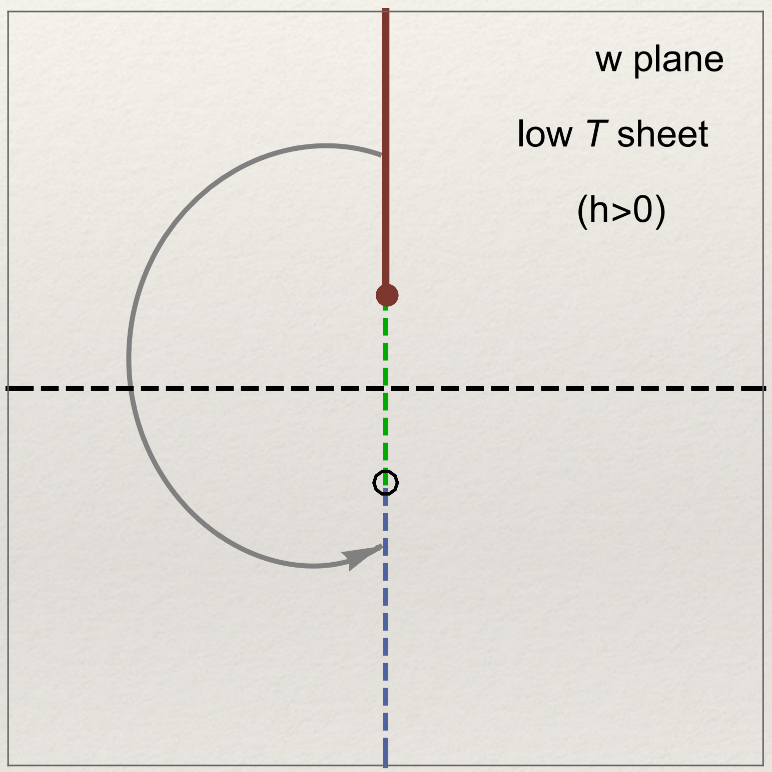
Uniformization



Uniformization: higher Riemann sheets

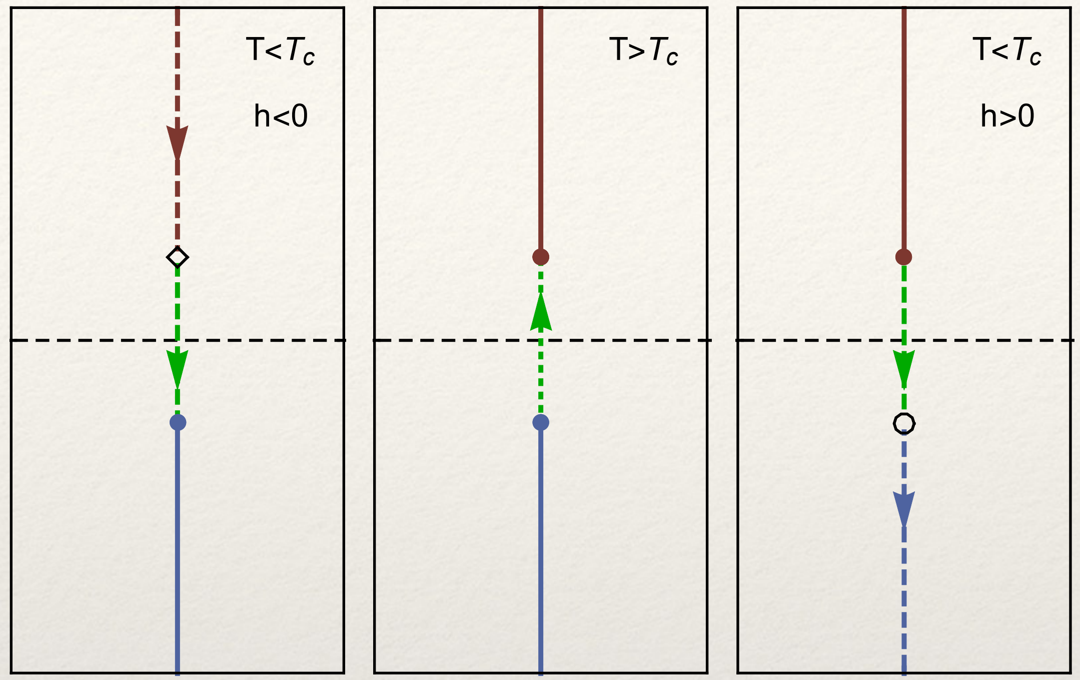


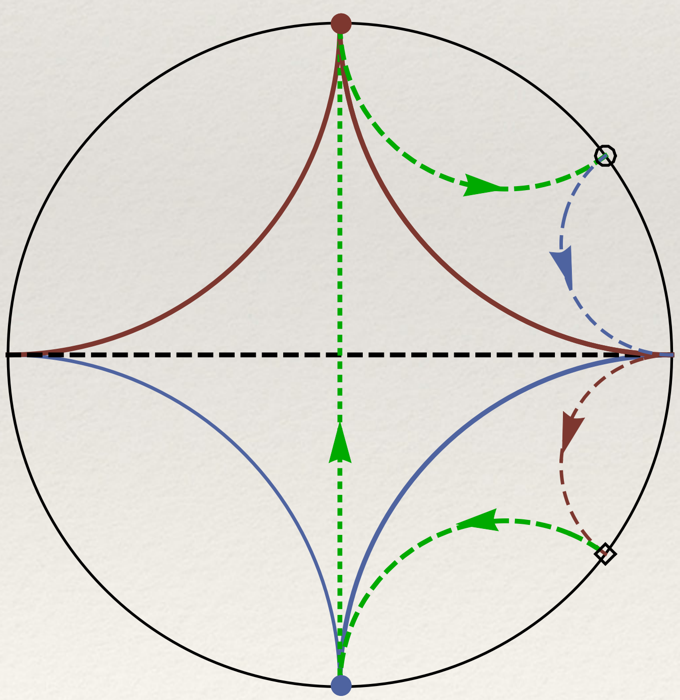


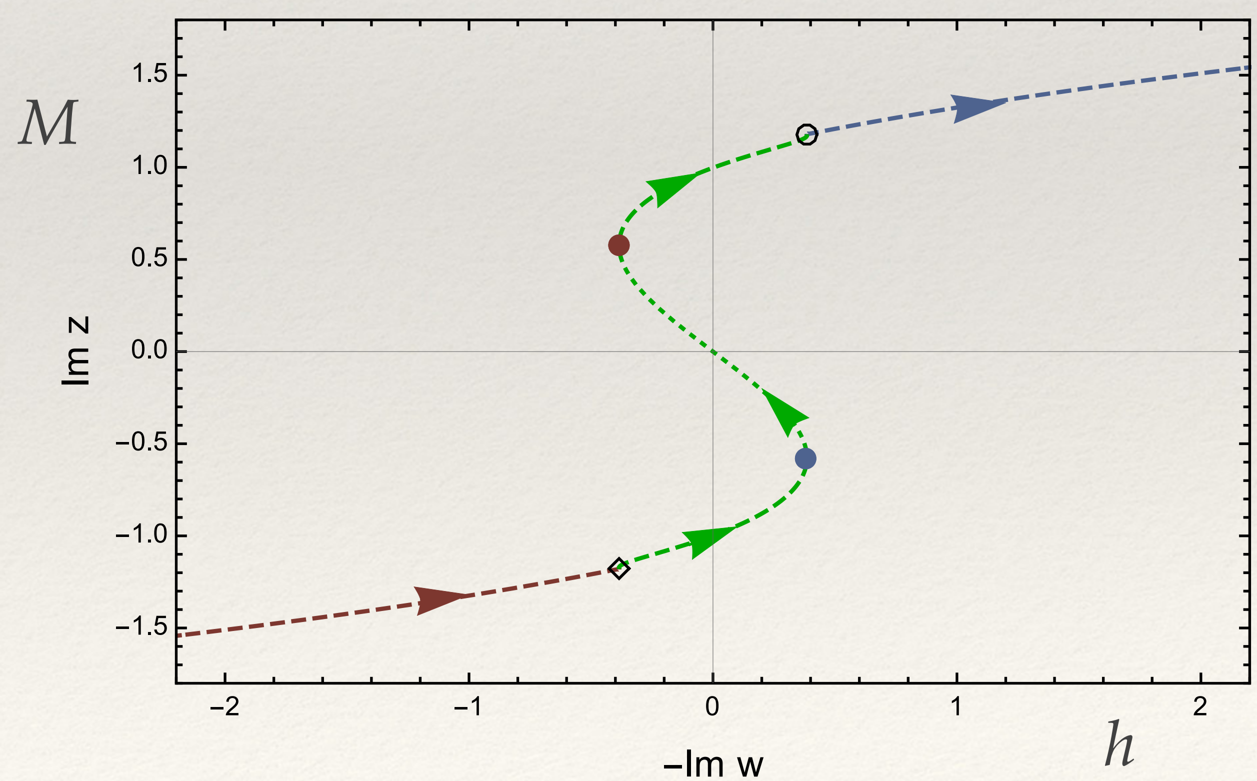


Moving within unit circle (smooth) ⟷ *Jumping through sheets*

Uniformization: higher Riemann sheets

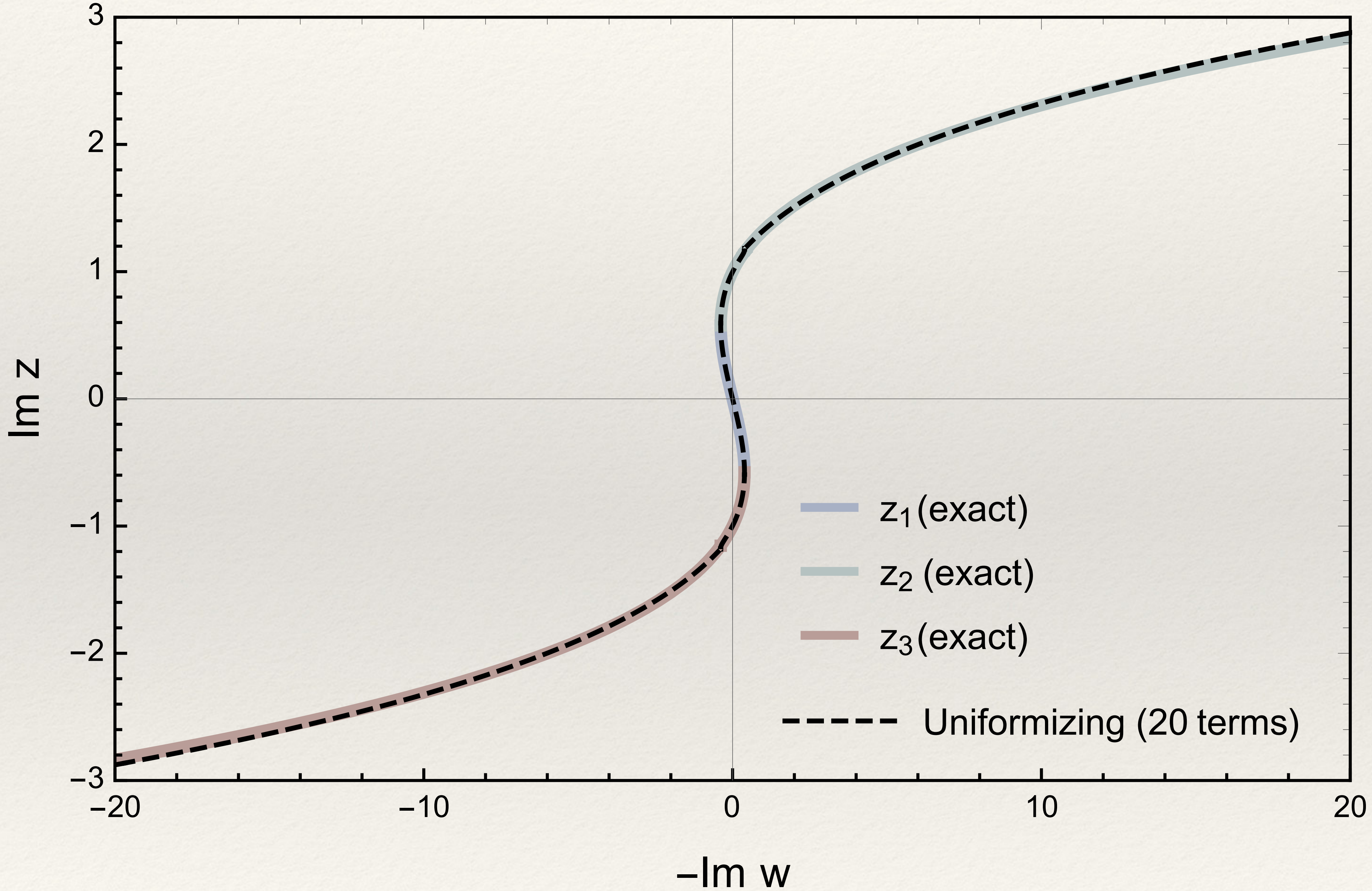






Uniformization

Low T



Conclusions

- Combined with conformal maps, Padé approximants provide a powerful tool to extract information from truncated Taylor series.

- In the crossover region by using this tool it is possible to pin down the location of the *Lee-Yang edge singularity* and also extract information on the *mapping parameters to critical Ising e.o.s.*

- Conformal Padé gives a significantly better approximation to the e.o.s than than the Taylor series, going beyond the radius of convergence .

- Uniformizing map allows one to reconstruct the e.o.s globally.
Physically: crossover to 1st order region.

- All of this comes with no more cost than the Taylor series!

Outlook

- Beyond mean field

- Numerical uncertainties in coefficients

- Extrapolation from imaginary μ, pairing with other resummation schemes [with V. Skokov, F. Rennecke]
  [e.g. Ratti et al '21-22, Mukherjee et al '22,…]

- Singularity elimination