Turbulence modelling

A game we have no choice but to play…

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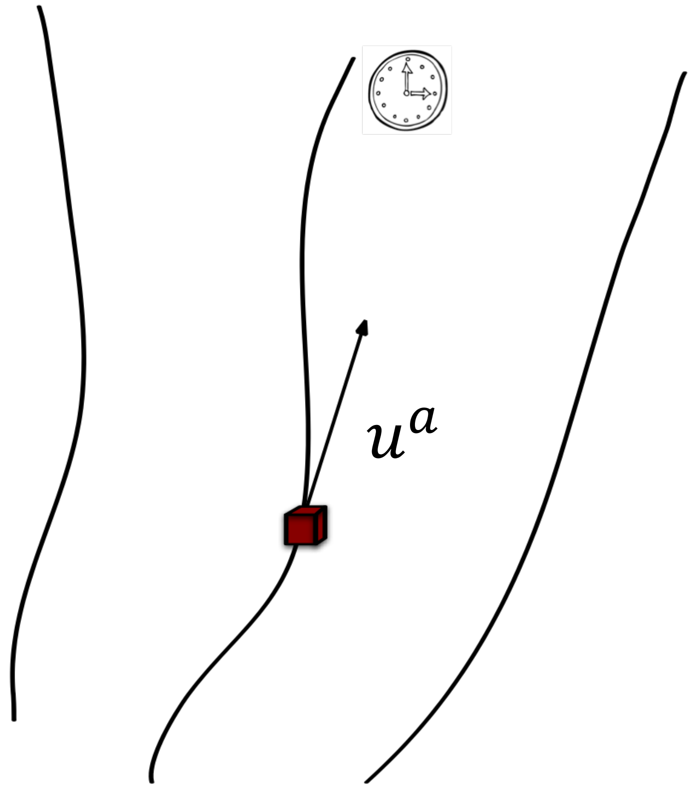
Neutron star merger dynamics, essentially an exercise in relativistic fluid dynamics.

Easy to formulate: Need to solve

$$G_{ab} = \frac{8\pi G}{c^4} T_{ab}$$

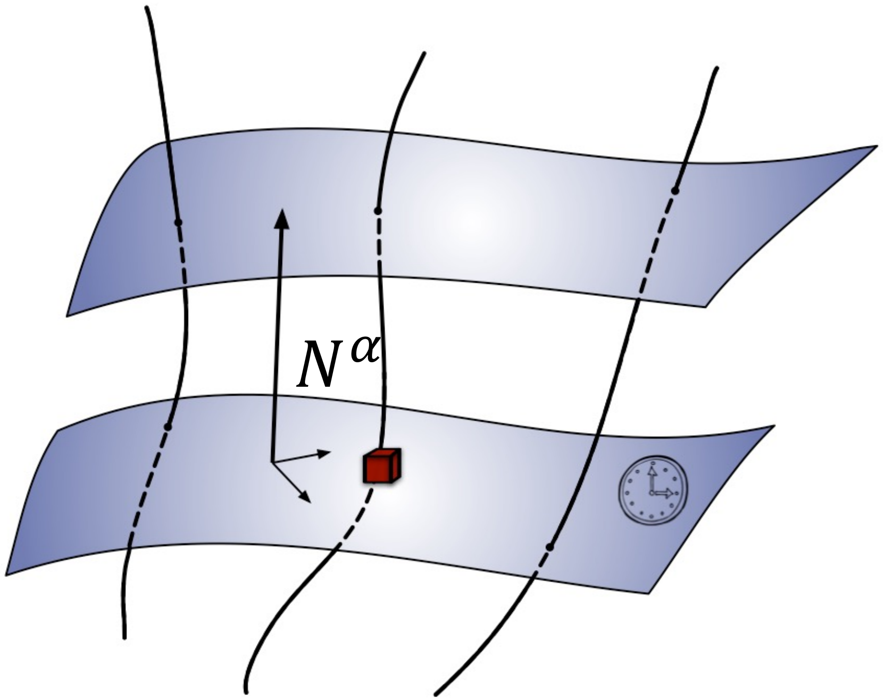
for the spacetime metric g_{ab} and the fluid four velocity u^a, making the input physics as "realistic" as possible.



The four-velocity describes the motion of individual fluid elements through spacetime.

In a coordinate frame moving with the fluid, u^a measures the progression of (proper) time and allows us to "fibrate" spacetime.

But this fibration is not "useful" for simulations…



For simulations we need a **foliation** of spacetime. Split space and time in such a way that we can tell a computer to march forwards one step at a time:

$$u^a = \gamma(N^a + v^a) \quad \text{with} \quad N^a v_a = 0$$

Key point: The variables we use for the evolution are not the ones from the microphysics. We need to "translate" back and forth.

As a starting point, consider a **perfect fluid** (no heat flow etcetera) for which we have

$$T^{ab} = (p + \varepsilon)u^a u^b + p g^{ab}$$

with energy density ε and pressure p.

To close the system, we need an **equation of state**; a closure relation $p = p(\varepsilon, \dots)$ which incorporates the appropriate micro-physics.

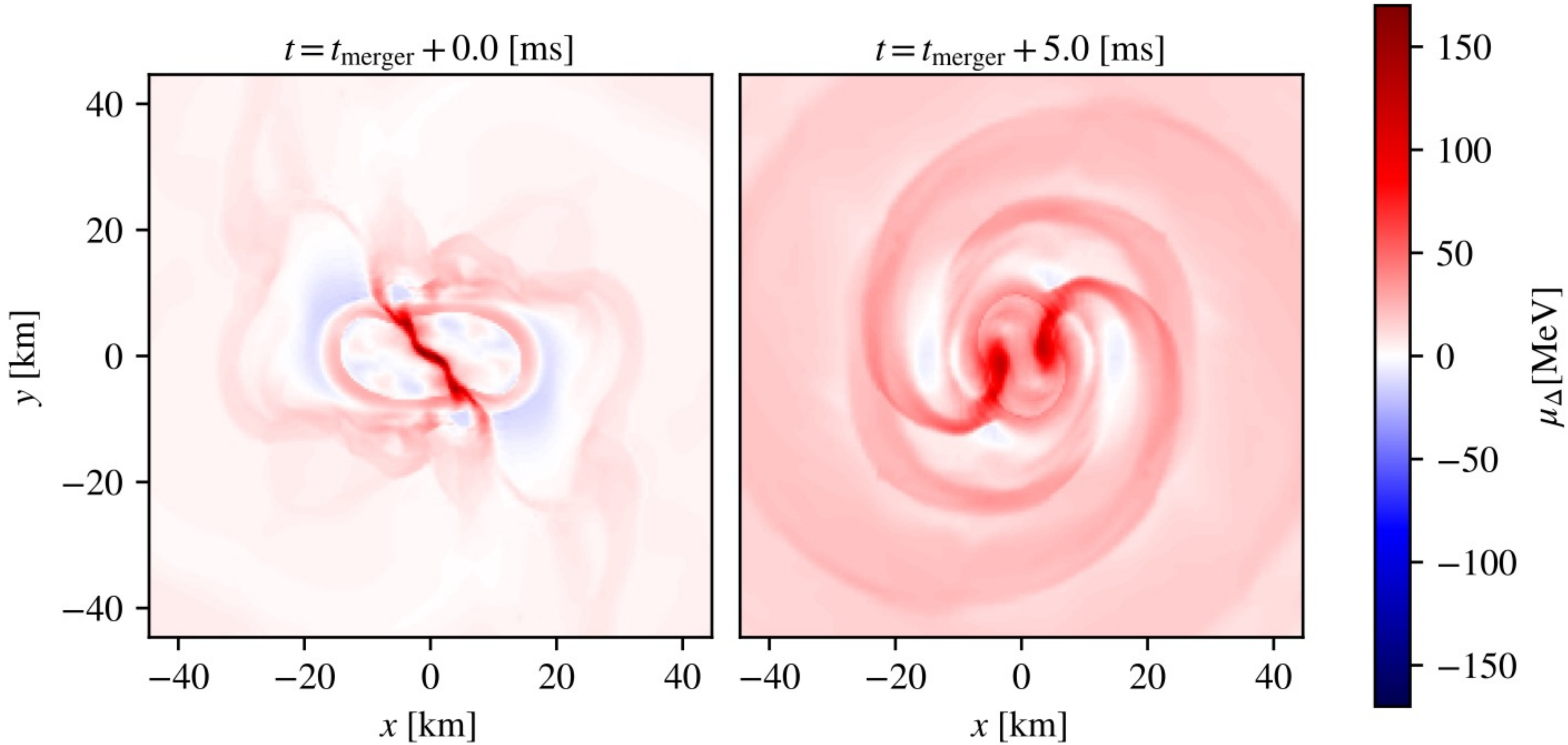
Example: For a cold ideal fluid composed of npe matter we have the thermodynamical relation

$$\begin{aligned} p + \varepsilon &= n_\mathrm{n}\mu_\mathrm{n} + n_\mathrm{p}\mu_\mathrm{p} + n_\mathrm{e}\mu_\mathrm{e} \\ &= n_\mathrm{n}\mu_\mathrm{n} + n_\mathrm{e}(\mu_\mathrm{p} + \mu_\mathrm{e}) \\ &= n_\mathrm{b}\mu_\mathrm{n} + n_\mathrm{e}(\mu_\mathrm{p} + \mu_\mathrm{e} - \mu_\mathrm{n}) \\ &= n_\mathrm{b}\mu_\mathrm{n} \end{aligned}$$

charge neutrality $n_\mathrm{p} = n_\mathrm{e}$

baryon number density $n_\mathrm{b} = n_\mathrm{n} + n_\mathrm{p}$

equilibrium $\mu_\mathrm{n} = \mu_\mathrm{p} + \mu_\mathrm{e}$



[Hammond+]

For npe matter, use $\beta = \mu_{\mathrm{n}} - \mu_{\mathrm{p}} - \mu_{\mathrm{e}}$ to encode the deviation from (cold) beta-equilibrium (…but "warm" equilibrium is different.)

Key message: The matter may be far from equilibrium.

Challenge: Merger dynamics should be within reach of the next-generation gravitational-wave detectors and we (naturally!) want to extract as much physics from these signals as possible.

Requires **robust simulations with a reliable physics implementation** (beyond the equilibrium equation of state!).

We are, unfortunately, quite far from this.

Where we actually are: Assuming a 3-parameter model $p = p(n, \varepsilon, Y_{\mathrm{e}})$ and stepping up the complexity, we may;

- assume that reactions are fast enough that the matter remains in equilibrium, or
- slow enough that the composition is frozen,
- try to add the relevant neutrino aspects,
- add whatever other physics we may be interested in…

For a **reactive** system we need to evolve

$$u^a \nabla_a Y_\mathrm{e} = \Gamma_\mathrm{e}/n$$

which may seem fairly innocent, but is problematic.

Focussing on the numerical implementation, the main issue relates to what can/cannot be resolved in a simulation.

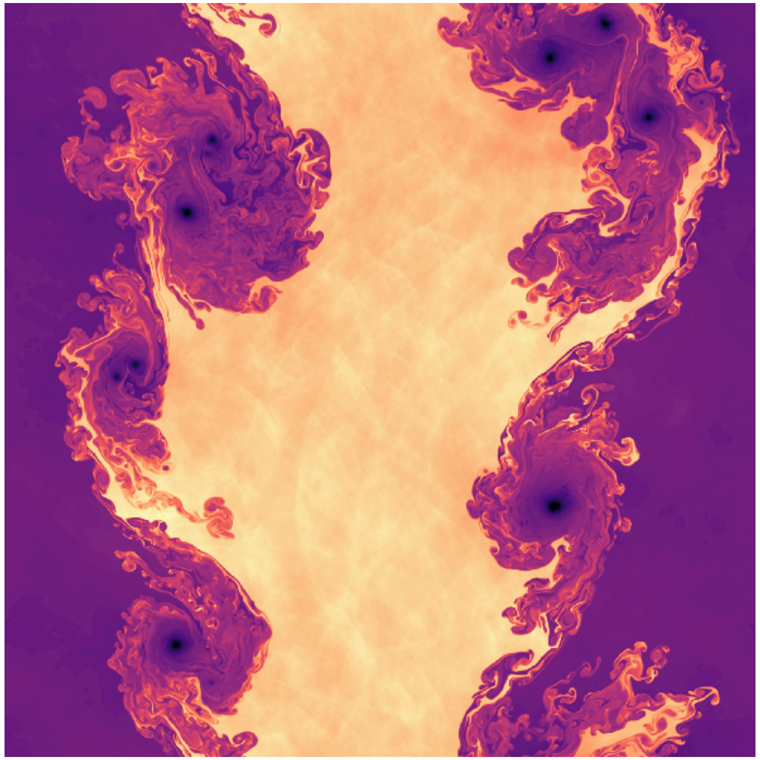
- When reactions are slow enough that they can be resolved, they should be resolved - we have to evolve the reactive system.
- When reactions are fast enough that they cannot be resolved, they may still leave an imprint on the dynamics. We need to figure out how to approximate this.

In reality, the parameter space of a neutron star merger will have regions where each assumption holds (and there will be "grey" areas in between…)

Need to worry about unresolved **small-scale dynamics** (=turbulence).

Drastic scale discrepancy:

- neutron star core: m.f.p. ~0.1mm
- best resolution in simulations ~10m



Strategy: Account for unresolved features through Large Eddy Filtering scheme (some kind of sub-grid model).

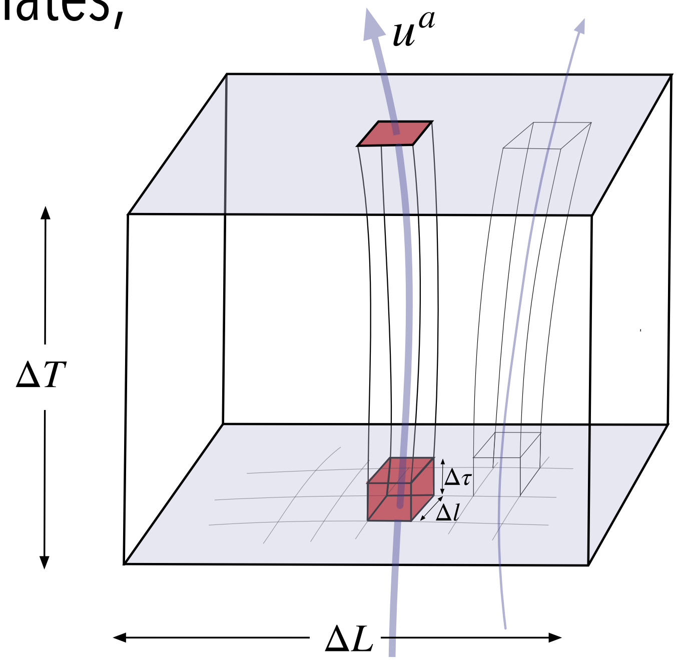
Current "proof of principle" simulations assume filtering in the foliation, rather than the "physical" fields (in the fluid frame).

This is problematic:

- lose connection with micro-scale physics
- introduces filtered metric

Covariant framework based on Fermi coordinates;

- local analysis tied to the fluid
- filtering operation explicitly defined
- compatible with the Einstein equations
- explicit link to thermodynamics
- "physical interpretation" of closure terms



$$\bar{n} = -\bar{u}_a \langle n^a \rangle \Longrightarrow \nabla_a \langle n^a \rangle = \nabla_a (\overline{n u^a}) = 0$$

Filtering of an ideal fluid leads to a non-ideal stress-energy tensor:

$$\overline{T}^{ab} = (\bar{\varepsilon} + \bar{p})\bar{u}^a \bar{u}^b + \bar{p} g^{ab} + 2\bar{u}^{(a} q^{b)} + s^{ab}$$

Dissipative terms depend on "residuals" represented by some **closure scheme**.

Coefficients must be tuned to resolved simulations.

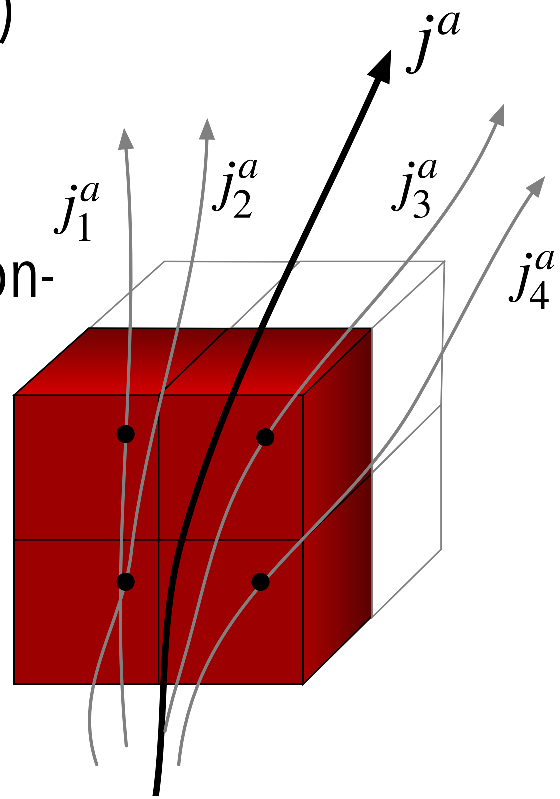
Filtering (explicit/implicit) also affects the thermodynamics:

$$\overline{p} + \overline{\varepsilon} = \overline{n}\overline{\mu} + \mathcal{M}$$

In effect, we are not (ever?) simulating the actual equation of state from microphysics. This could be problematic, as the aim is match simulations against observations to constrain the nuclear physics.

For charged flows, the filtered system is unlikely to be charge neutral (even though the micro-scale model is)

$$\langle j^a \rangle = \langle \sigma u^a \rangle + \langle J^a \rangle \Longrightarrow \overline{\sigma} = -\overline{u}_a \langle J^a \rangle$$



Effectively turns ideal magnetohydrodynamics into non-ideal electromagnetism.

A very different problem…

Understanding/modelling this is key for instabilities and dynamo precursors.

take home points

To match the precision of upcoming observations we need to (continue to) improve numerical simulations.

1. Reactions are difficult to implement as the timescales vary over many orders of magnitude in the different regions of parameter space explored in a merger.
2. Need to represent “unresolved” features (turbulence and phase transitions) through a suitable averaging/filtering scheme. This inevitably introduces non-ideal magneto-fluid aspects.
3. Must pay more attention to poorly understood “systematics”.