# Non-Gaussian Fluctuations in Fluid

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Based on work with Basar, Stephanov and Yee

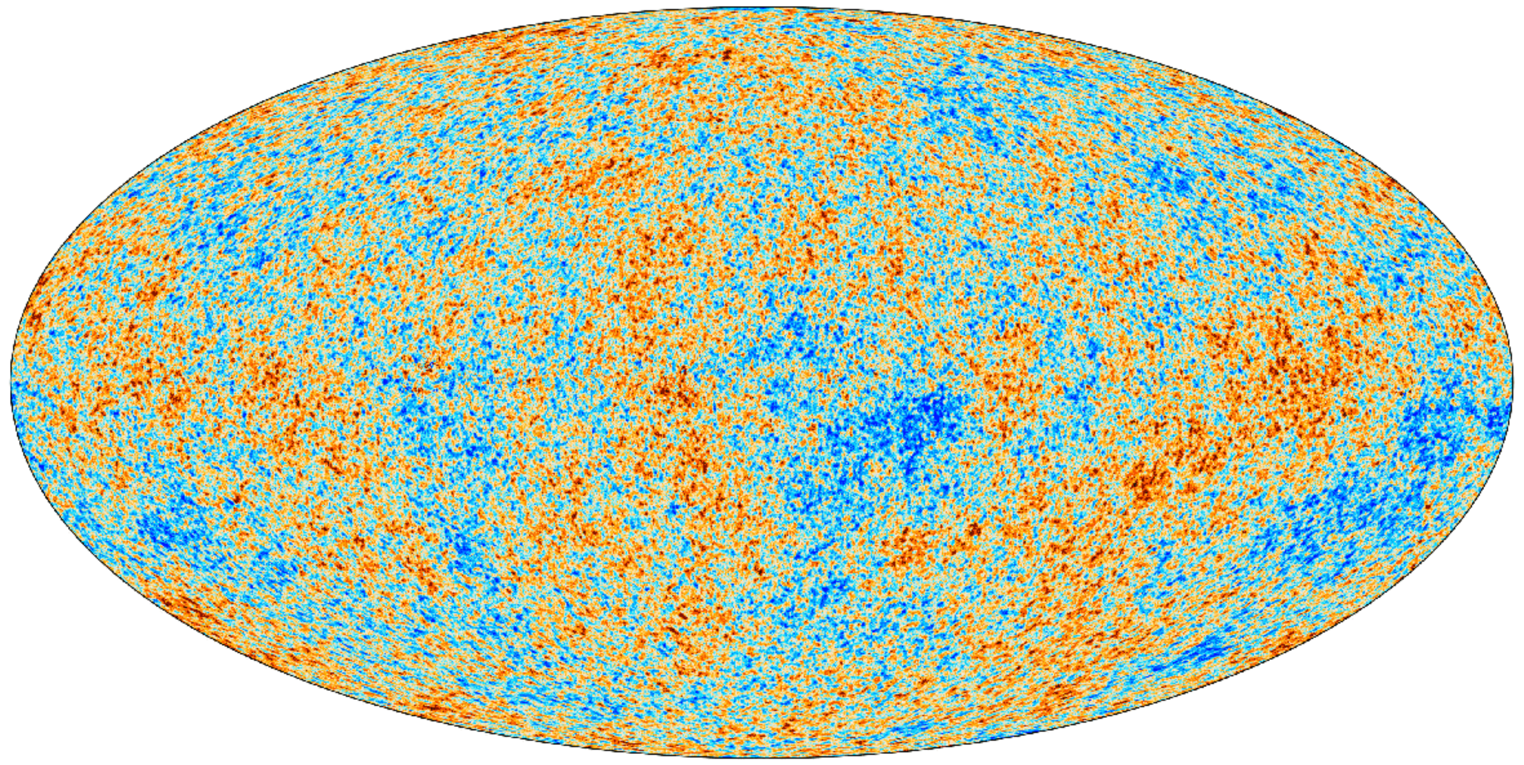
INT workshop on criticality and chirality

Aug 22 2023, Seattle

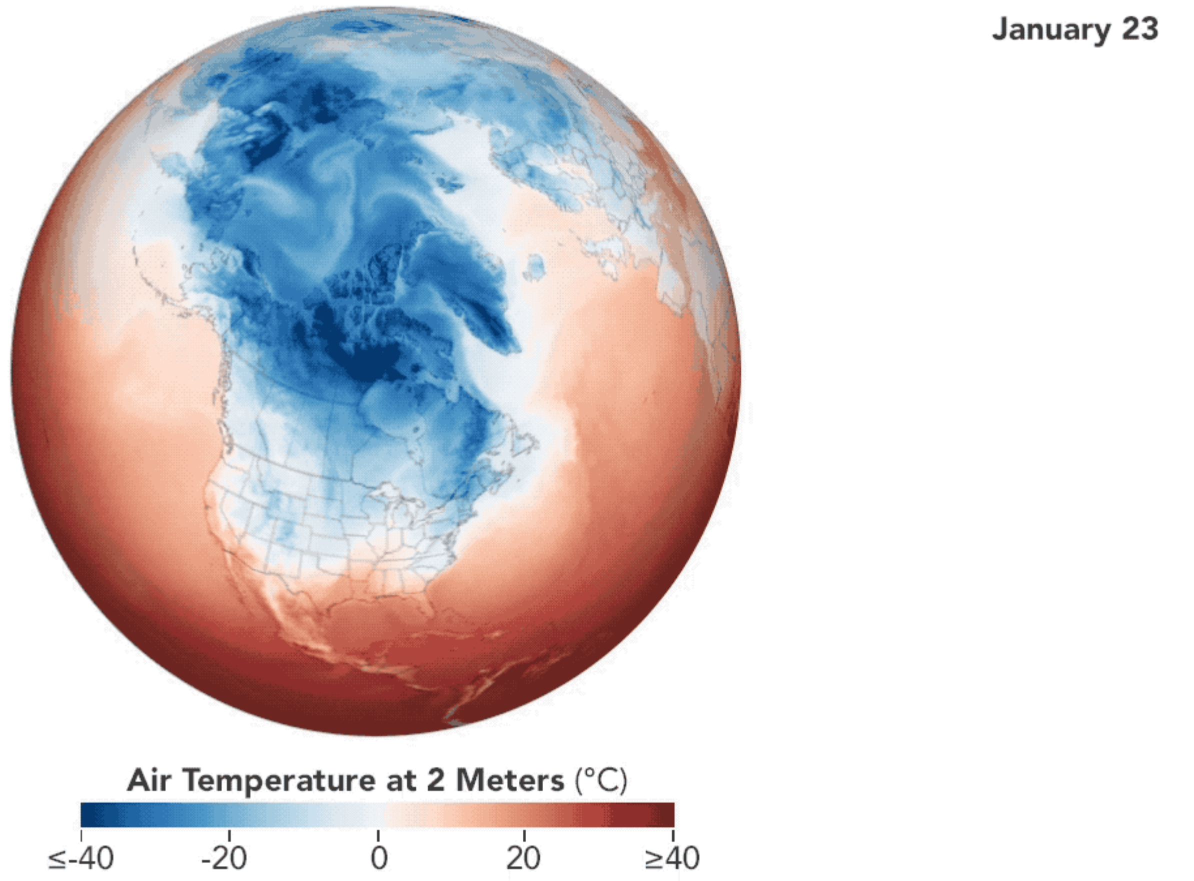
NCBJ

# Fluctuations on all length scales

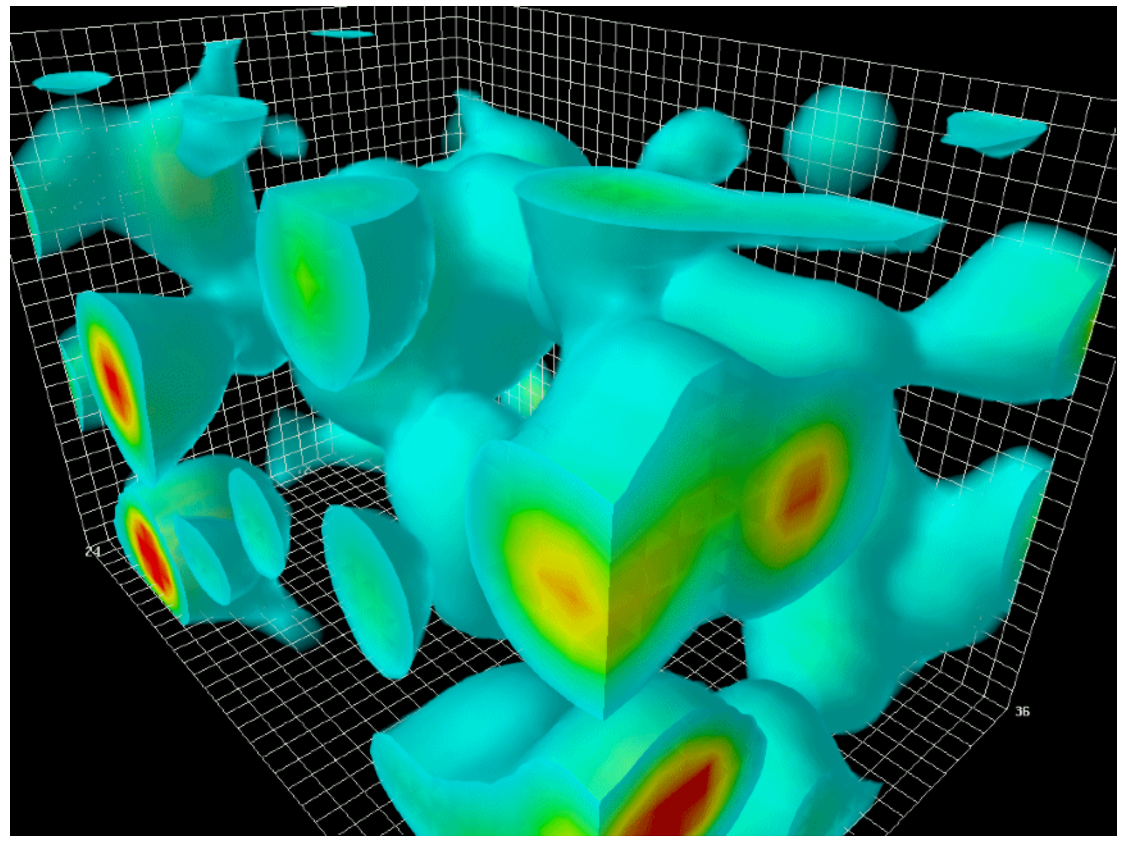
- Fluctuations are ubiquitous phenomena emerging on all length scales.



CMB



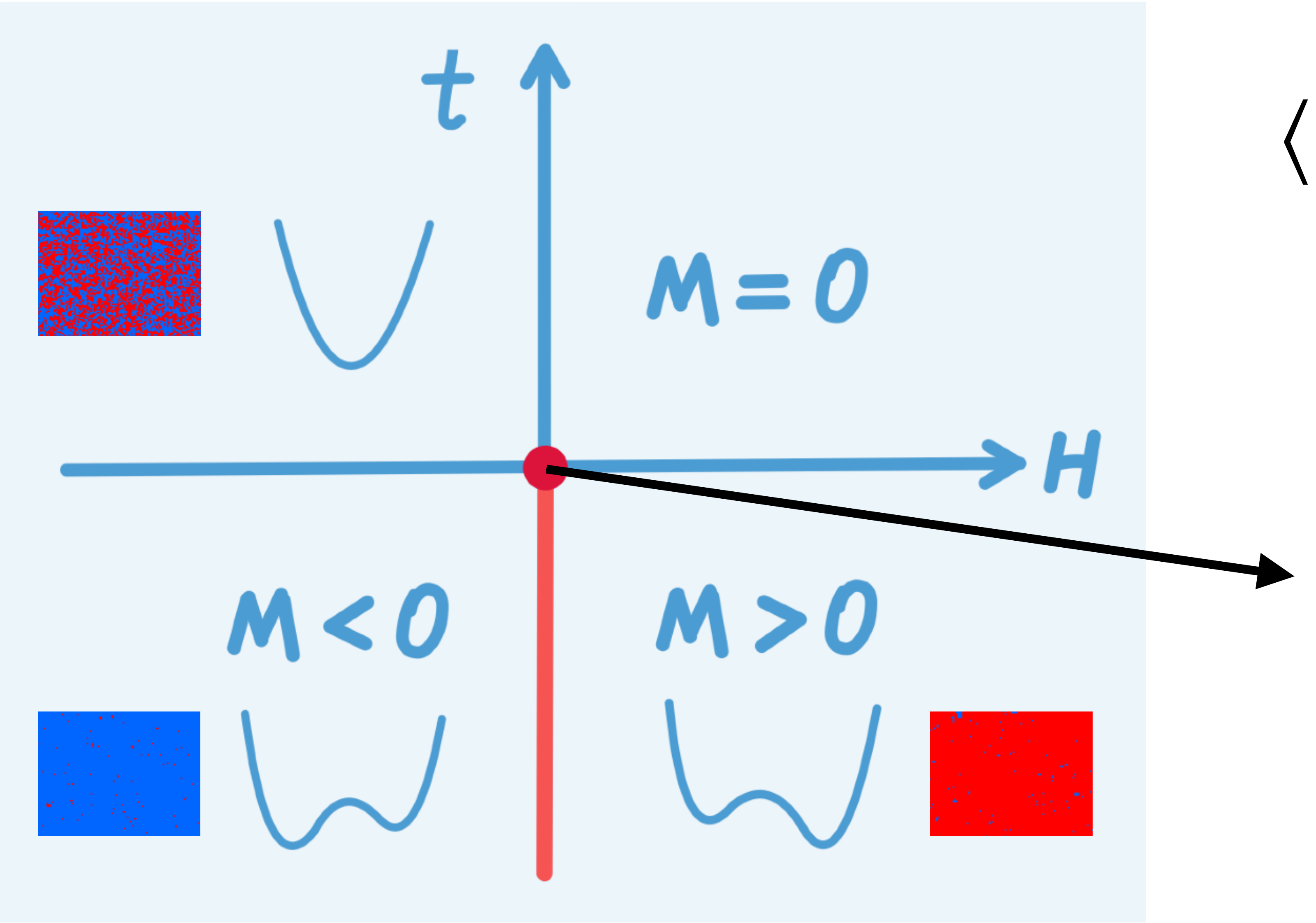

Atmosphere



Quantum fluctuations

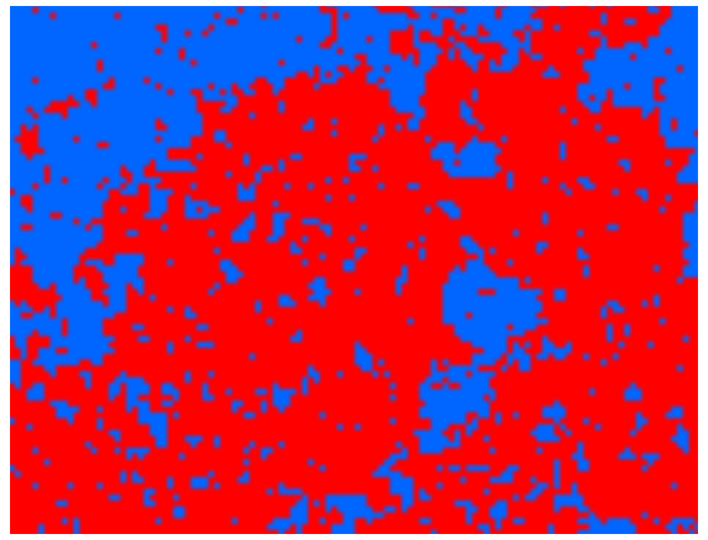
# Thermal fluctuations and critical point

- Thermal fluctuations can be described by EOS dealing with *large* number of DOFs *in equilibrium*.
- Fluctuation correlation length $\xi$ diverges at the critical point.

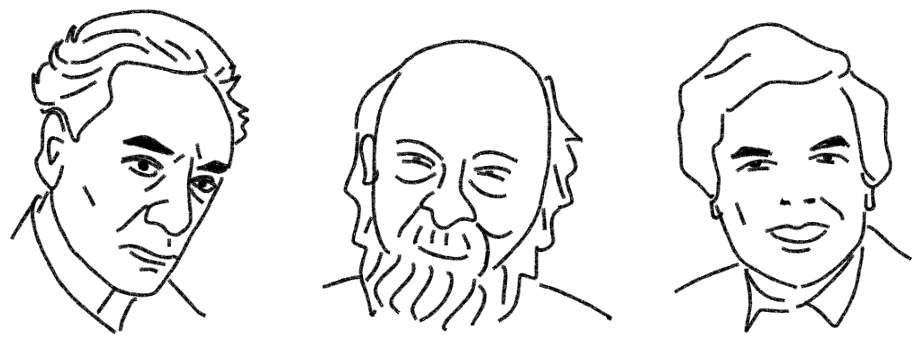



Ising phase diagram

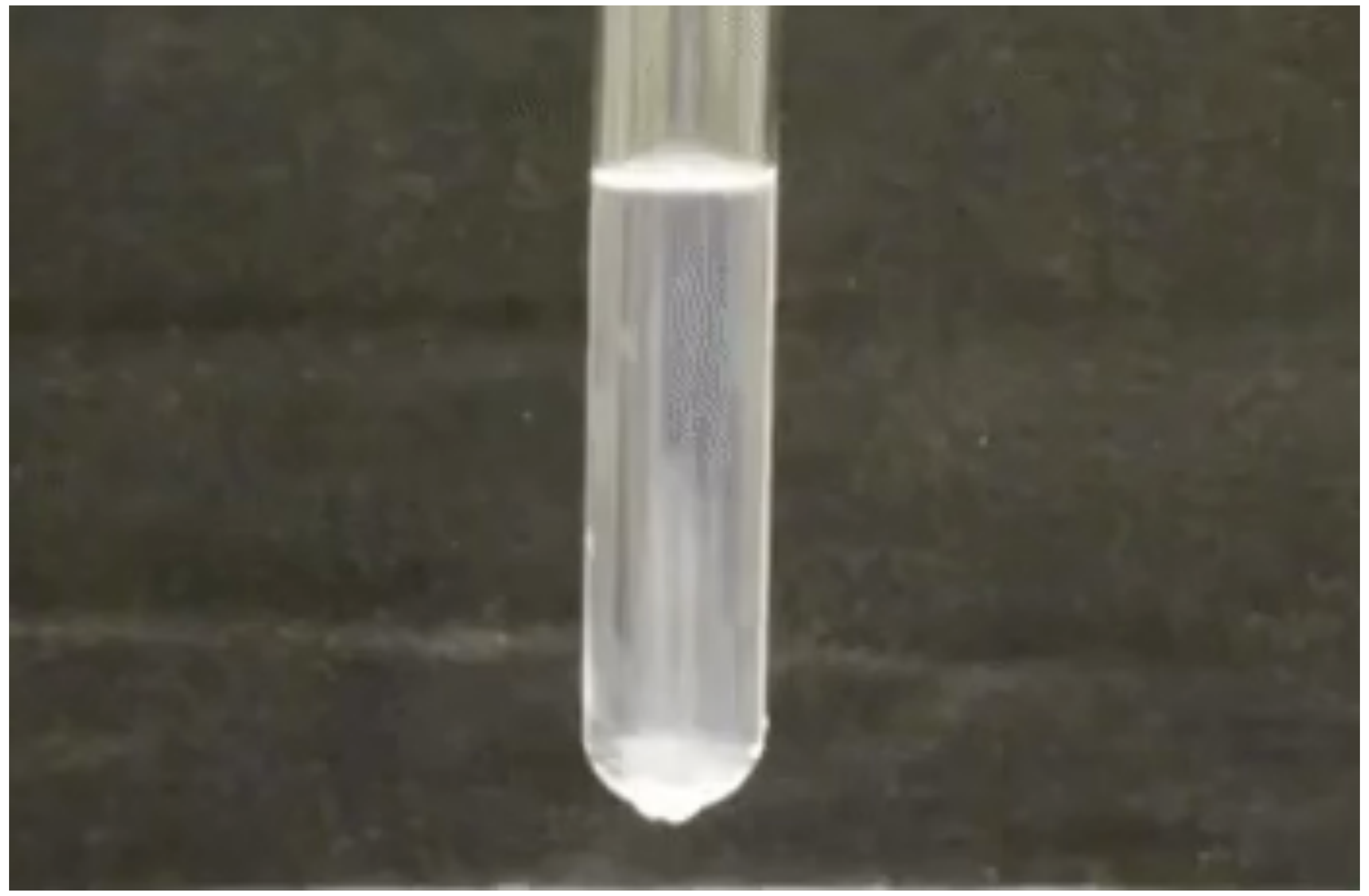
$$\langle \phi(x_1)\phi(x_2)\rangle \sim e^{-\frac{|x_1 - x_2|}{\xi}}$$



$\xi \to \infty$



Landau Kadanoff Wilson



Critical opalescence: $\xi \leftrightarrow \lambda_{\text{light}}$

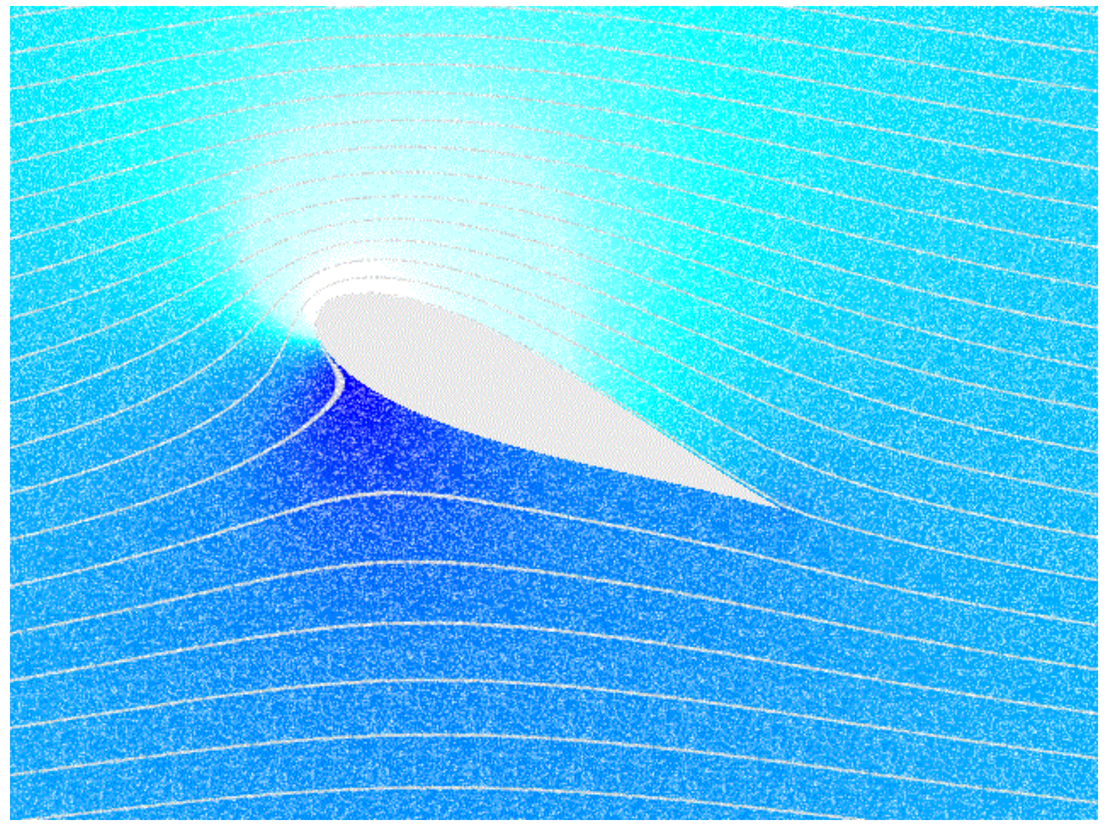
See also Skokov's talk

# Fluctuations in hydrodynamics

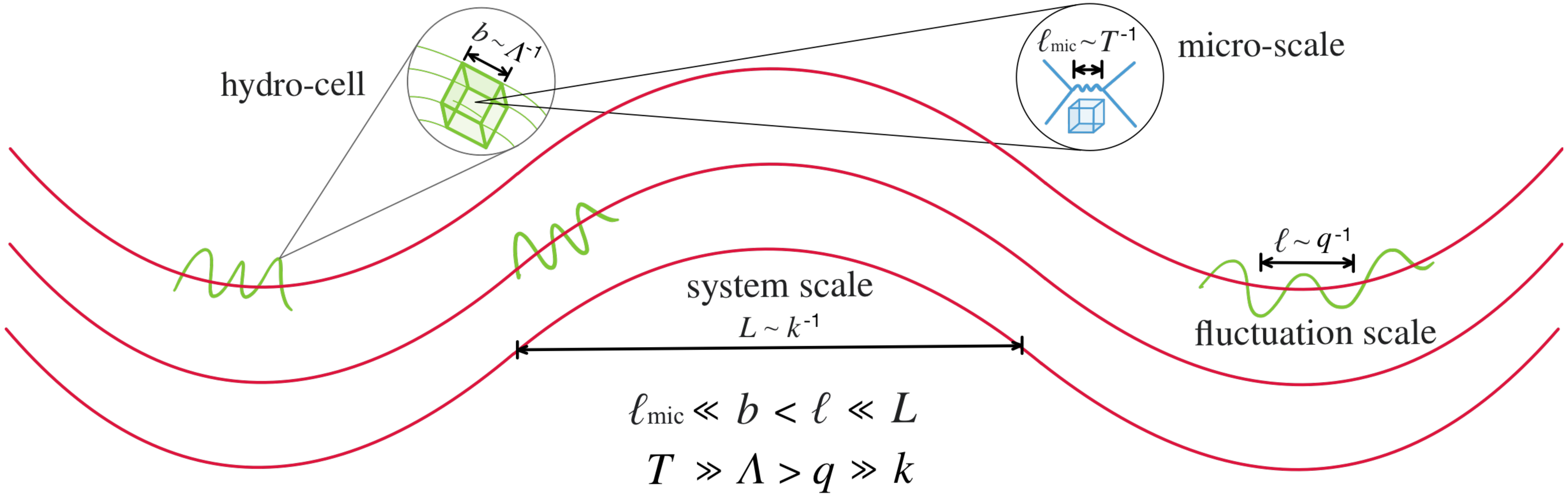
- Hydrodynamic fluctuations: *locally thermalized* thermal fluctuations comoving with fluid, may *not equilibrate* at large scales, described by a set of conservation equations

$$\partial_t \psi = \nabla \cdot (J[\psi])$$

$\psi = (n, \epsilon, \pi_i)$ conserved current

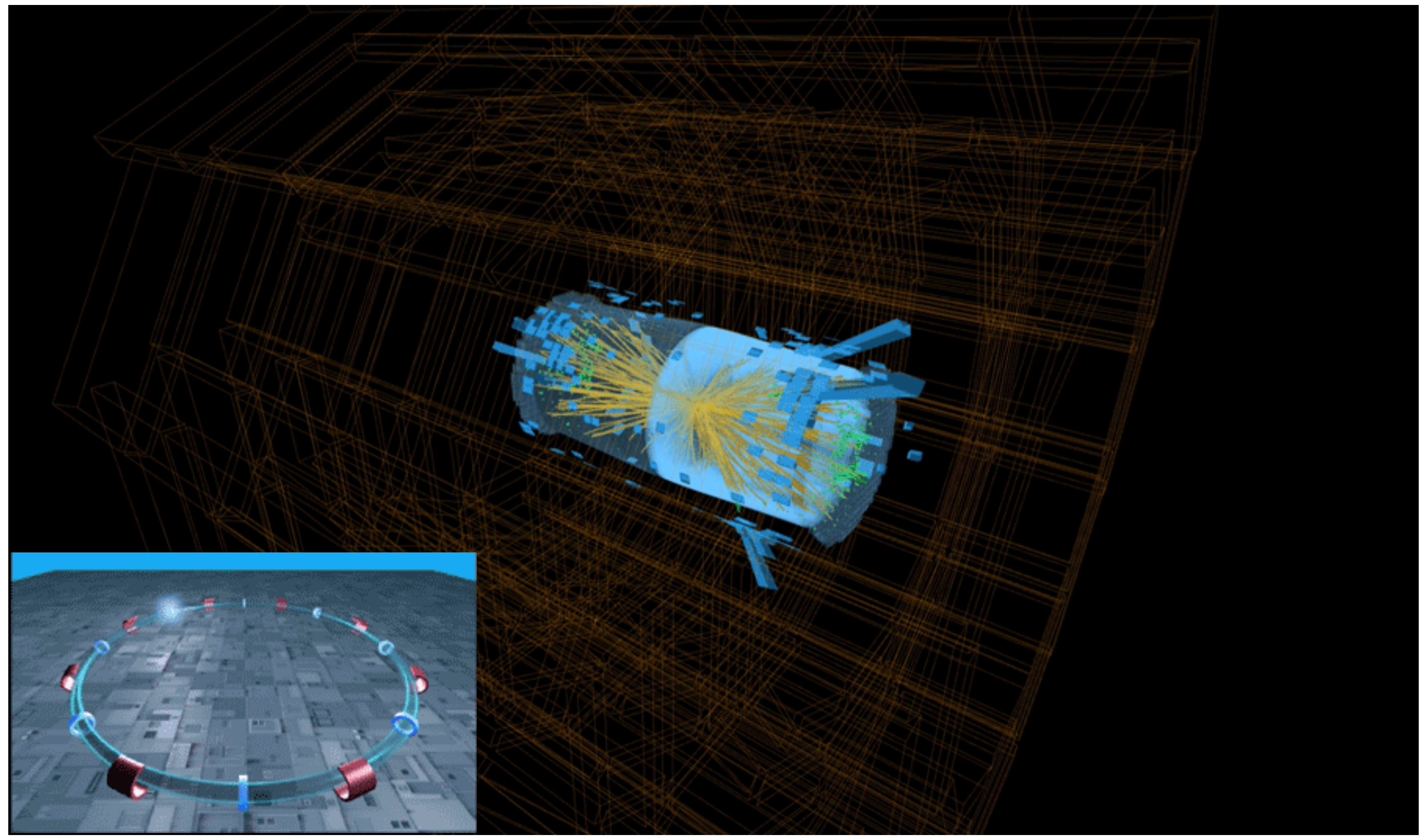


Flow around a wing
(Wikipedia)

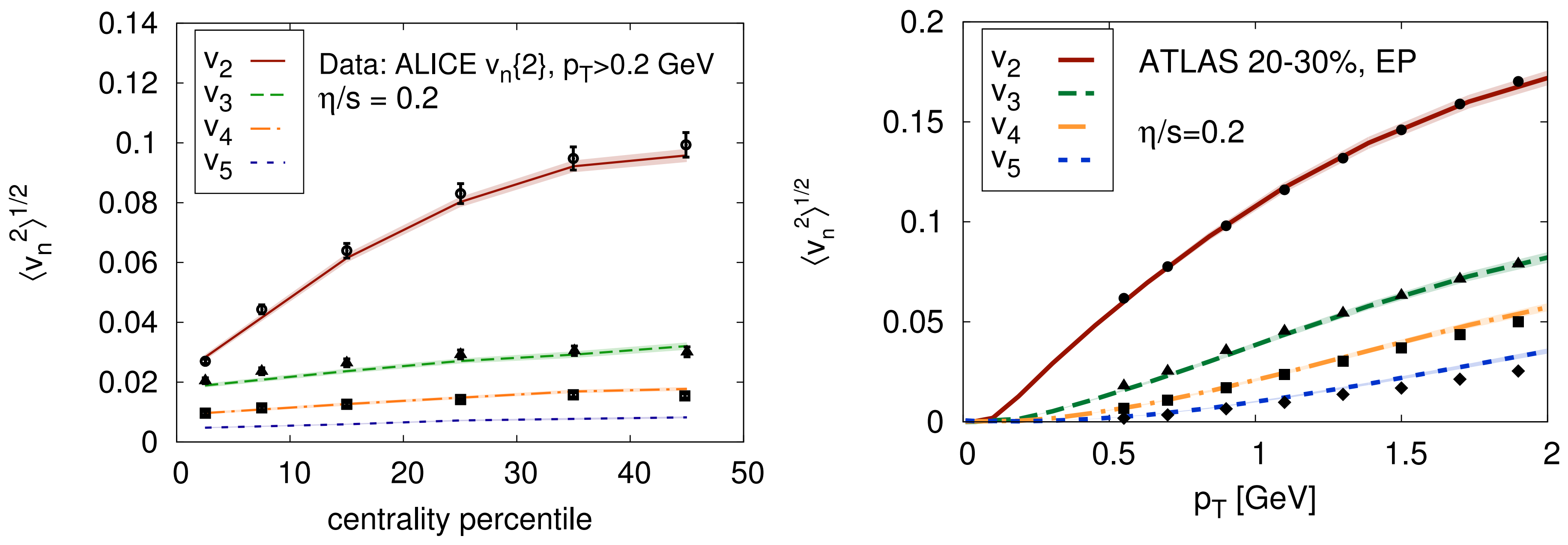


# Hydrodynamic description of QGP

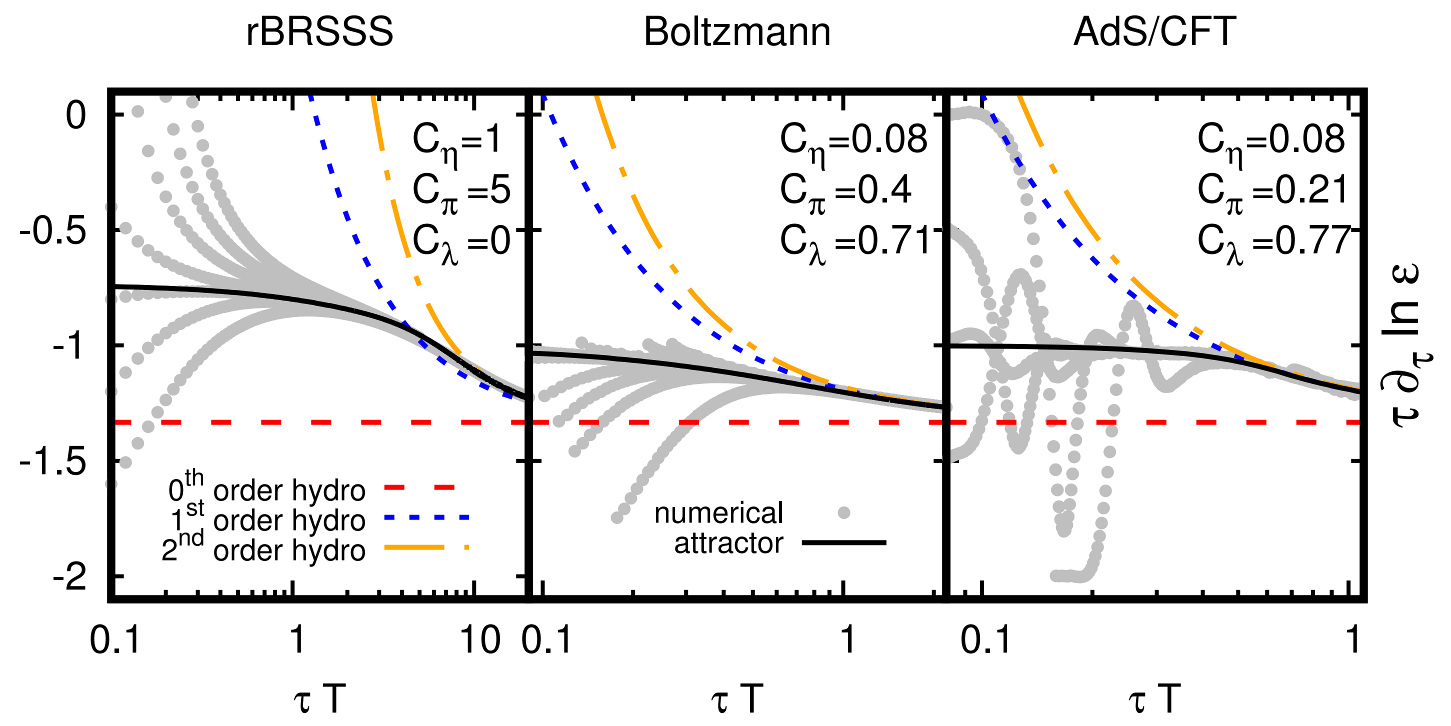
- Quark-Gluon Plasma in heavy-ion collisions: *small* enough for fluctuations to be important; and *large* enough for hydrodynamics to be applicable.



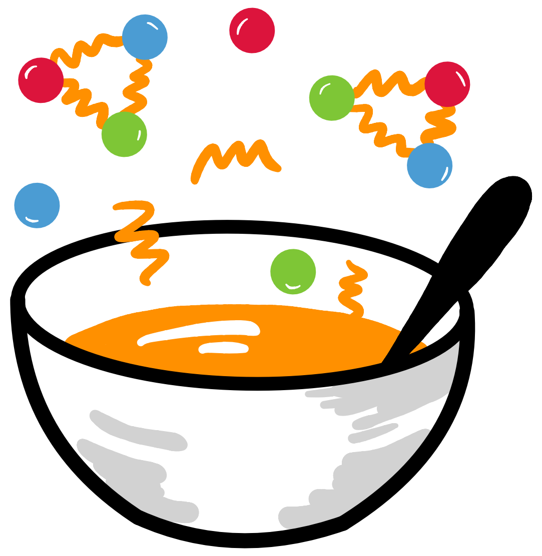
Collision event simulation at LHC (CERN)




Analysis for flow collectivity manifests QGP as a *perfect fluid* C. Gale et al, 1301.5893




Hydrodynamic *attractor* W. Florkowski et al, 1707.02282, P. Romatschke et al, 1712.05815



Size of created fire balls $\sim 10$ fm

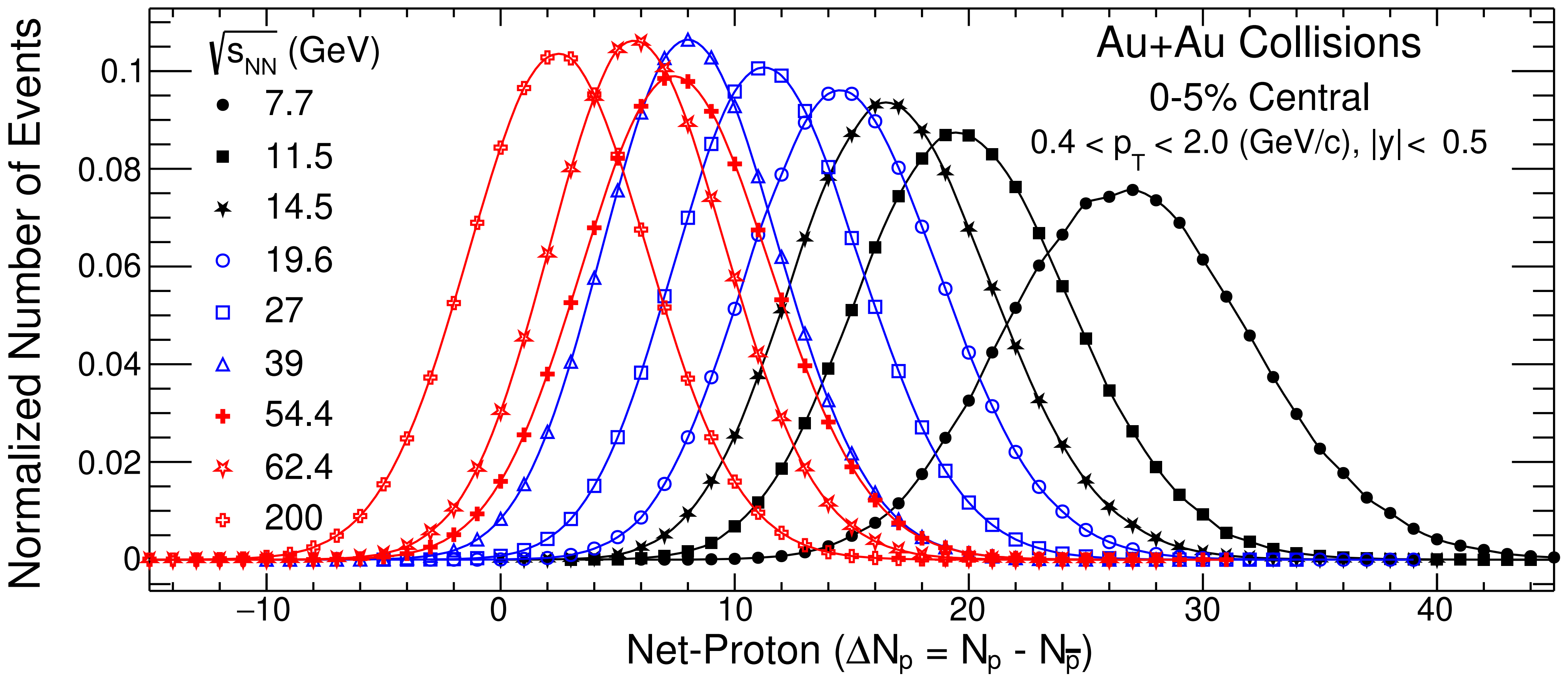
Number of particles $\sim 10^2 - 10^4$

Early Universe is more like a soup

# Stochastic process in HIC

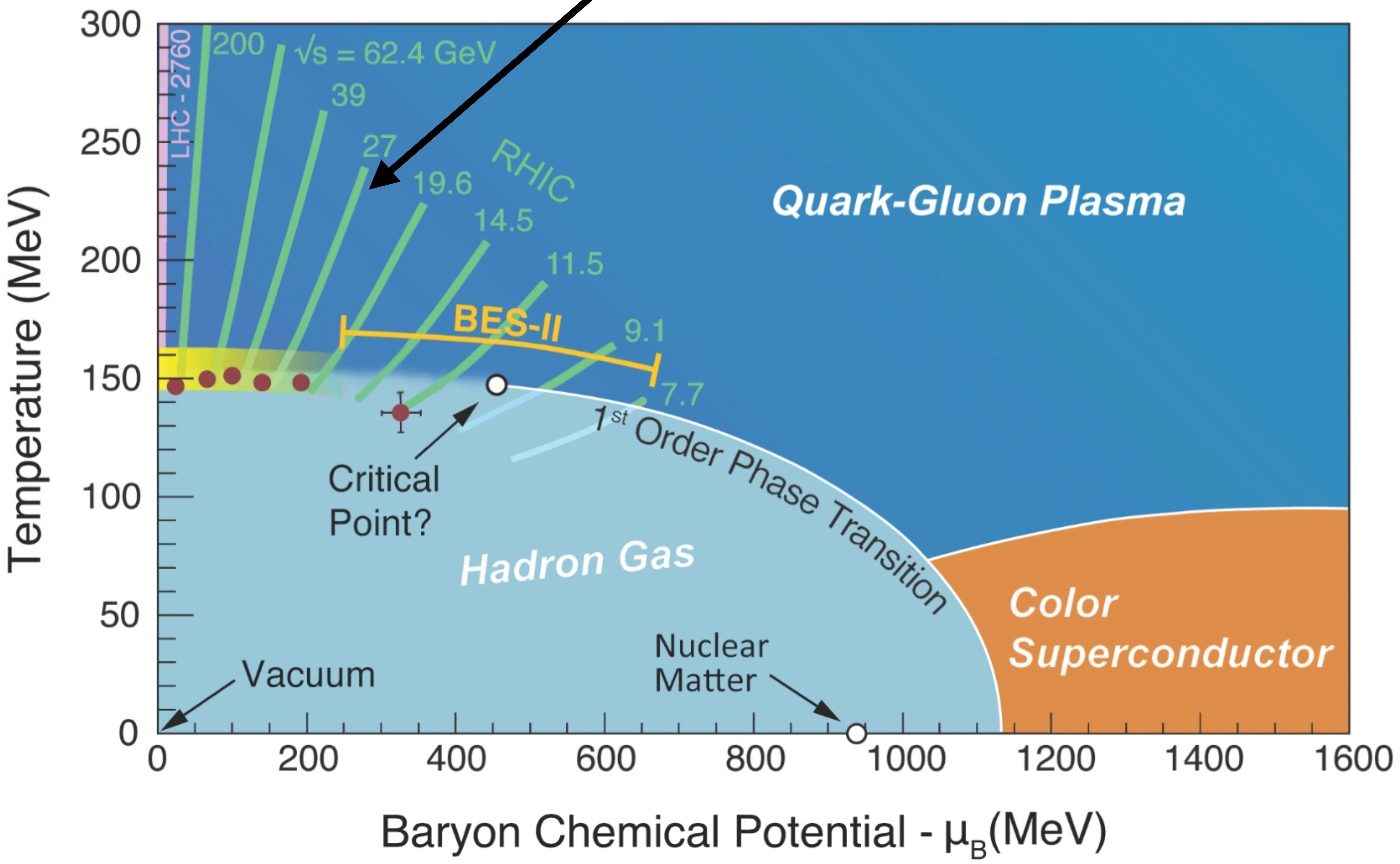
- Observables fluctuate event-by-event.
- Beam Energy Scan: 7.7-200 GeV (phase I), and 3.3-19.6 GeV (phase II).

One collision events →




Events number vs net-proton yields STAR

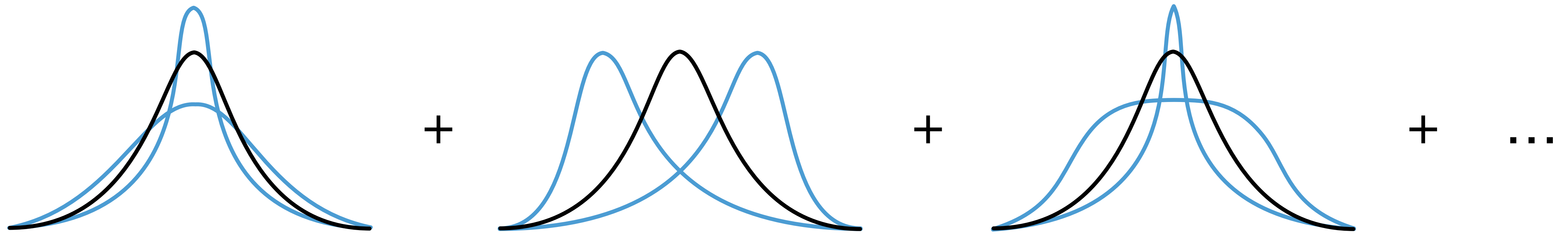
Trajectories of millions of collision events at fixed beam energy




QCD phase diagram

# Probability distribution for fluctuating variables

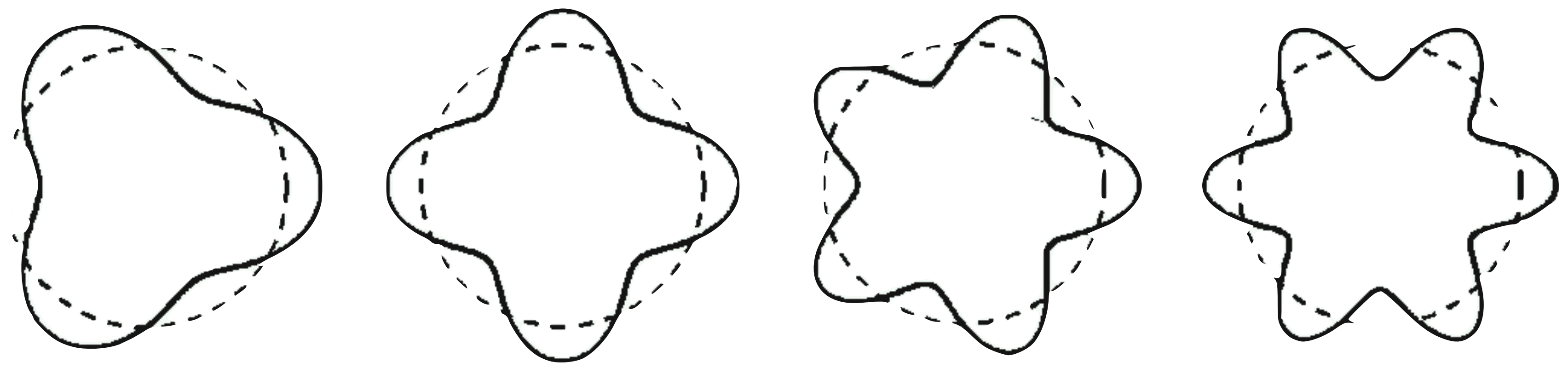
- Fluctuations can be described by probability distribution $P[\psi]$ and its associated *cumulants* (or correlation functions).



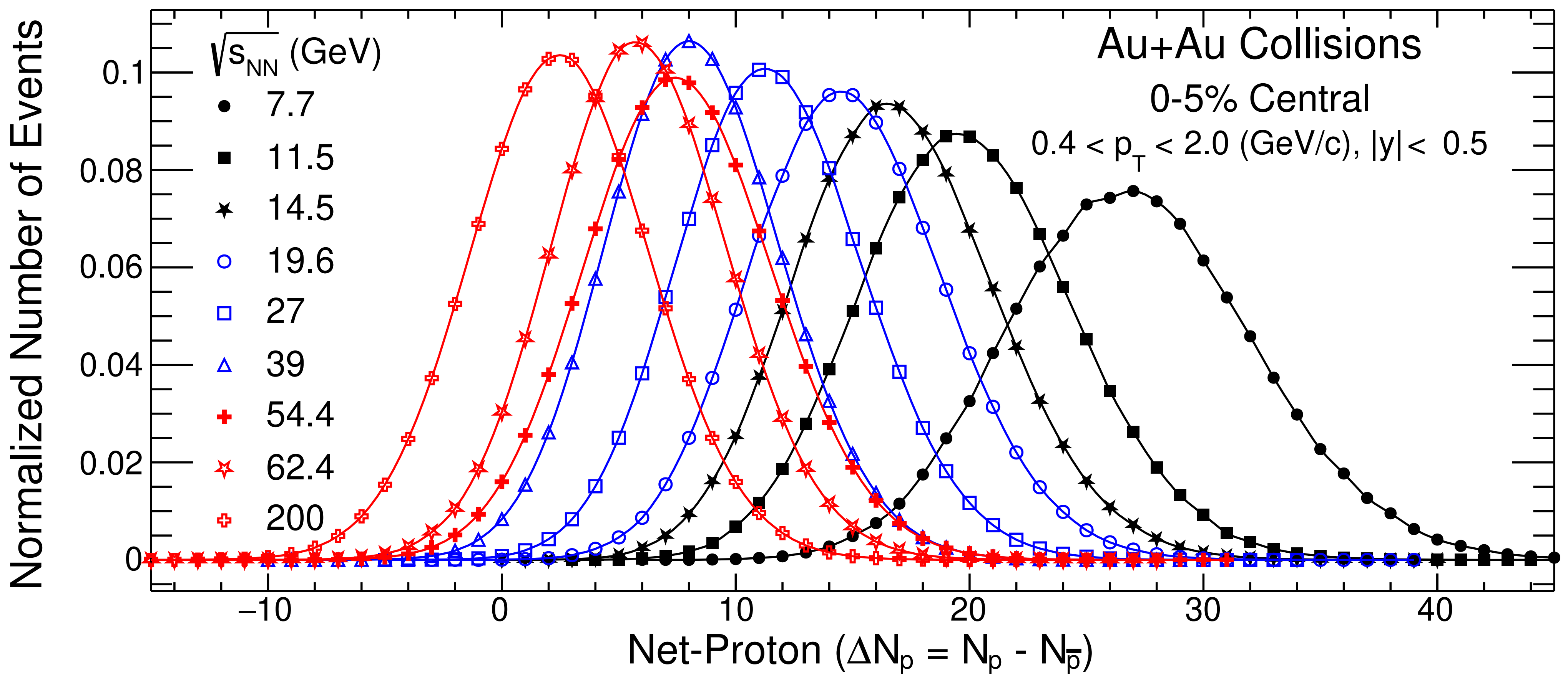

variance
↓
width

skewness
↓
lopsidedness

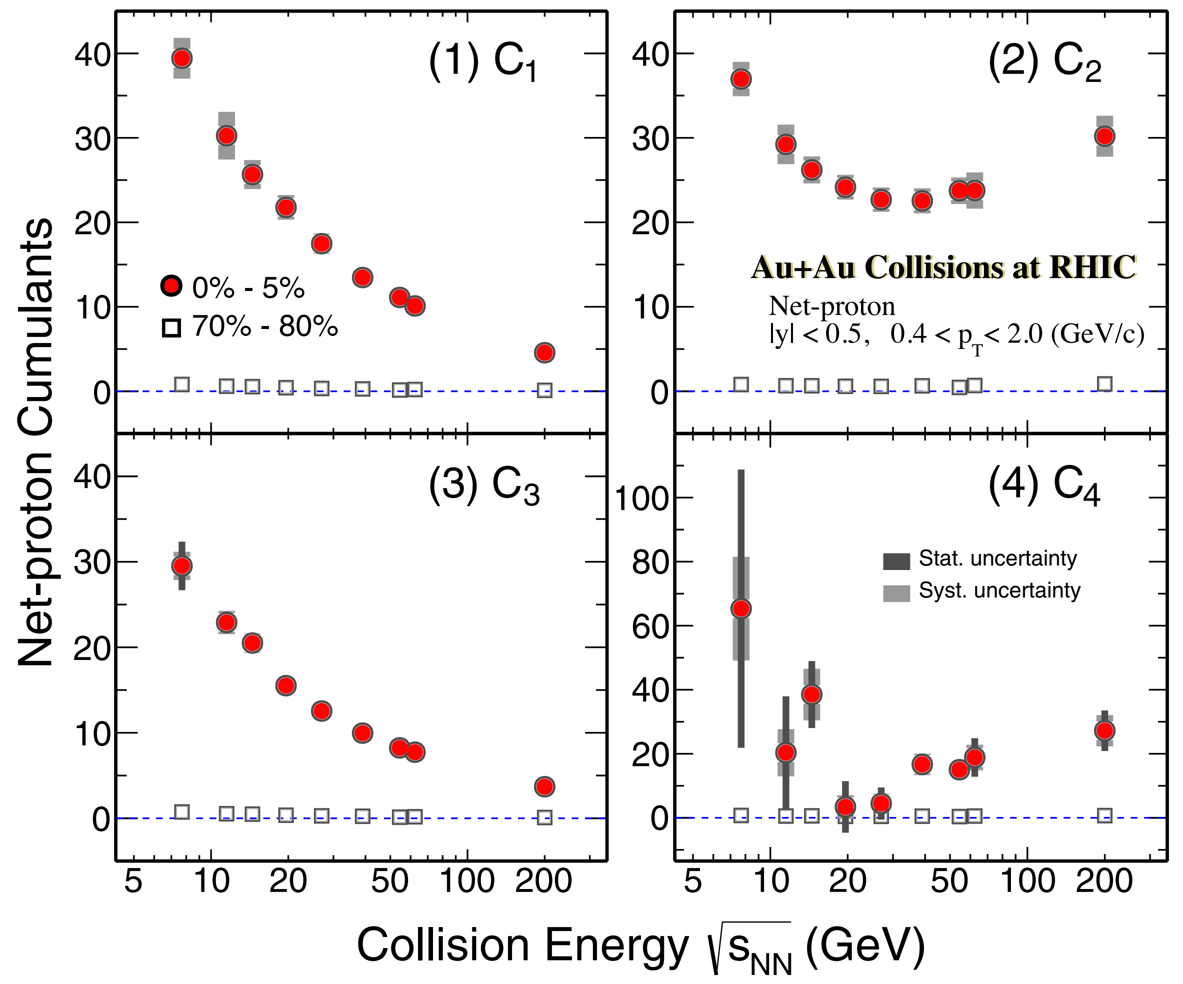
kurtosis
↓
sharpness



Geometry eccentricity expansion




Events number vs net-proton yields STAR

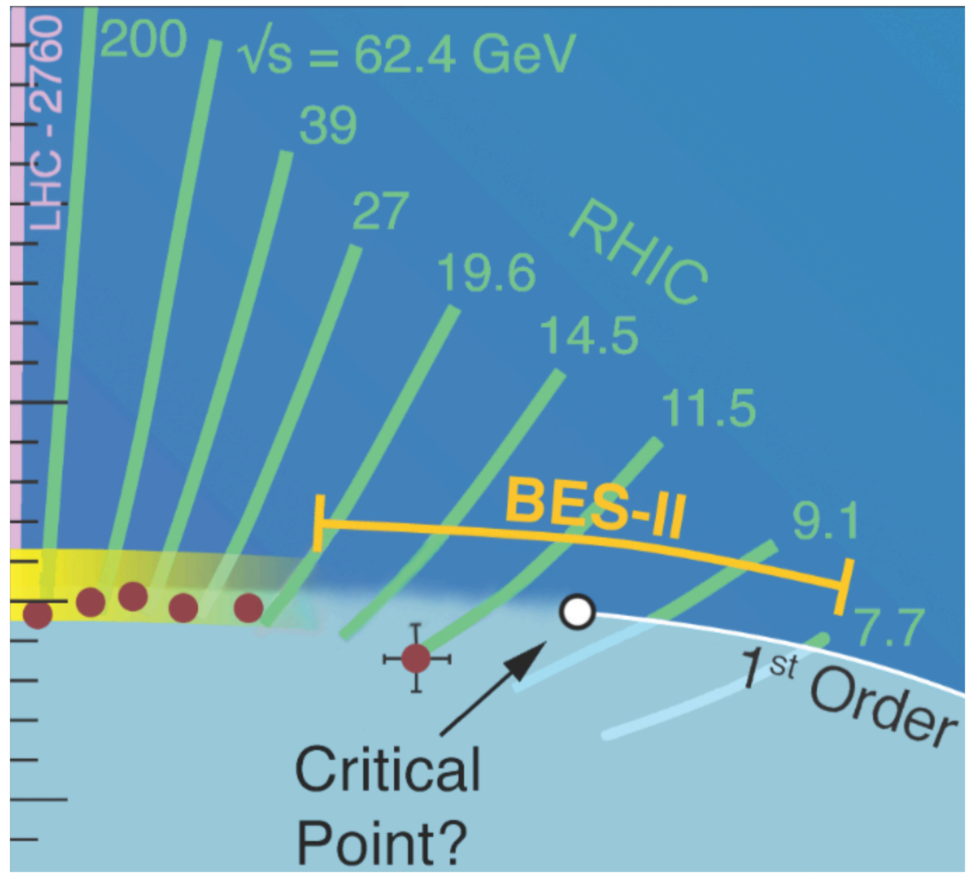



Net-proton cumulants vs energy

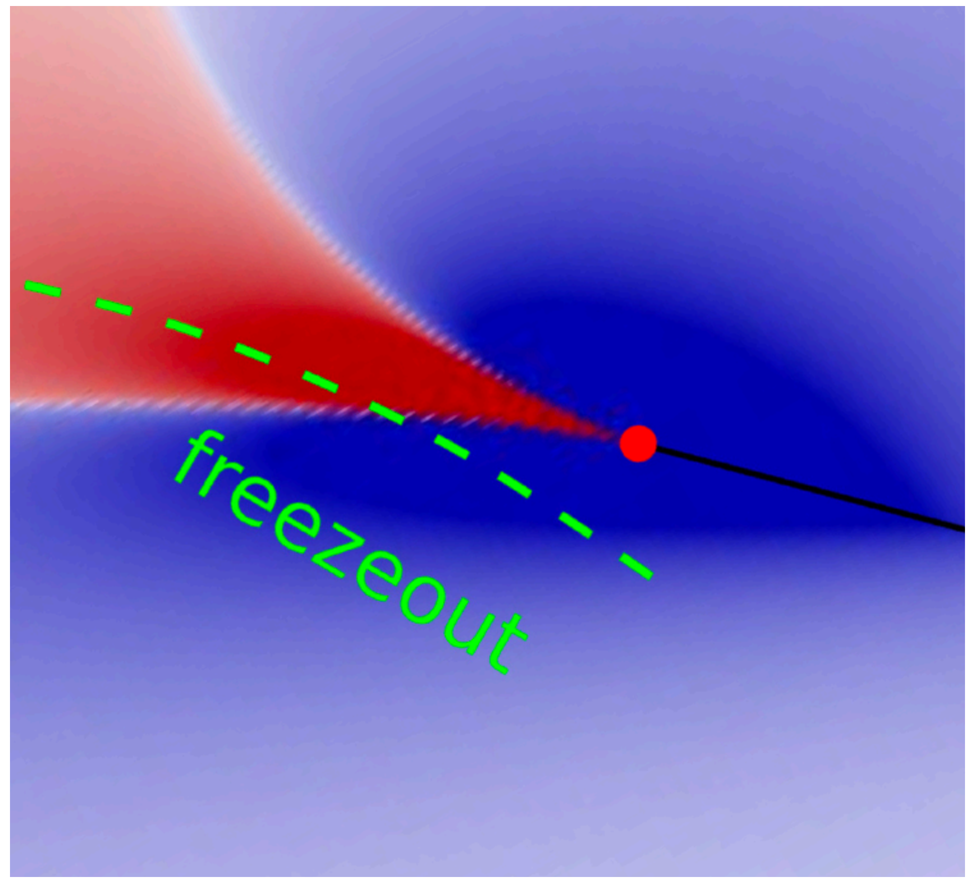
# Cumulants and critical point

- In thermodynamics, higher-order cumulants are more sensitive to the correlation length $\xi$ which diverges at critical point. Stephanov 0809.3450, 1104.1627

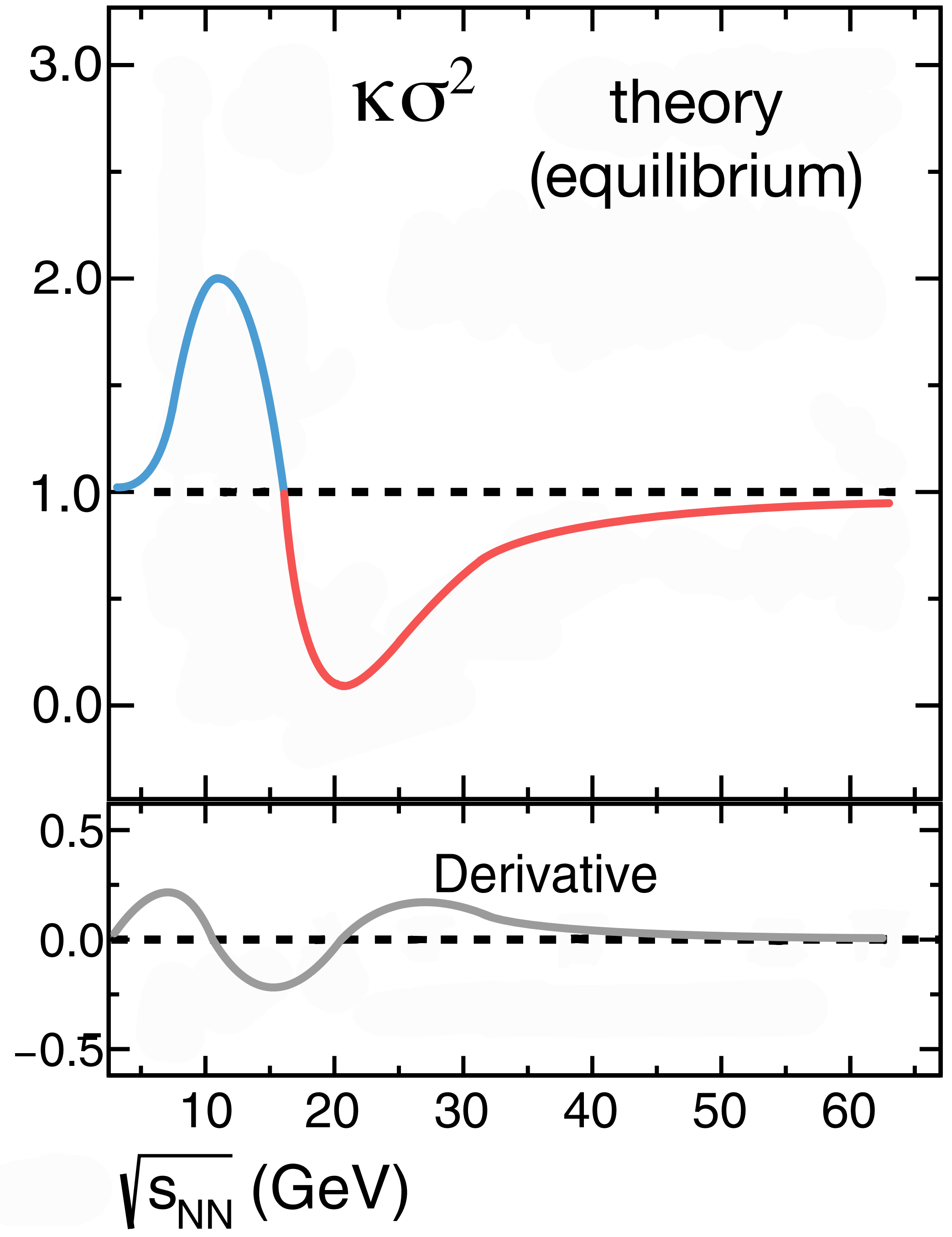
$$n\text{-th cumulants} \sim \xi^{\frac{5n-6}{2}}$$



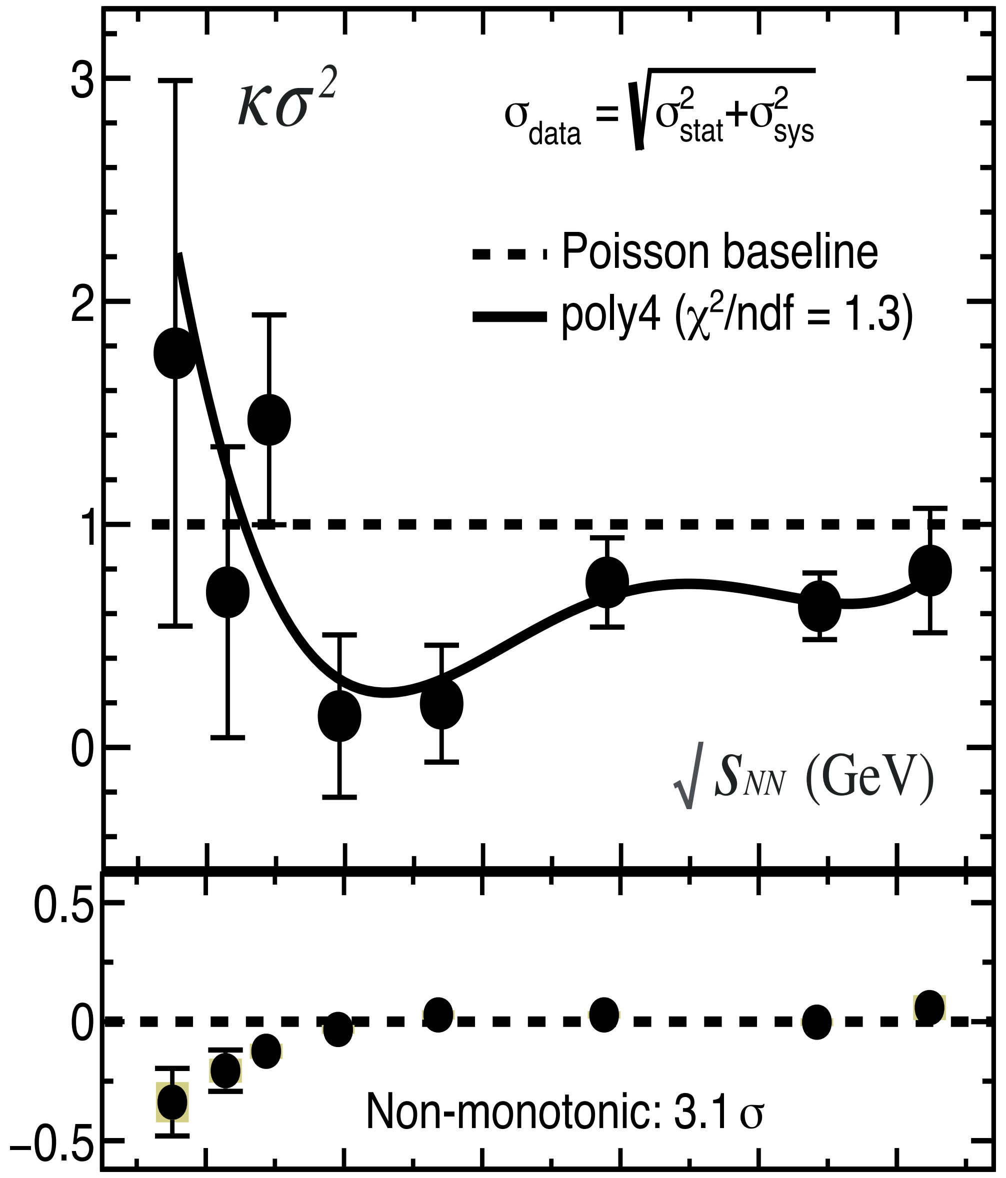

Critical point region




4th cumulant contours




Theoretical prediction




Experimental result (STAR BES-I)

Non-monotonic energy dependence

# Various theoretical approaches

- **PDEs** (bottom-up like)

  Starting from phenomenological equations with required properties

  e.g., Langevin equations in *stochastic* description, Fokker-Planck (FP) equations in *deterministic* description.

  Akamatsu et al, 1606.07742
  Nahrgang et al, 1804.05728
  Singh et al, 1807.05451
  Chattopadhyay et al, 2304.07279
  …

- **EFTs** (top-down like)

  Starting from effective action with first principles

  e.g., Martin-Siggia-Rose (MSR), Schwinger-Keldysh (SK), Hohenberg-Halperin (HH), n-particle irreducible (nPI), etc.

  Glorioso et al, 1805.09331
  Jain et al, 2009.01356
  Sogabe et al, 2111.14667
  Chao et al, 2302.00720
  …

# Stochastic and deterministic description

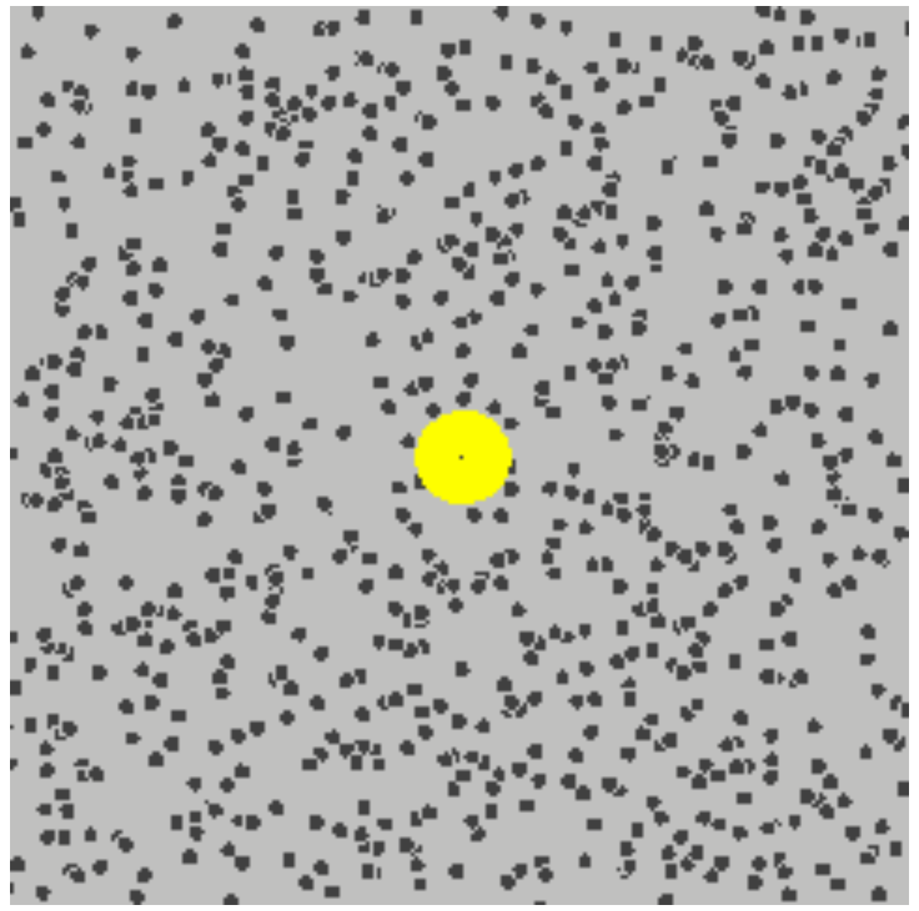
- Langevin equation
  (Newton's equation + noise):

$$\partial_t \breve{\psi}_i = F_i[\breve{\psi}] + \eta_i$$

$$\langle \eta_i(x_1)\,\eta_j(x_2)\rangle = 2Q_{ij}\,\delta^{(3)}(x_1 - x_2)$$

Newton

Langevin



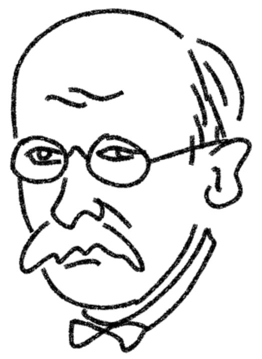
Brownian motion

- Fokker-Planck equation
  (probability evolution equation):

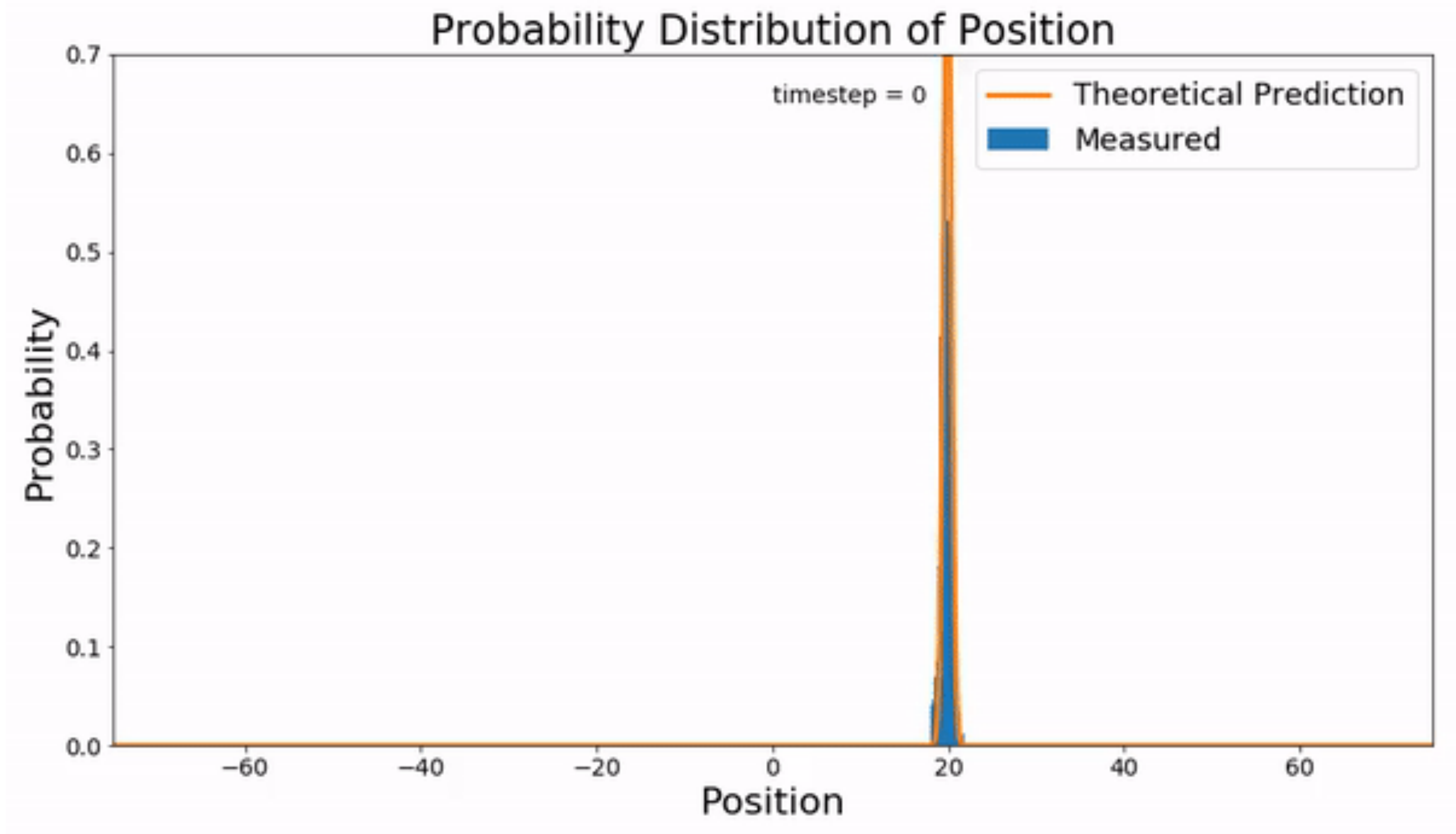
$$\partial_t P[\psi] = \frac{\partial}{\partial \psi}\big(\mathbf{flux}[\psi]\big)$$

$$\mathbf{flux}[\psi] = -FP + \frac{\partial}{\partial \psi}(QP)$$

Fokker



Planck

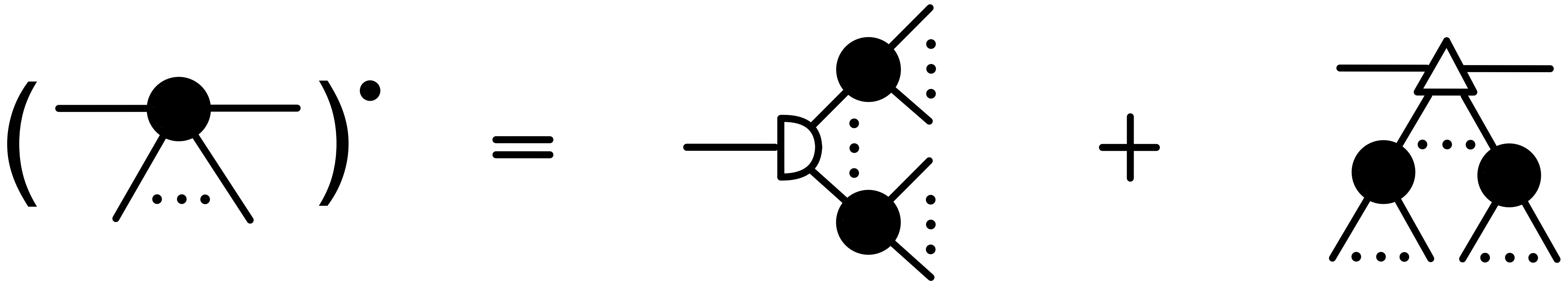



(Wikipedia)

# Correlator evolution equations

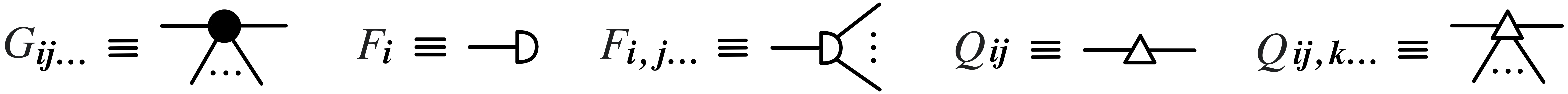
- Evolution equations for n-pt correlators $G_n \equiv \langle \underbrace{\phi \ldots \phi}_{n} \rangle \equiv \int d\psi \, P[\psi] \, \underbrace{\phi \ldots \phi}_{n}$
  where $\phi \equiv \psi - \langle \psi \rangle$ : XA et al, 2009.10742, 2212.14029

$$\partial_t G_n = \mathscr{F} \left[ \psi, G_2, G_3, \ldots, G_n, G_{n+1}, \ldots, G_\infty \right]$$



(with all combinatorial configurations)

useful for higher order cumulants




Stochastic:

*One* equation

*Millions* of samples

⟷

Deterministic:

*One* sample

*Millions* of equations

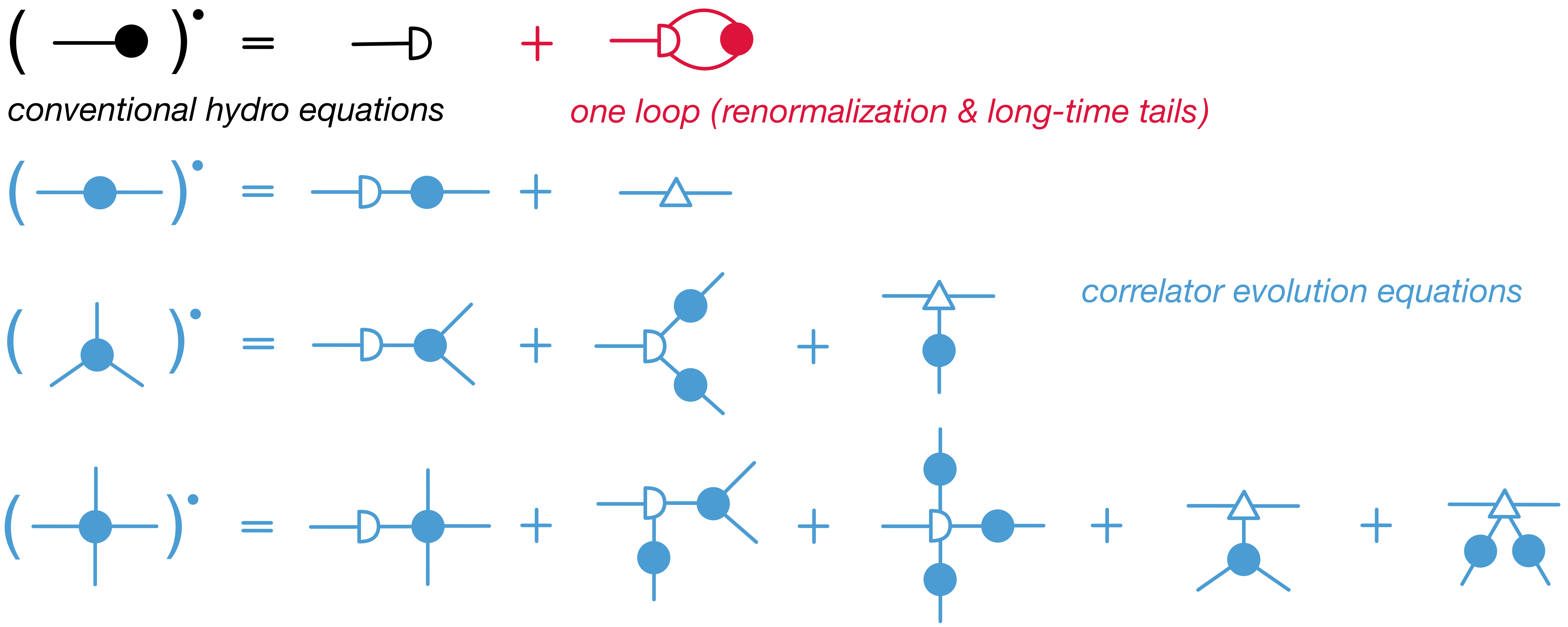
# Truncated evolution equations

- Power counting in loop expansion parameters $\varepsilon \sim (q\xi)^3$ (inverse number of uncorrelated cells in fluctuation scales): XA et al, 2009.10742

$$G_n \sim \varepsilon^{n-1}, \qquad F \sim 1, \qquad Q \sim \varepsilon\,.$$

- Correlator evolution equations can be truncated and iteratively solved:

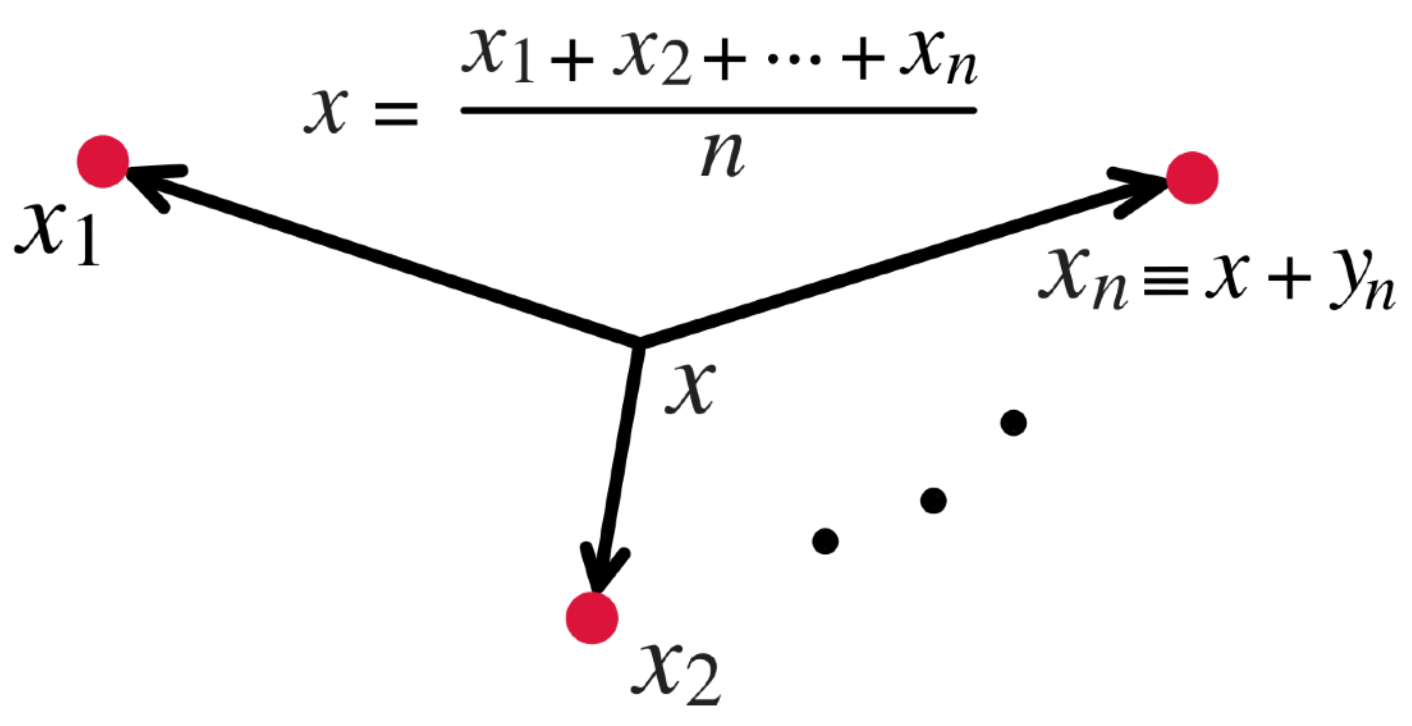
$$\partial_t G_n = \mathscr{F}\,[\psi, G_2, G_3, \ldots, G_n] + \mathcal{O}(\varepsilon^n)$$



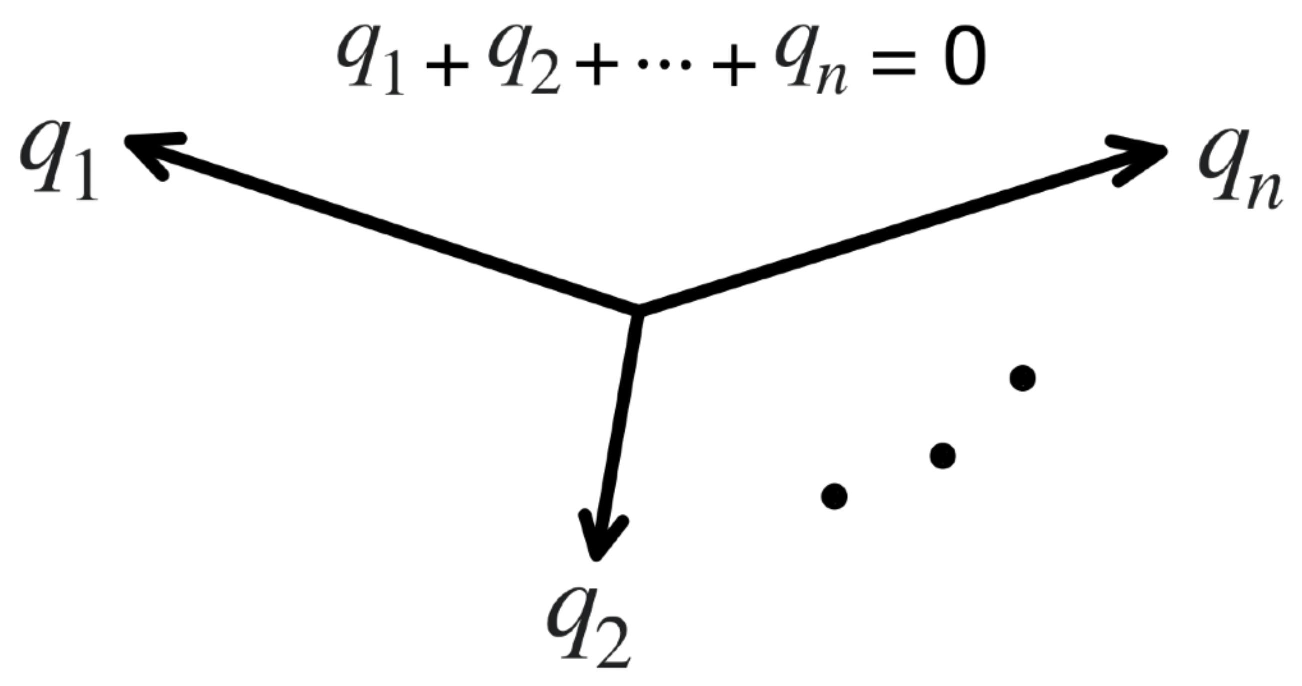
# Multi-point Wigner function

- For fluctuation fields, we introduced $n$-pt Wigner function XA et al, 2009.10742

$$W_n(x; q_1, \ldots, q_n) = \int d^3y_1 \ldots d^3y_n \, e^{-(iq_1y_1 + \ldots + iq_ny_n)} \, \delta^{(3)}\left(\frac{y_1 + \ldots + y_n}{n}\right) G_n(x; y_1, \ldots, y_n)$$




$y$-space




$q$-space

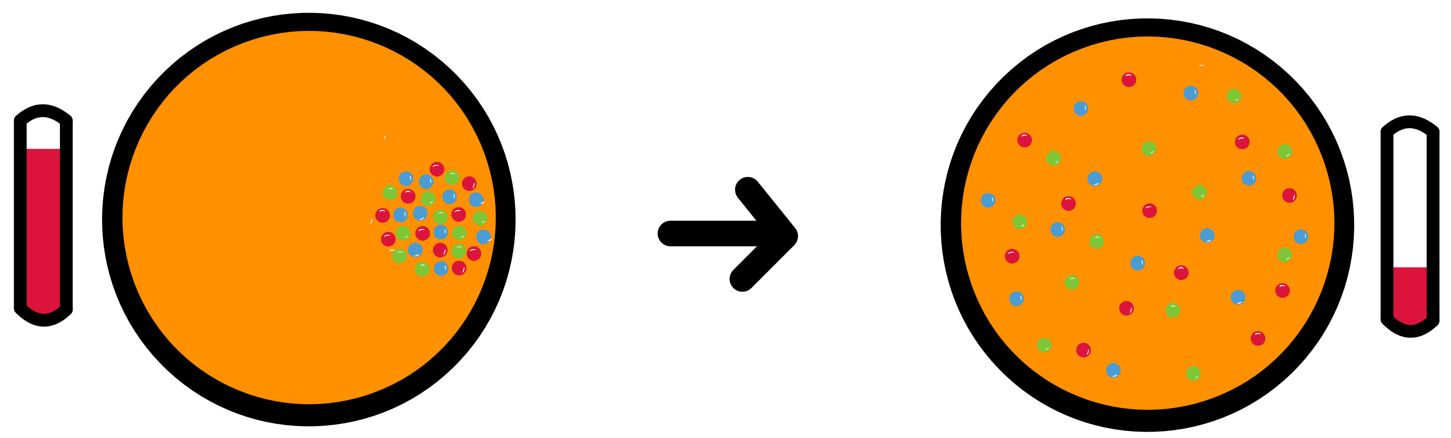
# Application: charge diffusion near critical point

- Simple charge diffusion problem: XA et al, 2009.10742

$$\partial_t n = \nabla \lambda \nabla \alpha + \eta, \qquad \langle \eta(x)\eta(y) \rangle = 2\,\nabla^{(x)} \lambda\, \nabla^{(y)}\, \delta^{(3)}(x-y)$$

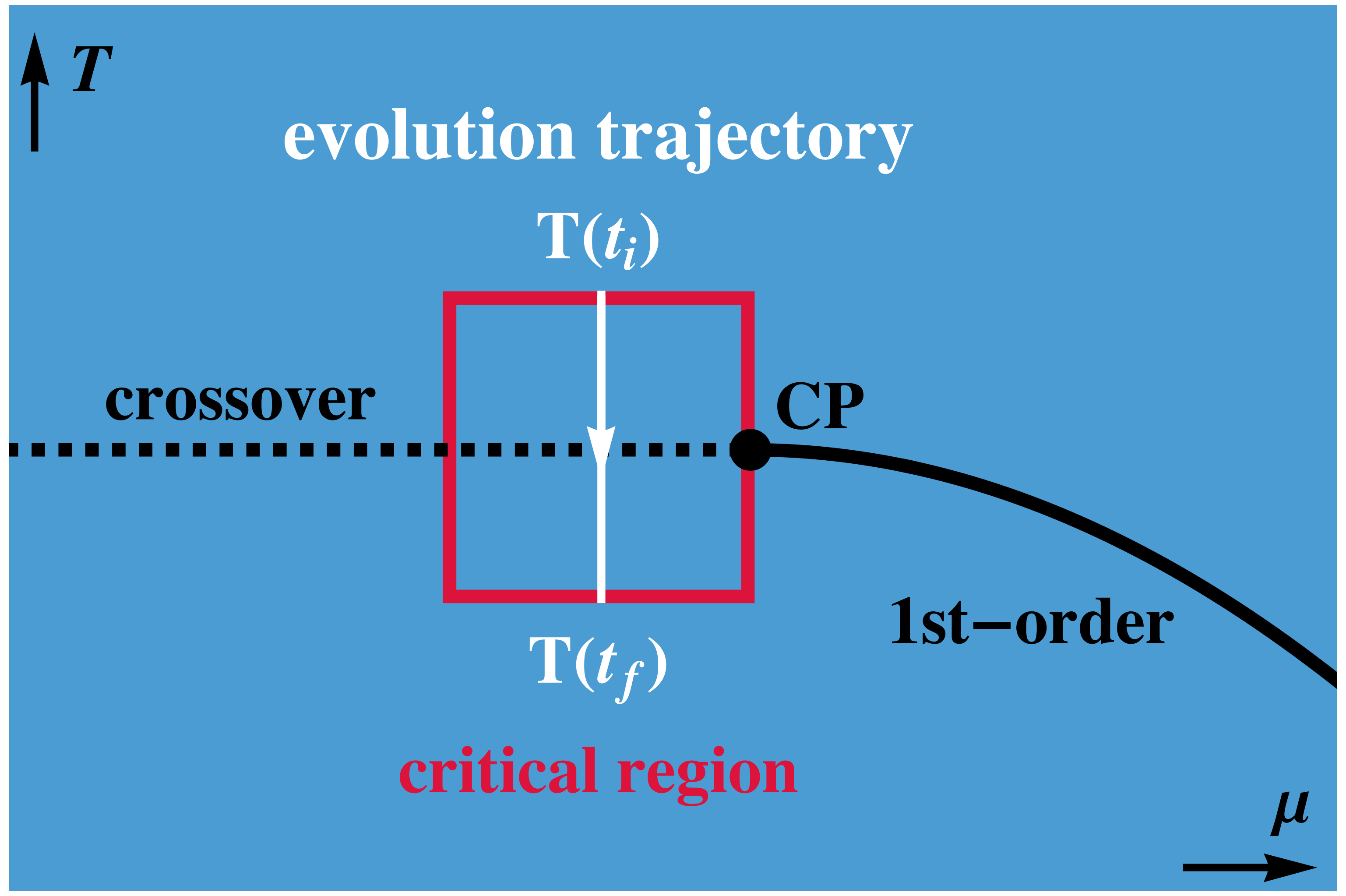
| quantities | general | diffusive charge |
|---|---|---|
| variable | $\psi_i$ | $n(\boldsymbol{x})$ |
| variable index | $i, j, k$, etc. | $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$, etc. |
| Onsager matrix | $Q_{ij}$ | $\boldsymbol{\nabla}_{\boldsymbol{x}} \lambda \boldsymbol{\nabla}_{\boldsymbol{y}}\, \delta^{(3)}_{\boldsymbol{xy}}$ |
| drift force | $F_i$ | $\boldsymbol{\nabla}_{\boldsymbol{x}} \lambda \boldsymbol{\nabla}_{\boldsymbol{x}} \alpha$ |

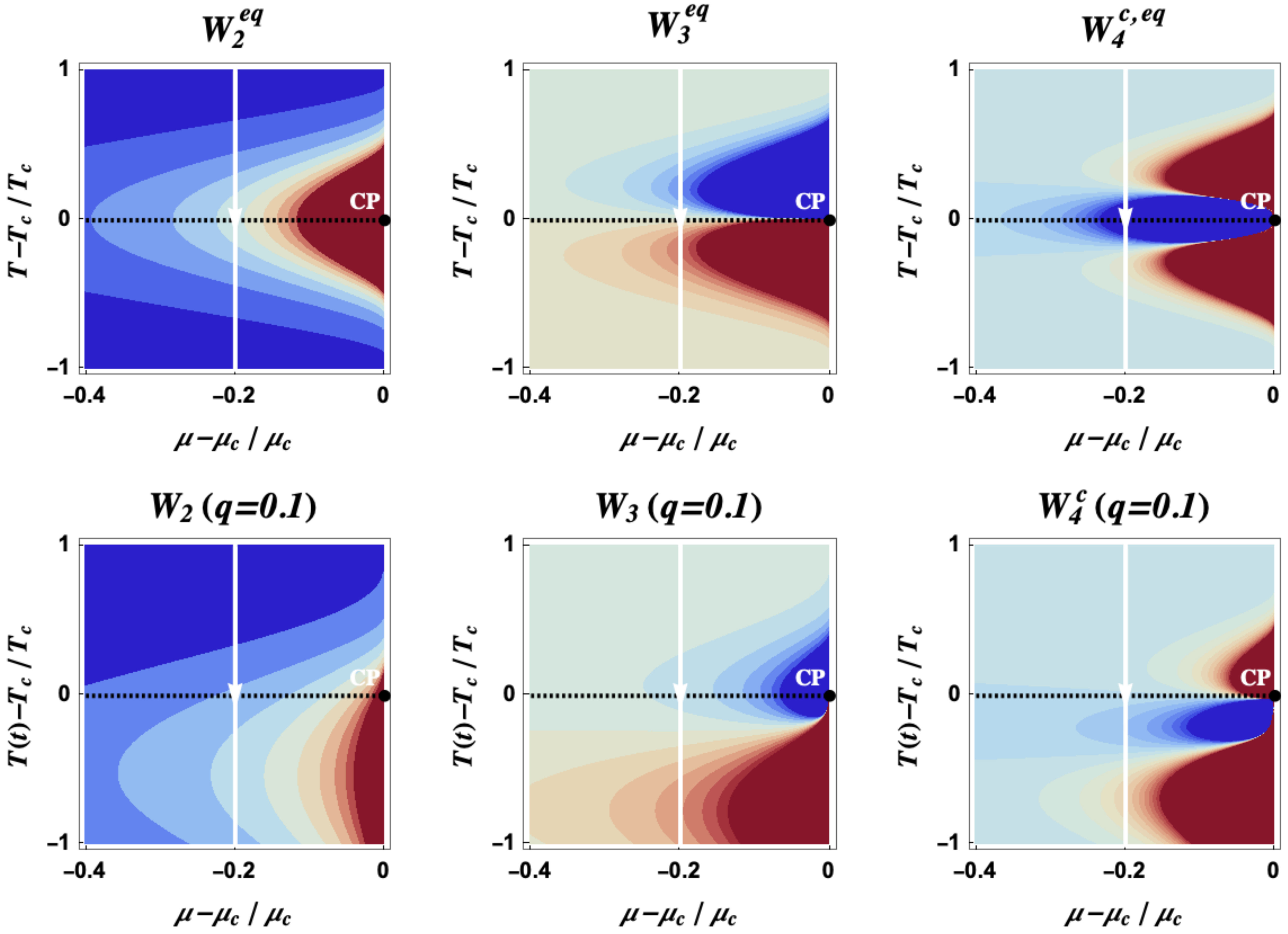
$n \equiv$ charge density; $\lambda \equiv$ conductivity; $\alpha \equiv$ chemical potential

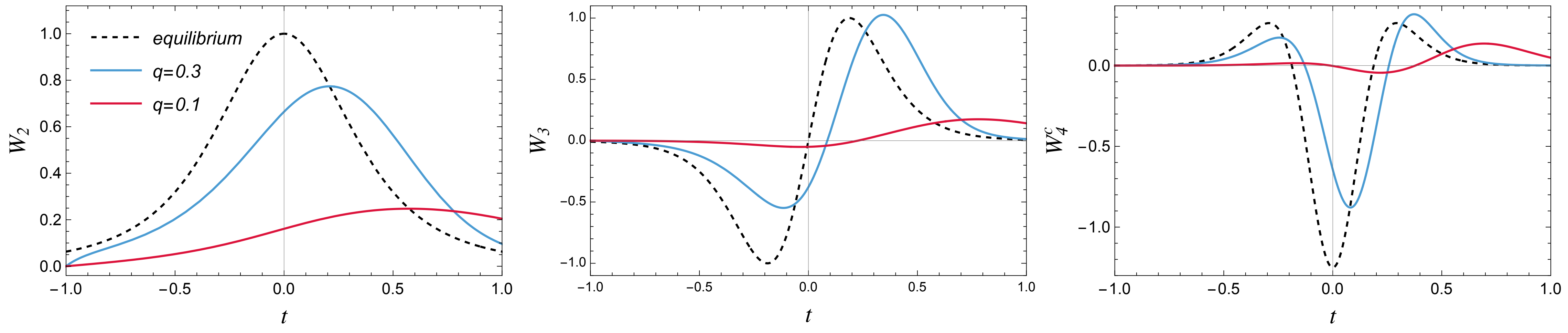


# Application: charge diffusion near critical point

- Charge diffusion near QCD critical point: memory effect XA, 2209.15005

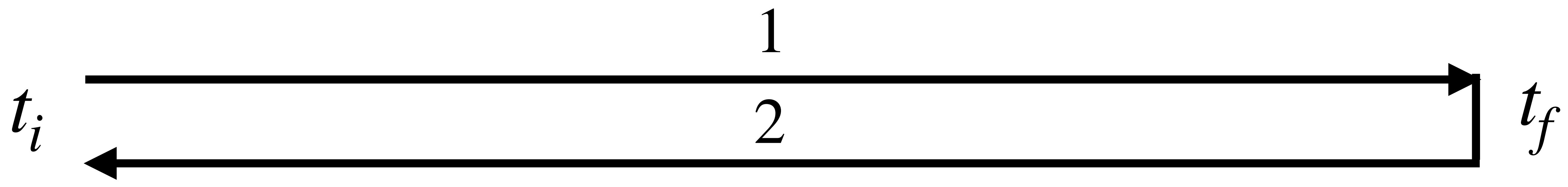


# Schwinger-Keldysh approach

- SK formalism




Schwinger Keldysh

$$Z = \int \mathcal{D}\psi_1 \mathcal{D}\psi_2 \mathcal{D}\chi_1 \mathcal{D}\chi_2 \, e^{\, i I_0(\psi_1, \chi_1) - i I_0(\psi_2, \chi_2)} = \int \mathcal{D}\psi_1 \mathcal{D}\psi_2 \, e^{\, i \int_\tau \mathcal{L}_{\rm EFT}}$$

- The effective Lagrangian is constructed following *fundamental symmetries*:

Glorioso et al, 1805.09331; Jain et al, 2009.01356

$$\mathcal{L}_{\rm EFT}(\psi_r, \psi_a) = \psi_a Q^{-1}(F - \dot{\psi}_r) + i\psi_a Q^{-1}\psi_a \quad \text{where} \quad \psi_r = \frac{1}{2}(\psi_1 + \psi_2), \quad \psi_a = \psi_1 - \psi_2$$

$$P[\psi] = \int_{\psi_r = \psi(t)} \mathcal{D}\psi_r \mathcal{D}\psi_a \, J(\psi_r) \, e^{\, i \int_{-\infty}^{t} d\tau \mathcal{L}_{\rm EFT}} \longrightarrow \partial_t P[\psi] = \frac{\partial}{\partial \psi}\left(\textbf{flux}[\psi]\right)$$

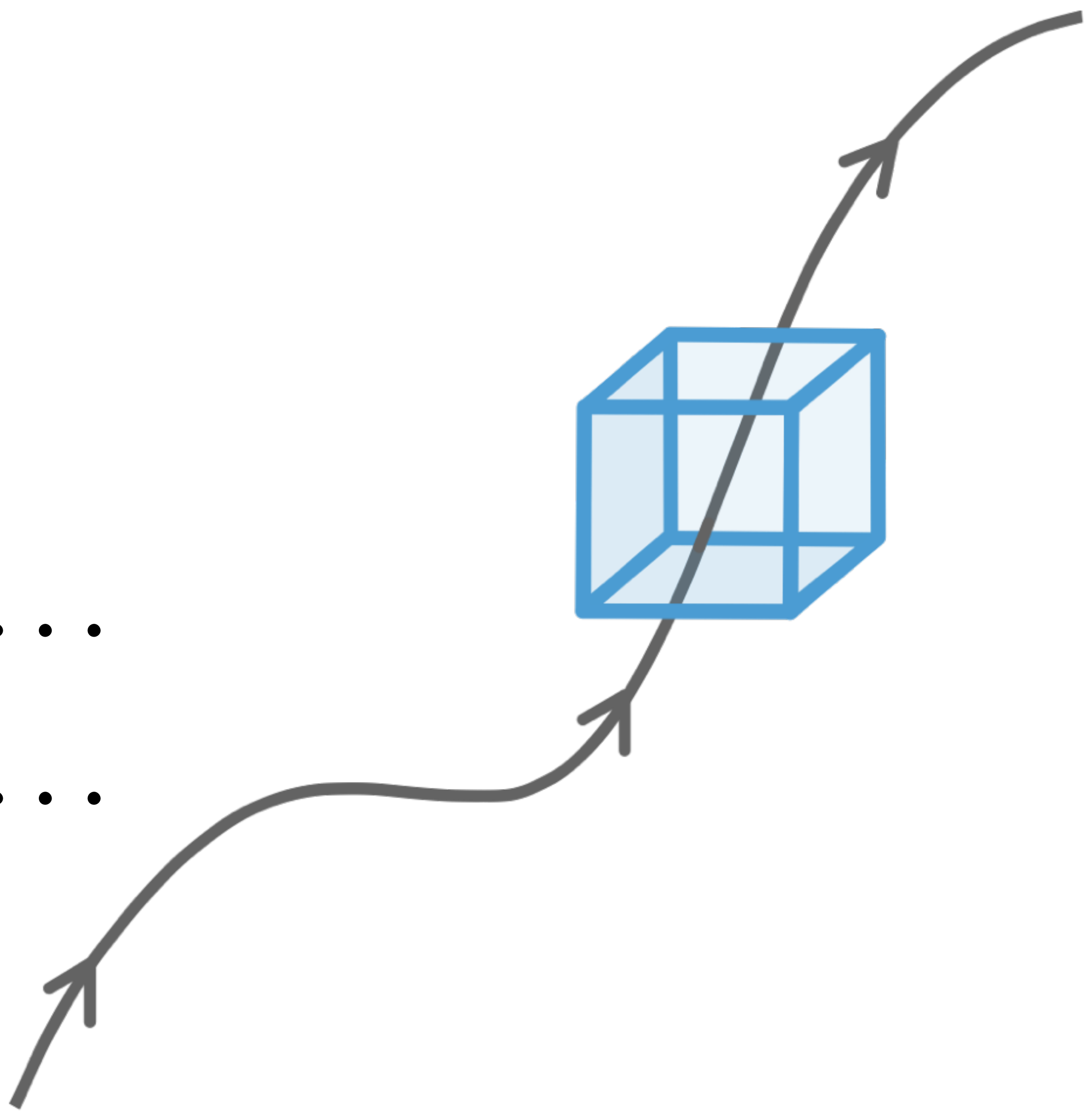
XA et al, in progress

# Relativistic dynamics

## Eulerian specification

more often used in non-relativistic theory

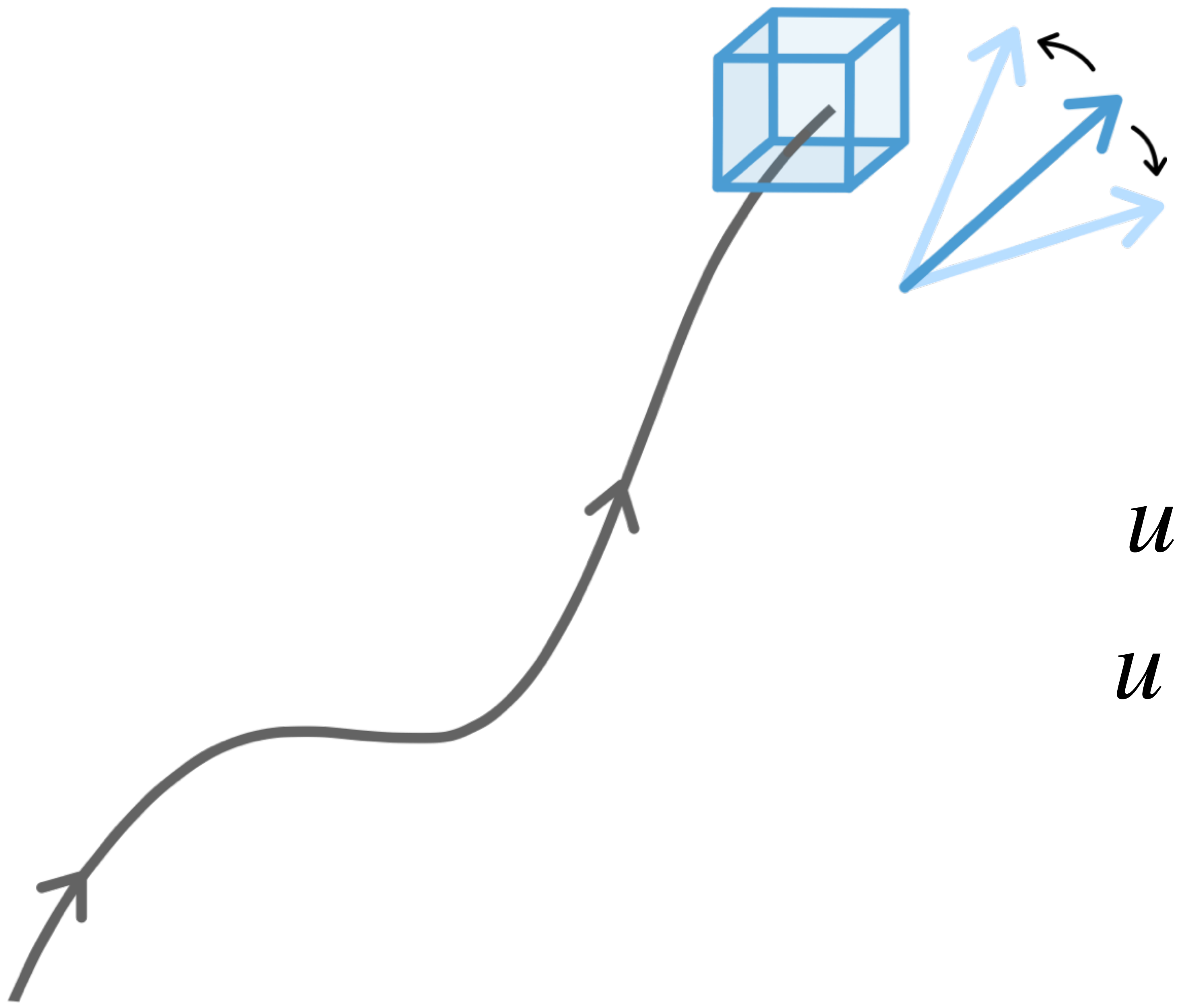
$$(\partial_t + v \cdot \nabla)\, \psi_i = \ldots$$
$$(\partial_t + v \cdot \nabla)\, G_n = \ldots$$



There is a global time for every observer.
All n-pt correlators $G_n$ can be measured
at the same time of the same lab frame.

## Lagrangian specification

more convenient for relativistic theory



$$u = u(\psi)$$

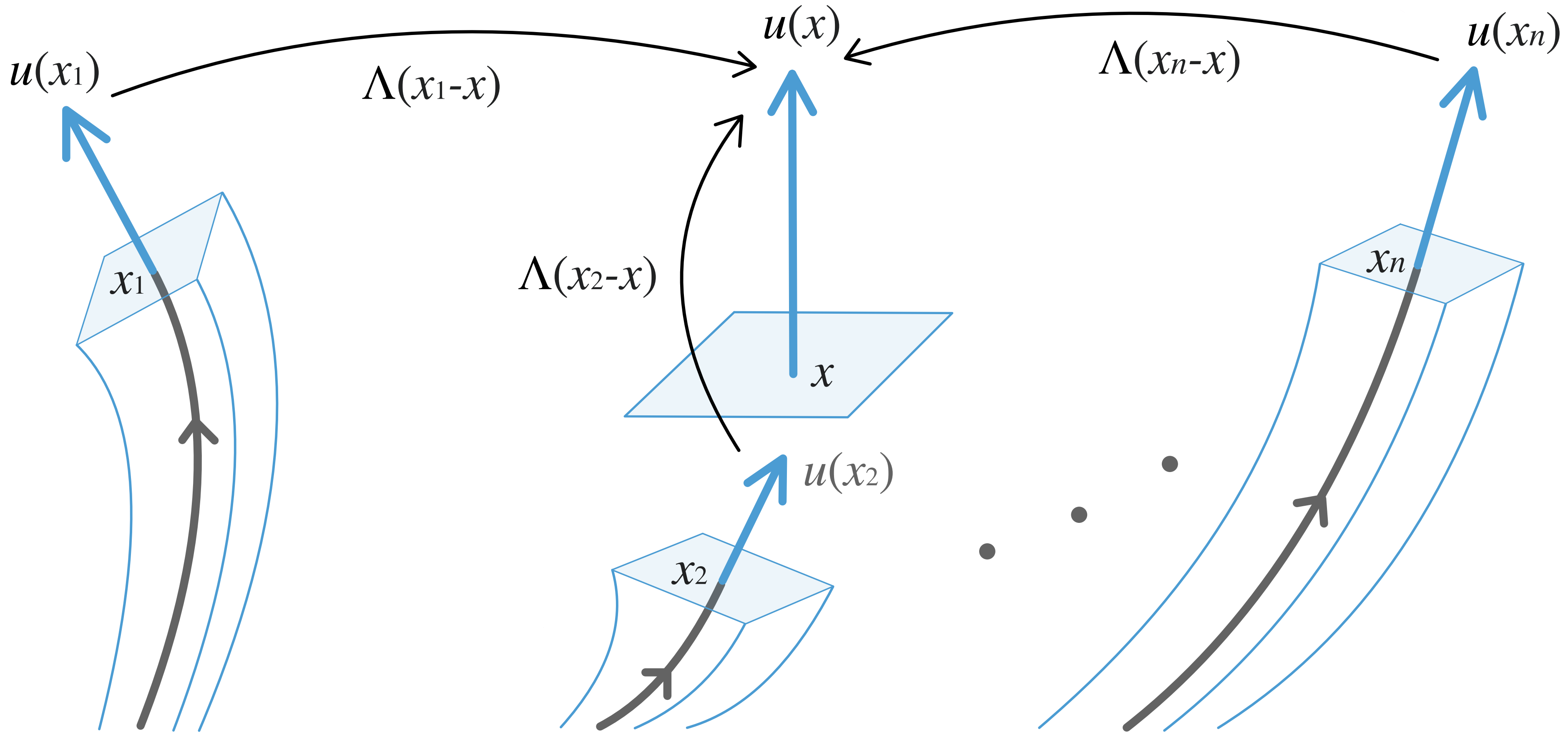
$$u \cdot \partial \psi_i = \ldots$$
$$u \cdot \partial G_n = \ldots$$

Each fluid cell has its own clock (proper time).
How to define the analogous *equal-time*
correlator $G_n$ in relativistic theory?

# Confluent formulation: correlator and derivative

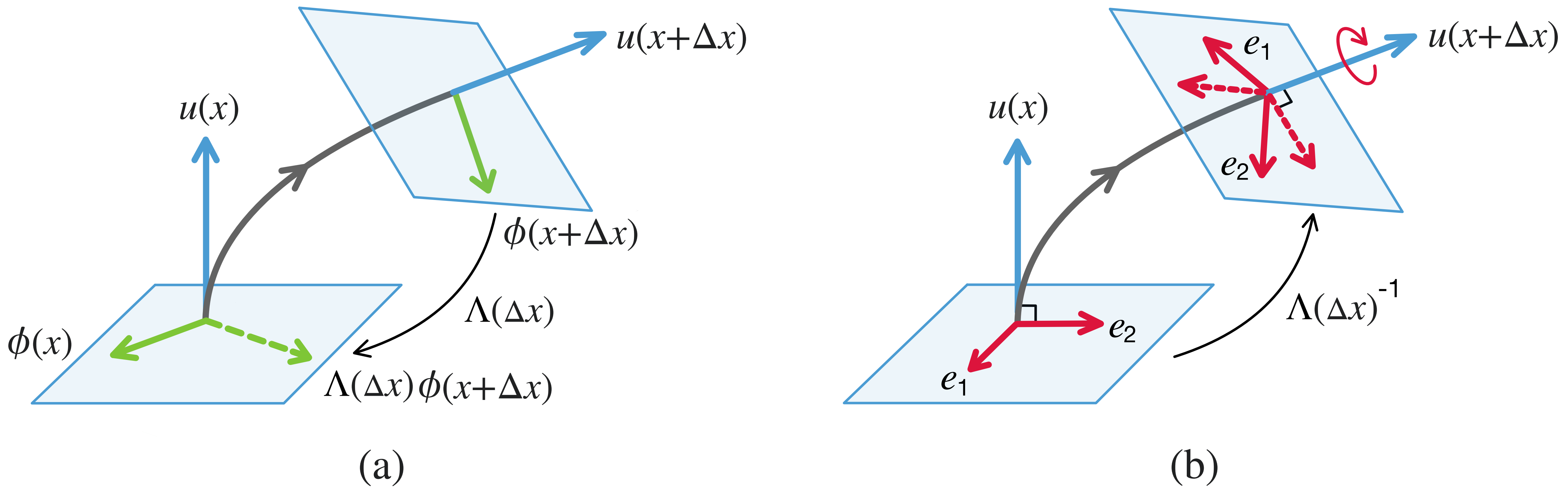
- Confluent formulation: covariant description for the comoving fluctuations.
  XA et al, 2212.14029

**Confluent correlator $\bar{G}$**




boost all fields (measured at their own local rest frame) to one common frame (chosen at their midpoint)

**Confluent derivative $\bar{\nabla}$**



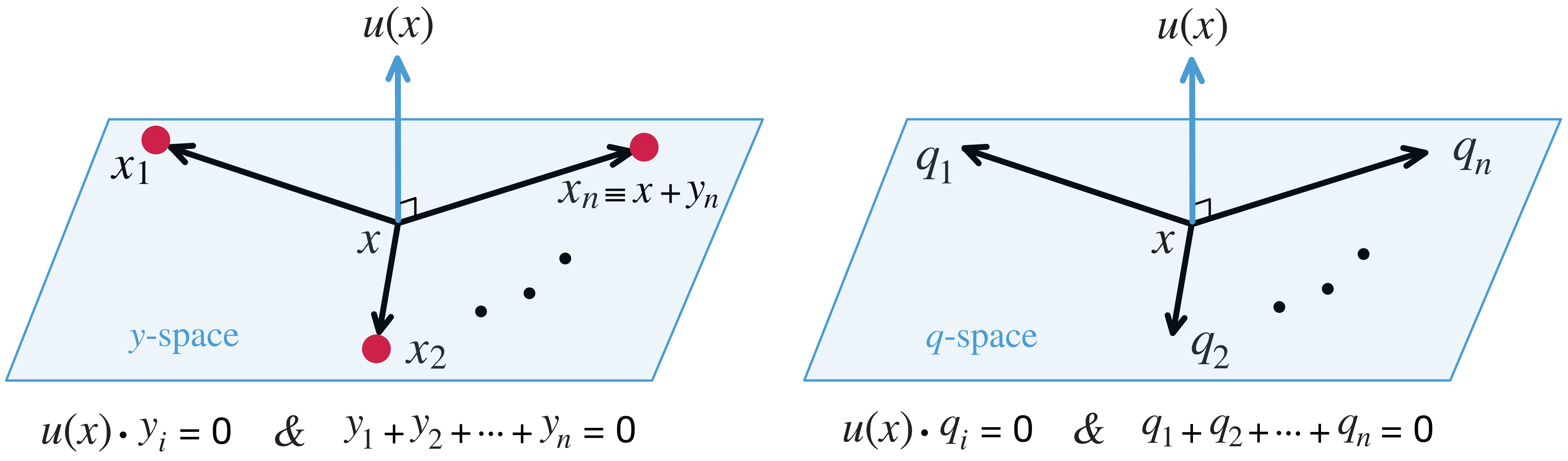

as the $n$ points move, the frame at midpoint moves accordingly, the difference of a given field before and after the movement is calculated in one same frame, with the equal-time constraint preserved by introducing the local triad $e_a^\mu$ with $a = 1,2,3$

# Confluent formulation: Wigner function

- The confluent $n$-pt Wigner transform is performed from $y^a = e^a_\mu y^\mu$ to $q^a$

XA et al, 2212.14029

$$W_n(x; q_1^a, \ldots, q_n^a) = \int \prod_{i=1}^{n} \left(d^3 y_i^a \, e^{-i q_{ia} y_i^a}\right) \delta^{(3)}\left(\frac{1}{n}\sum_{i=1}^{n} y_i^a\right) \bar{G}_n(x + e_a y_1^a, \ldots, x + e_a y_n^a)$$




(a) y-space

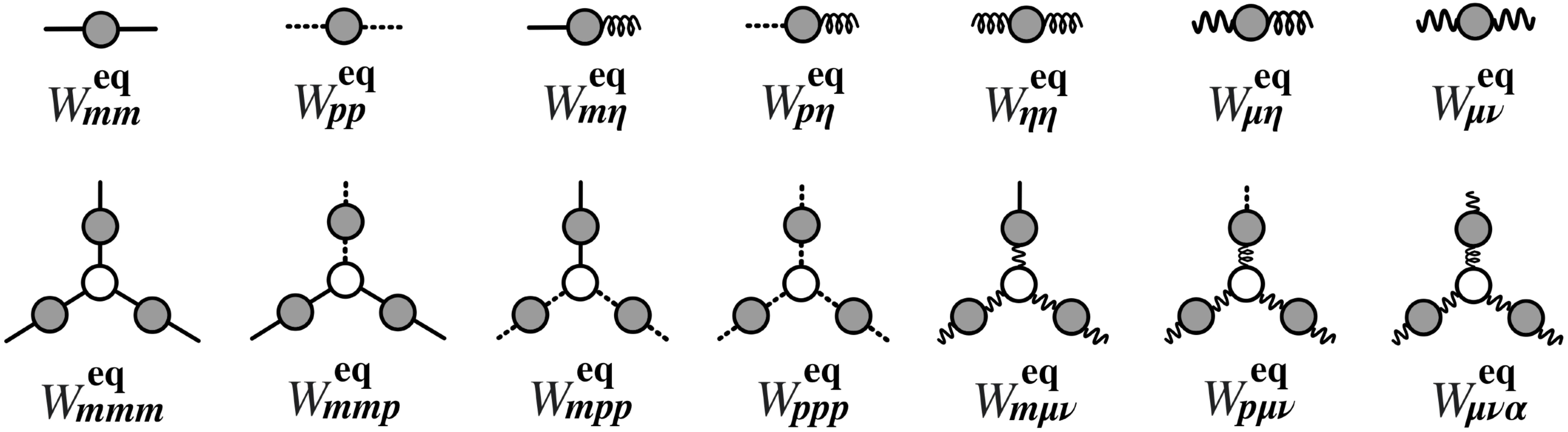
(b) q-space

# Fluctuation evolution equations

- Fluctuation evolution equations in the *impressionistic* form: XA et al, in progress

$$\mathscr{L} W_n = i c_s q (W_n - \dots) + \gamma q^2 (W_n - \dots) + k W_n + \dots \quad \text{where} \quad \mathscr{L} = u \cdot \bar{\nabla}_x + f \cdot \nabla_q$$

for which the solutions match thermodynamics with entropy $S(m, p, u_\mu, \eta)$, where $m = s/n$ and $\eta$ is a Lagrangian multiplier for $u^2 = -1$.

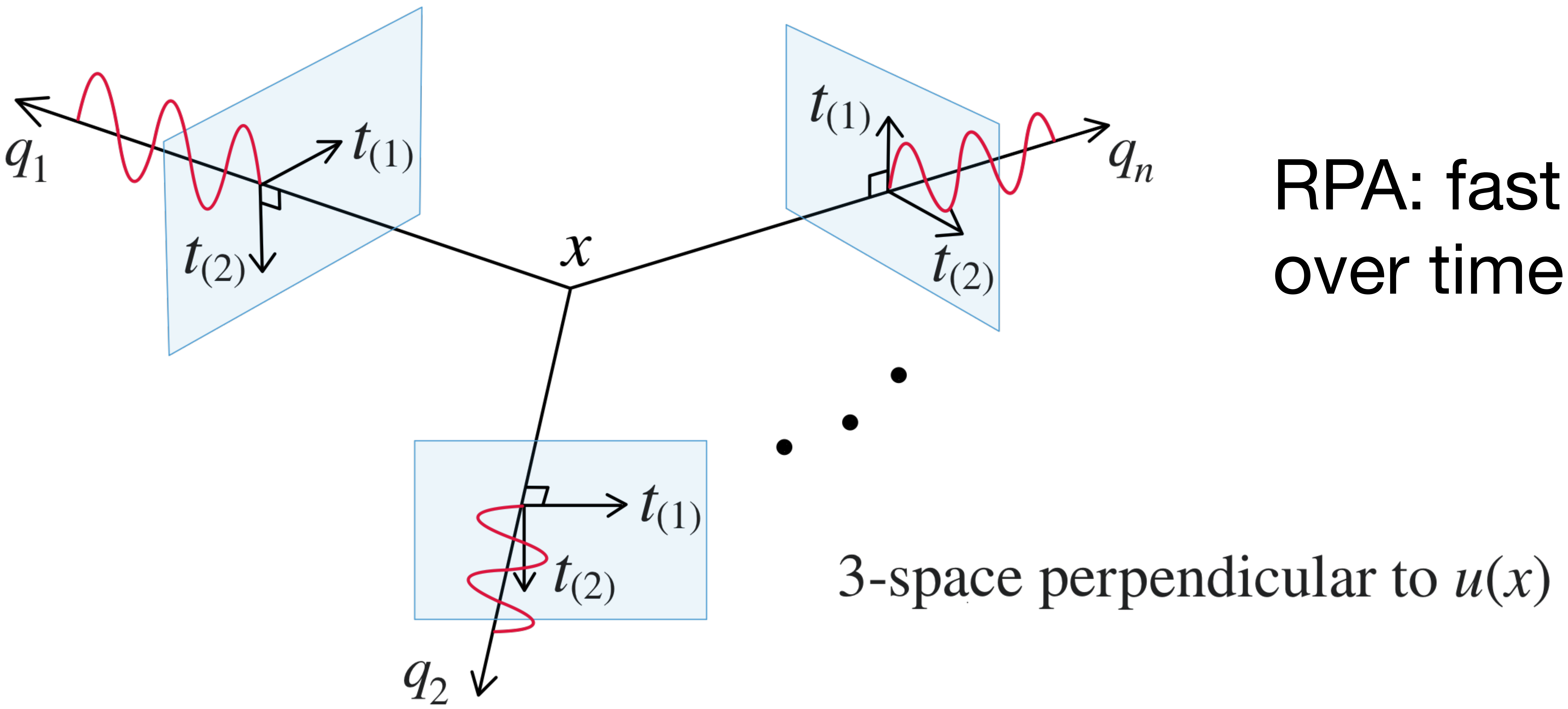



NB: given $\phi = (\delta m, \delta p, \delta u_\mu)$, one can derive 21+56+126=203 equations for the 2-pt, 3-pt and 4-pt functions——bite off more than one can chew!

# Rotating phase approximation

- $n$-pt functions are analogous to $n$-particle states lying in the Fock space. One can choose a set of new bases s.t. the ideal hydro equations are diagonalized:

$$\phi = \begin{pmatrix} \phi_m \\ \phi_p \\ \phi_\mu \end{pmatrix} = \begin{pmatrix} \delta m \\ \delta p \\ \delta u_\mu \end{pmatrix} \longrightarrow \Phi = \begin{pmatrix} \Phi_m \\ \Phi_\pm \\ \Phi_{(i)} \end{pmatrix} \sim \begin{pmatrix} \delta m \\ \delta p \pm c_s \hat{q} \cdot (w \delta u) \\ t_{(i)} \cdot \delta u \end{pmatrix} \quad i = 1,2$$




RPA: fast modes are averaged out over time scales $1/c_s q \ll \Delta t < 1/\gamma q^2$.

# Slow modes

- Under RPA, slow modes are $n$-pt functions $W_n(q_1, \ldots, q_n)$ subject to

$$\sum_{i=1}^{n} \lambda(q_i) = 0 \quad \text{where} \quad \lambda_\pm(q) = \pm\, c_s\,|\,q\,|\,, \quad \lambda_m(q) = \lambda_{(i)}(q) = 0\,.$$

E.g., $W_{+-}(q_1, q_2)$ is a slow mode since $\lambda_+(q_1) + \lambda_-(q_2) = c_s(\,|\,q_1\,| - |\,q_2\,|\,) = 0.$

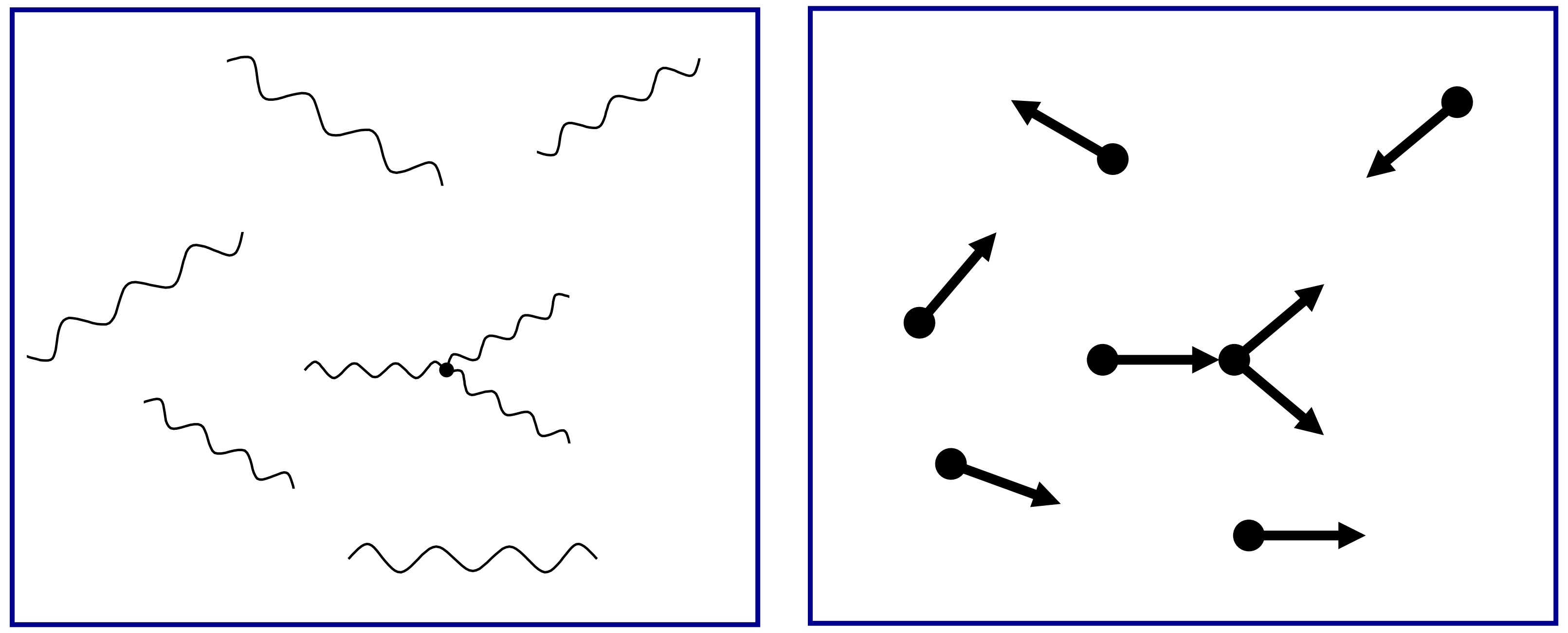
As a result, there are only 7+10+15=32 equations.

E.g., for 2-pt functions the slow modes are $W_{mm}, W_{m(i)}, W_{(i)(j)}, W_{+-}$.

- Phonon interpretation: XA et al, 1902.09517

$$\mathscr{L} W_{+-} = -\,\gamma_L\, q^2 \left( W_{+-} - \frac{T}{E} \right)$$

Bose-Einstein distribution at high T : $f = \dfrac{1}{e^{E/T} - 1} \approx \dfrac{T}{E}$



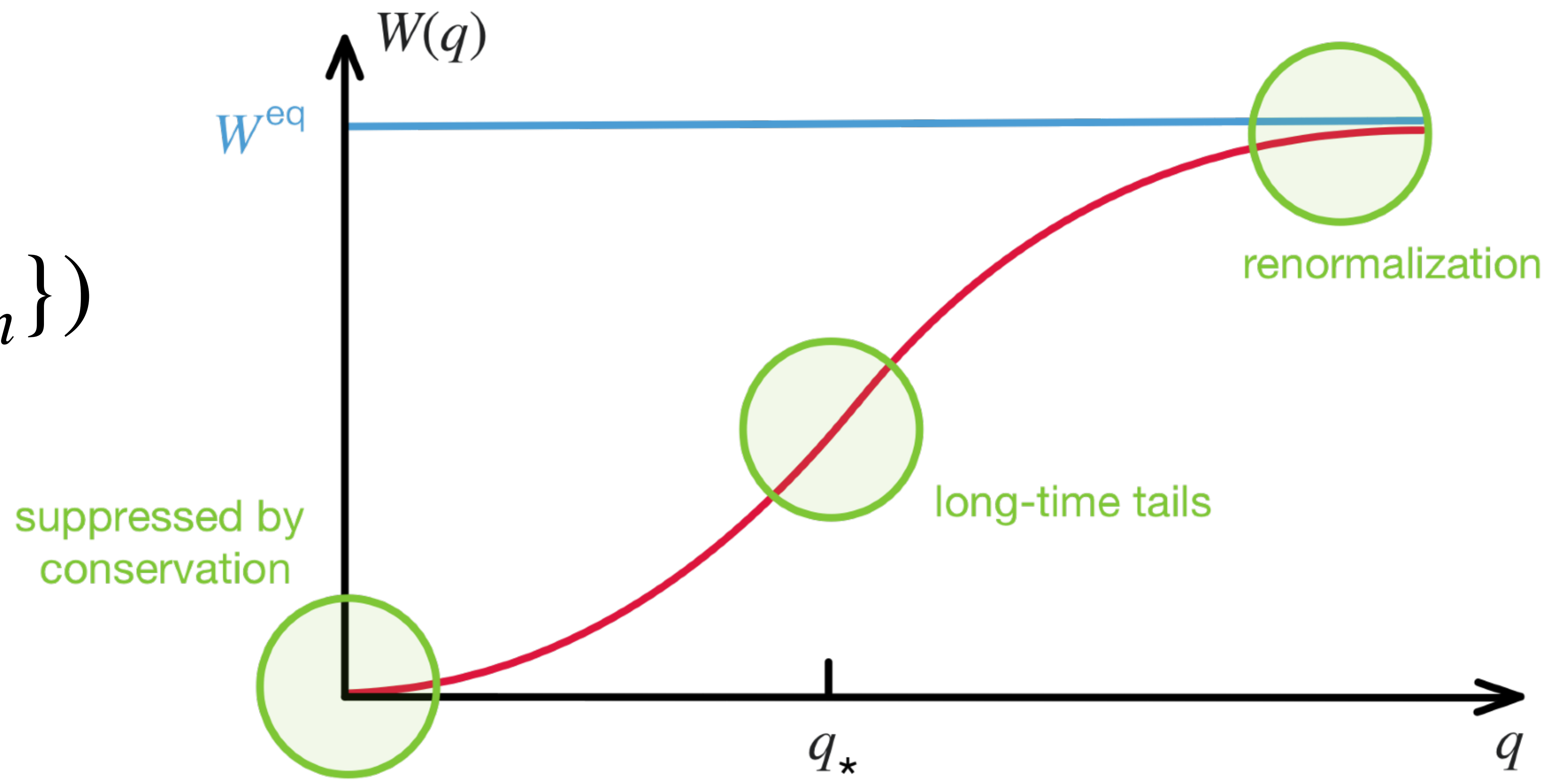
Kinetic picture (from Schaefer)

NB: there is no analogous phonon interpretation for non-Gaussian fluctuations.

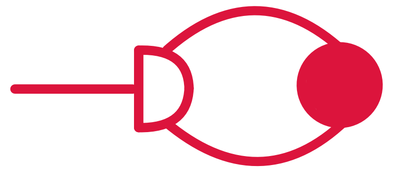
# Fluctuation feedback

- Hydrodynamic fluctuations renormalize bare quantities order by order in gradient expansion.

$$
\begin{aligned}
T_{\mu\nu}^{\text{physical}} &= T_{\mu\nu}^{(0)} + T_{\mu\nu}^{(1)} + T_{\mu\nu}^{(2)} + \ldots + \delta T_{\mu\nu}(\{G_n\}) \\
&= T_{\mu\nu}^{R(0)} + T_{\mu\nu}^{R(1)} + T_{\mu\nu}^{R(2)} \\
&\quad + \tilde{T}_{\mu\nu}^{(3/2)} + \tilde{T}_{\mu\nu}^{(3)} + \tilde{T}_{\mu\nu}^{(9/2)} + \ldots
\end{aligned}
$$




where $G_n(x) = \int d^3q_1 \ldots d^3q_n \delta^{(3)}(q_1 + \ldots + q_n) W_n(x, q_1, \ldots, q_n)$

Long-time tail due to $n$-pt functions is of order $\varepsilon^{n-1} \sim q^{3(n-1)} \sim k^{3(n-1)/2}$, the leading $k^{3/2}$ behavior results from 2-pt functions (via  ).

# Recap

- Various approaches for fluctuating hydrodynamics are developed, with their own advantages and disadvantages. See also Schaefer's talk

- Our framework for fluctuations dynamics now incorporates non-Gaussian fluctuations of fluid velocity.

# Outlook

- Need efforts to simulate the fluctuation equations with background.

- Need freeze-out prescription for the connection to observables. See Pradeep's talk

- More…