Tensor renormalization group approach to higher-dimensional quantum fields on a lattice

Shinichiro Akiyama [a), b)]

[a)] Endowed Chair for Quantum Software, University of Tokyo

[b)] Institute for Physics of Intelligence, University of Tokyo

2023.4.7 @ INT, University of Washington

Tensor Networks in Many Body and Quantum Field Theory

Contents

Research motivation

Tensor network & Lattice field theory

✔ **A method to investigate quantum many-body system expressing an objective function as a tensor contraction (= tensor network).** Orús, APS Physics 1(2019)538-550

✔ The natural application is QFT on a lattice, which gives us a finite-dimensional description of the original QFT.

Bañuls-Cichy, Rep. Prog. Phys. 83(2020)024401
Meurice-Sakai-Unmuth–Yockey, Rev. Mod. Phys. 94(2022)025005
Okunishi-Nishino-Ueda, J. Phys. Soc. Jap. 91(2022)062001

✔ TN method provides us with various ways to investigate lattice QFT.

- w/ the Hamiltonian formalism

  Describe a state vector as a TN, which is **variationally optimized**.

  Cf. **DMRG**, **TEBD**

  White, PRL69(1992)2863-2866, White, PRB48(1993)10345-10356
  Vidal, PRL91(2003)147902, Vidal, PRL98(2007)070201
  Cf. Talks in 4/3~4/6

- **w/ the Lagrangian formalism**

  Describe a path integral as a TN, which is **approximately contracted**.

  Cf. **TRG**, **TNR, Loop-TNR, GILT**

  Levin-Nave, PRL99(2007)120601
  Evenbly-Vidal, PRL115(2015)180405, Evenbly, PRB95(2017)045117
  Yang-Gu-Wen, PRL118(2017)110504
  Hauru-Delcamp-Mizera, PRB97(2018)045111

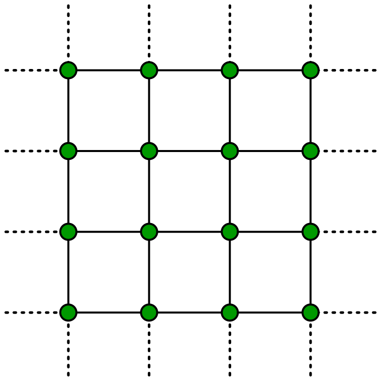
Advantages of the TRG approach

✔ Tensor renormalization group (TRG) approximately contract a given TN based on the idea of real-space renormalization group.

- No sign problem
- **The computational cost scales logarithmically w. r. t. system size**
- **Direct evaluation of the Grassmann integrals**
- **Direct evaluation of the path integral**

✔ **Applicability to the higher-dimensional systems**

- If the system is translationally invariant on a lattice, we can easily apply the TRG to contract the TN.
- TRG would give us valuable information for the future development of higher-dimensional TN algorithms.
  - PEPS, Fermionic PEPS, Tree TN, isoTNS, Fermionic isoTNS
  - Improvement of the TRG based on the removement of short-range correlations



Cf. Next talk by Ryo Sakai

Current status of (3+1)D TN calculations

Hamiltonian formalism	Lagrangian formalism
・QED at finite density Magnifico+	・Ising model SA+ ・Staggered fermion w/ strongly coupled U(N) Milde+ ・Complex ϕ^4 theory at finite density SA+ ・Nambu—Jona-Lasinio model at finite density SA+ ・Real ϕ^4 theory SA+ ・$\mathbb{Z}_2$ gauge-Higgs at finite density SA-Kuramashi

✔ So far, the (3+1)D TN calculations have been driven by the Lagrangian formalism w/ the TRG approach.

✔ Development of **parallel computing method** specialized for individual algorithms to reduce their execution time per process.

SA+, PoS(LATTICE2019)138

Yamashita-Sakurai, CPC278(2022)108423

Current status of higher-dimensional TRGs

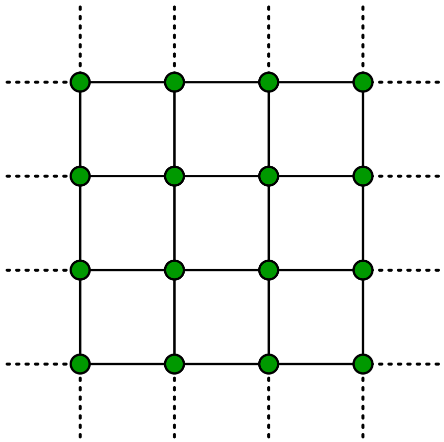
Algorithm	Cost	$d = 3$	$d = 4$
HOTRG Xie+, PRB86(2012)045139	$D^{4d-1}\ln L$	Ising Xie+, Potts model Wang+, free Wilson fermion Sakai+, $\mathbb{Z}_2$ gauge theory Dittirich+, Kuramashi-Yoshimura, U(1) gauge theory Judah Unmuth-Yockey	Ising model SA+, Staggered fermion w/strongly coupled U(N) Milde+
Anisotropic TRG (ATRG) Adachi-Okubo-Todo, PRB102(2020)054432	$D^{2d+1}\ln L$	Ising model Adachi+, SU(2) gauge Kuwahara-Tsuchiya, Real ϕ^4 theory SA+, Hubbard model SA-Kuramashi, $\mathbb{Z}_2$ gauge-Higgs SA-Kuramashi	Complex ϕ^4 theory SA+, NJL model SA+, Real ϕ^4 theory SA+, $\mathbb{Z}_2$ gauge-Higgs SA-Kuramashi
Triad RG Kadoh-Nakayama, arXiv:1912.02414	$D^{d+3}\ln L$	Ising model Kadoh-Nakayama, O(2) model Bloch+, $\mathbb{Z}_3$ (extended) clock model Bloch+, Potts models Raghav G. Jha	-

D: bond dimension, L: linear system size, d: spacetime dimension

TRG & Matrix product decomposition

Procedure of TRG approach

1) Represent the path integral as a tensor network.

$Z \quad \rightarrow$ 

- Some approximation is necessary for continuous degrees of freedom.

Cf. Meurice-Sakai-Unmuth–Yockey, Rev. Mod. Phys. 94(2022)025005
Meurice, "Quantum Field Theory, A quantm computation approach"

2) Take contractions approximately.

- Various algorithms are proposed.
- In 2D, we can also use other schemes to take contractions approximately.

Cf. iTEBD for 2D classical Ising model: Orús-Vidal, PRB78(2008)155117

TN rep. for 2d Ising model w/ PBC

Decompose nearest-neighbor interactions

$$Z = \Sigma_{\{\sigma=\pm 1\}} \Pi_{n,\mu} \exp[\beta J \sigma_n \sigma_{n+\hat{\mu}}] \quad \Rightarrow \quad Z = \mathrm{Tr}[\Pi_n T_{x_n y_n x'_n y'_n}]$$

$T_{x_n y_n x'_n y'_n}$ specifies the details of the model

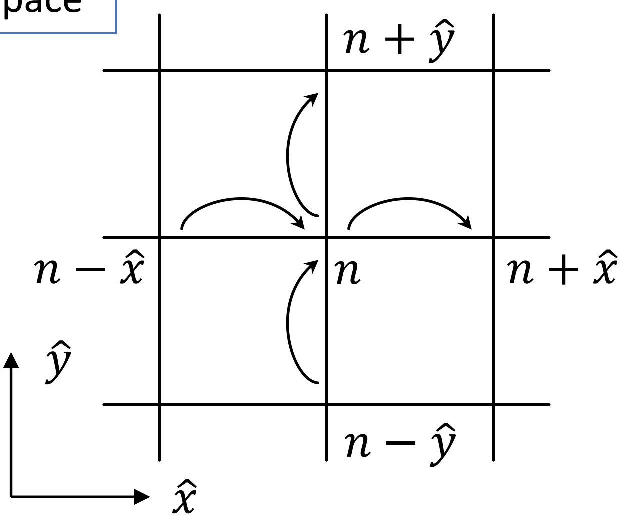
$$\exp[\beta J \sigma_n \sigma_{n+\hat{\mu}}] = \sum_{l_n} \sqrt{\lambda_{l_n}} U(\sigma_n, l_n) \sqrt{\lambda_{l_n}} U(\sigma_{n+\hat{\mu}}, l_n) = \sum_{l_n} W(\sigma_n, l_n) W(\sigma_{n+\hat{\mu}}, l_n)$$

$$W(a,b) \coloneqq \sqrt{\lambda_b} U(a,b)$$

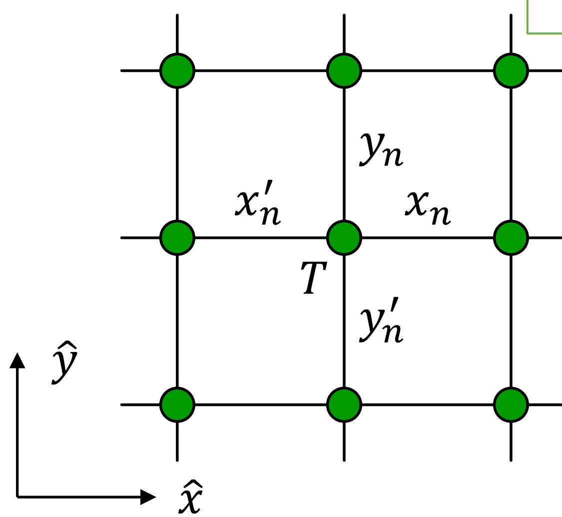
$$T_{x_n y_n x'_n y'_n} \coloneqq \sum_{\sigma_n = \pm 1} W(\sigma_n, x_n) W(\sigma_n, y_n) W(\sigma_n, x'_n) W(\sigma_n, y'_n)$$

$$x'_n \coloneqq x_{n-\hat{x}}, \; y'_n \coloneqq y_{n-\hat{y}}$$

Real Space

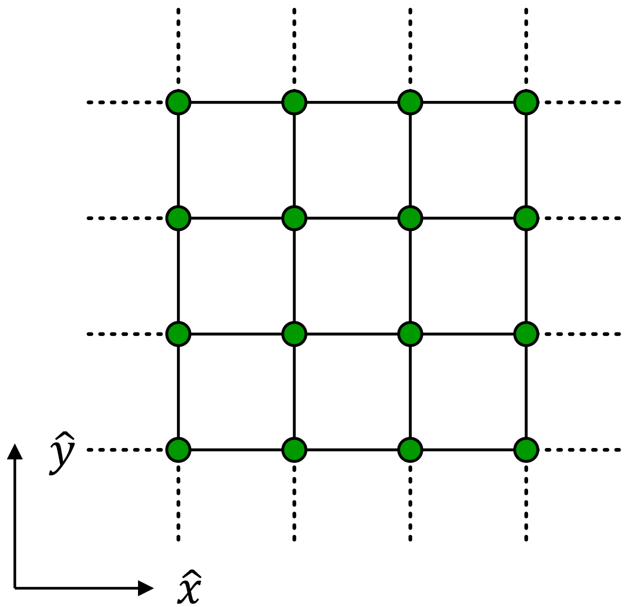


TN rep. for Z



Basic concept of TRG algorithm

We cannot perform the contractions in TN rep. exactly (too many d. o. f.)



Idea of real-space renormalization group

Iterate a simple transformation **w/ approximation** and we can investigate thermodynamic properties

+

Information compression w/ the Singular Value Decomposition (SVD)

$$A_{ij} = \Sigma_k U_{ik} \sigma_k V_{jk} \approx \Sigma_{k=1}^{D} U_{ik} \sigma_k V_{jk}$$

w/ $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\min(m,n)} \geq 0$

(A: $m \times n$ matrix, U: $m \times m$ unitary, V: $n \times n$ unitary)

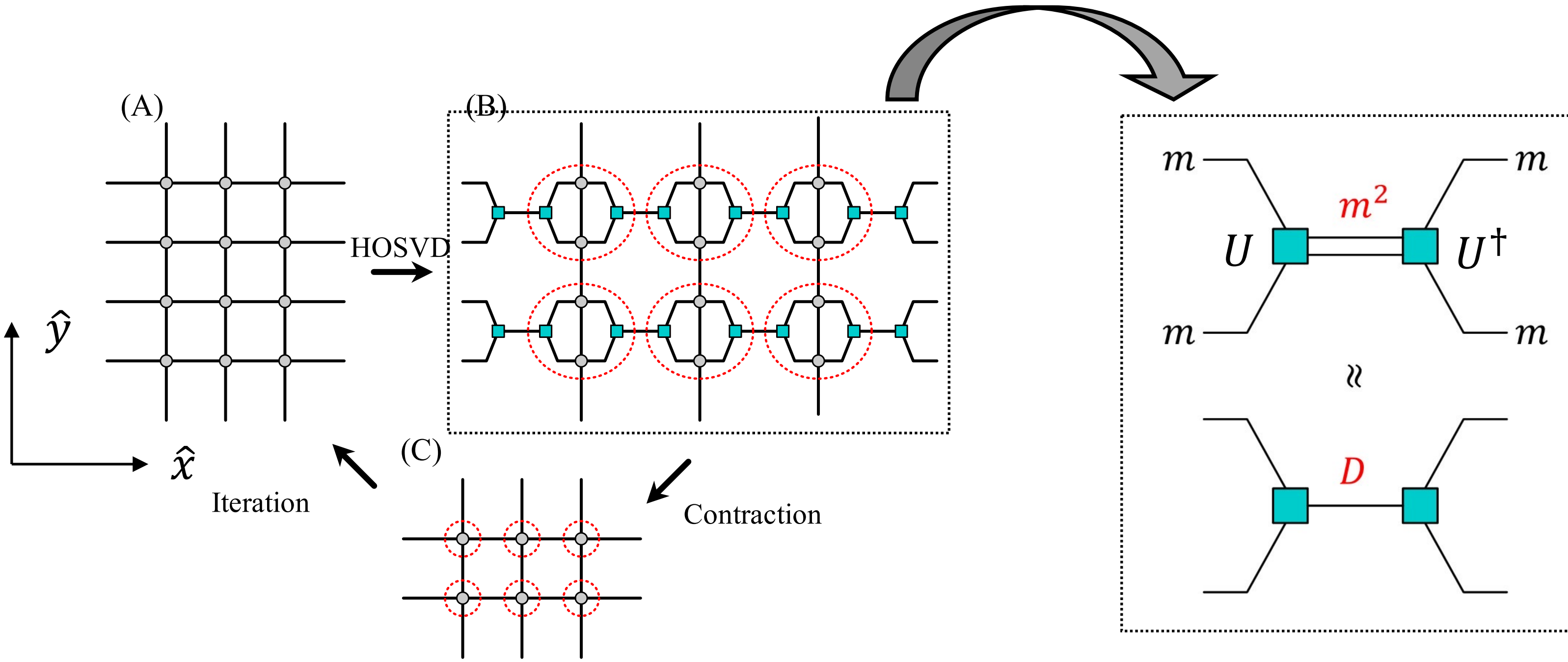
↓

TRG employs the SVD to reduce d. o. f. and perform the tensor contraction approximately

Higher-order TRG (HOTRG)

Xie+, PRB86(2012)045139

✔ Applicable to any d-dimensional lattice



Sequential coarse-graining along with each direction

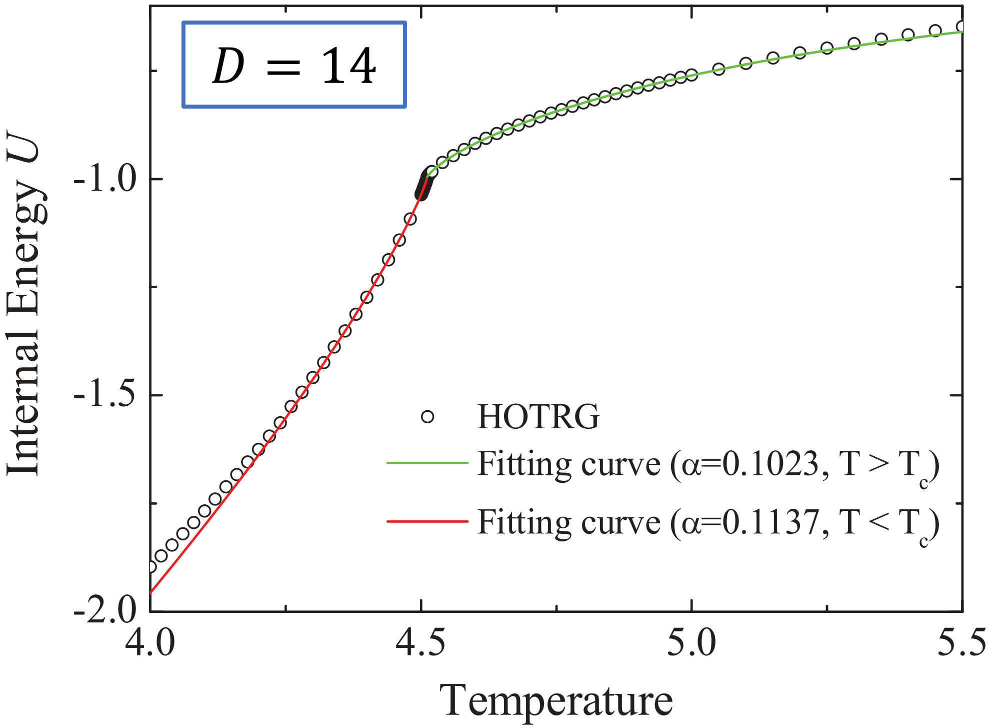
***D*: bond dimension**

✔ # of tensors are reduced to half.
Iterating this CG n times, we can approximately contract 2^n tensors.

Cf. Talk by James Osborn

Example: 3d Ising model w/ HOTRG

Xie+, PRB86(2012)045139



Critical point

Method	T_c
HOTRG (D = 16, from U)	4.511544
HOTRG (D = 16, from M)	4.511546
Monte Carlo[37]	4.511523
Monte Carlo[38]	4.511525
Monte Carlo[39]	4.511516
Monte Carlo[35]	4.511528
Series expansion[40]	4.511536
CTMRG[12]	4.5788
TPVA[13]	4.5704
CTMRG[14]	4.5393
TPVA[16]	4.554
Algebraic variation[41]	4.547

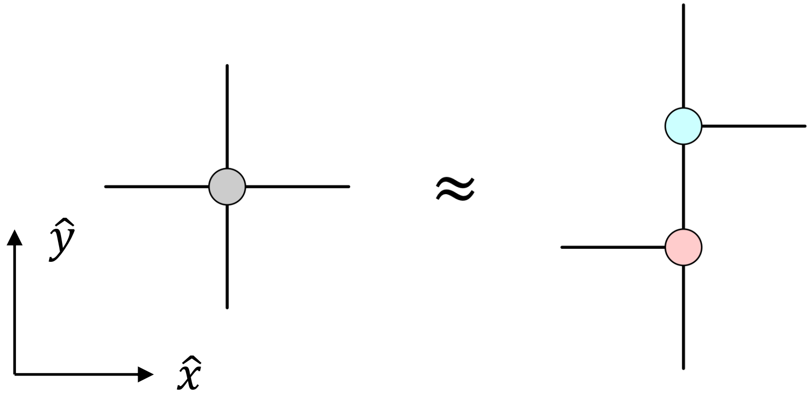
Good agreement with
the Monte Carlo results

Anisotropic TRG (ATRG)

Adachi-Okubo-Todo, PRB102(2020)054432

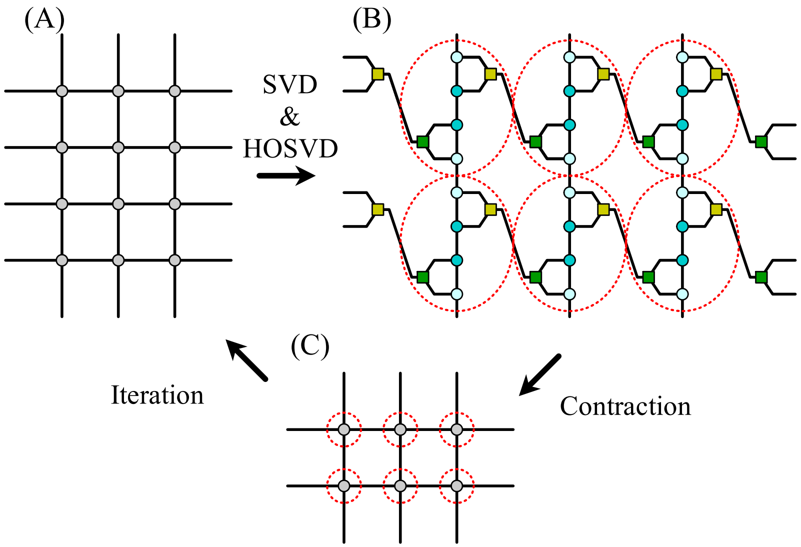
✔ Applicable to any d-dimensional lattice

✔ Accuracy with the fixed computational time is improved compared with the HOTRG.



$$T_{xyx'y'} \approx \sum_{i=1}^{D} A_{xyi} B_{x'y'i}$$

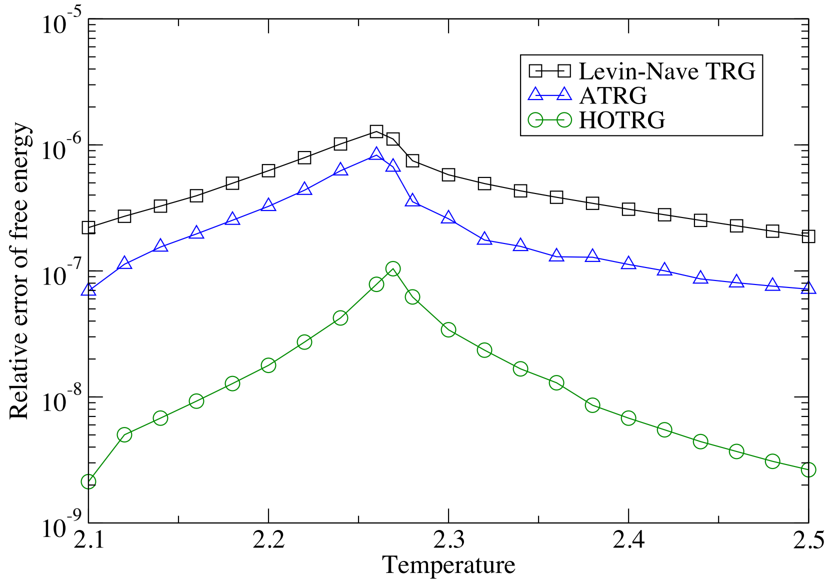
ATRG considers the block-spin transformation within lower-rank tensors. (Memory: $O\left(D^{2d}\right) \rightarrow O\left(D^{d+1}\right)$)



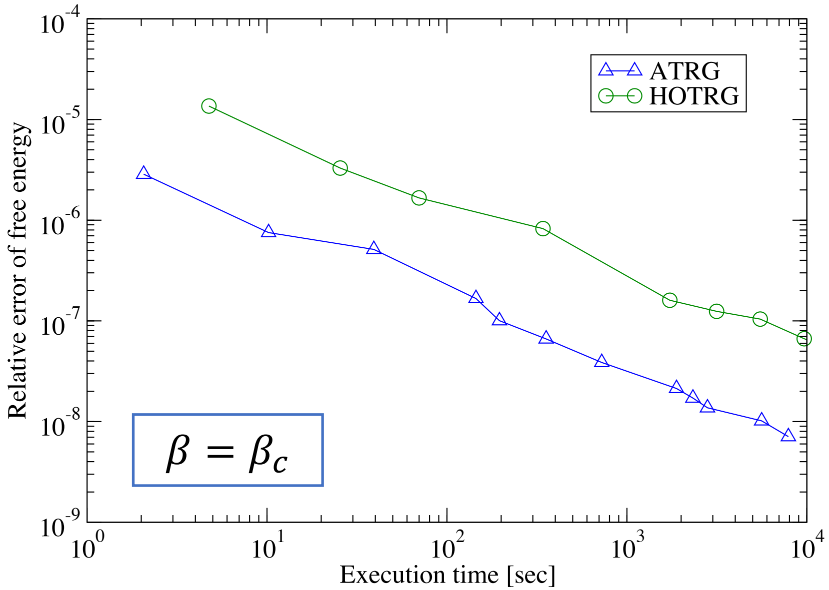
# of tensors are reduced to half

ATRG for 2d Ising model

Comparison of three types of TRG w/ $D = 24$



Relative error vs execution time



✔ HOTRG & ATRG improve the accuracy of the original (LN-)TRG at the same D. The exact solution is well reproduced.

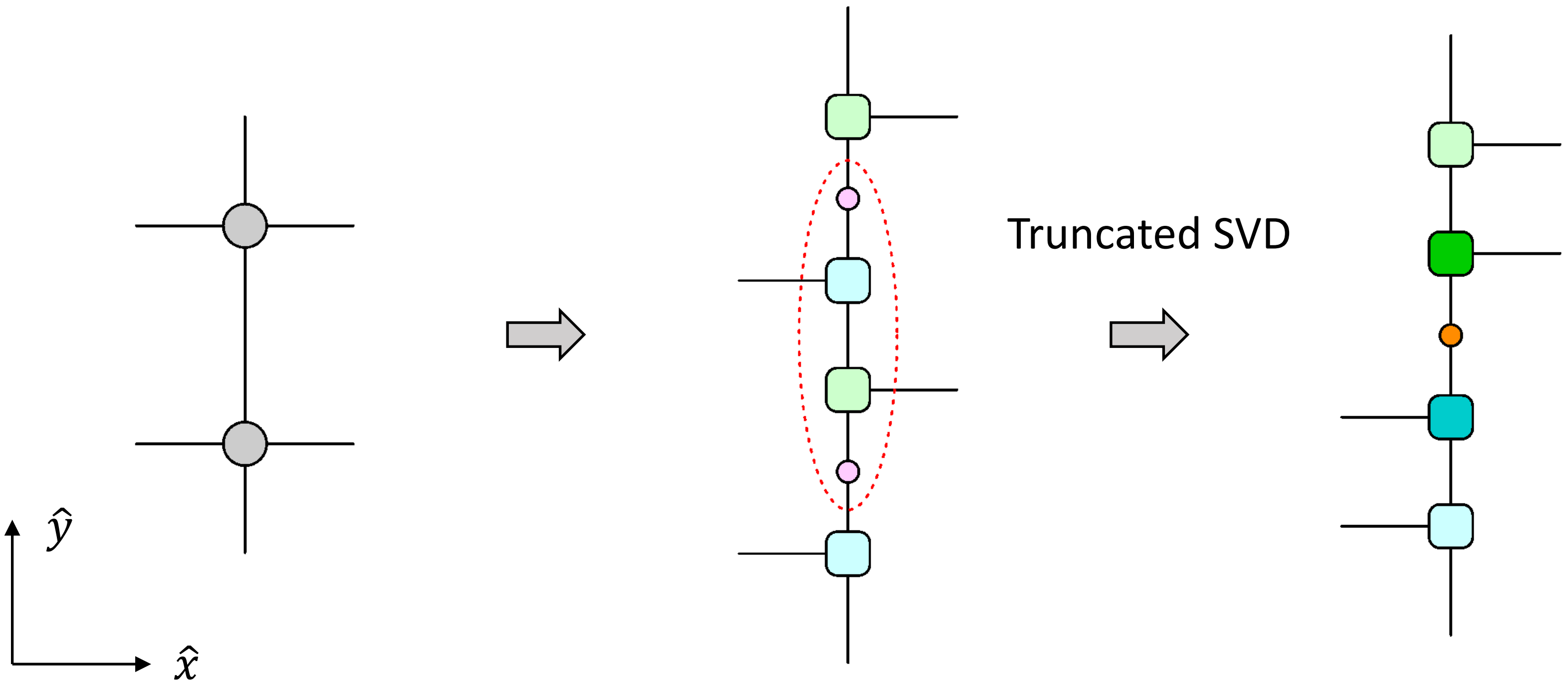
✔ **ATRG shows better performance than the HOTRG at the same execution time.**

Canonical form in ATRG 1/2

✔ Canonical form is an MPS constructed by tensors w/ orthogonality conditions.

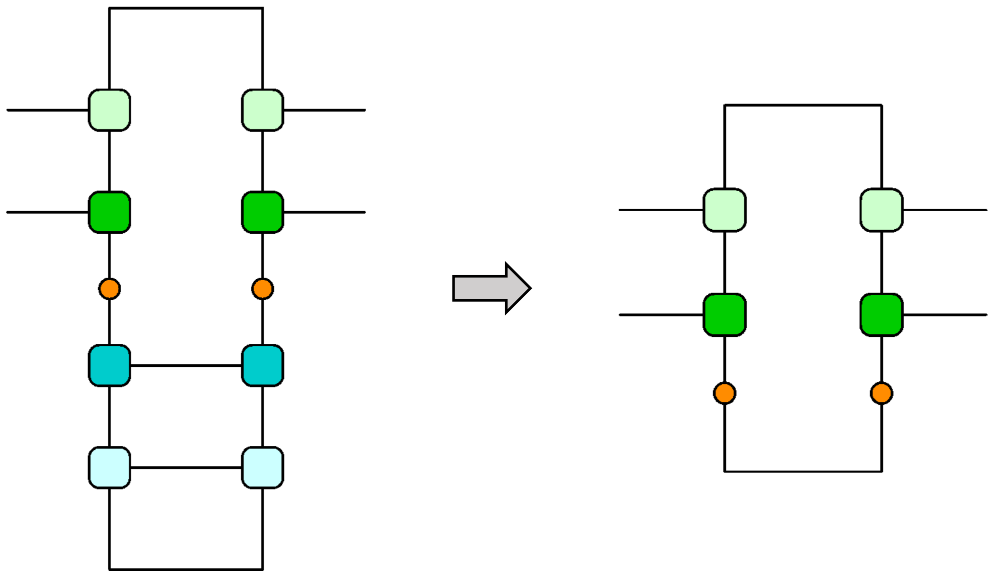
Schollwöck, Annals of Physics 326(2011)96-192
Cf. Talk by Pooja Siwach

✔ ATRG converts two adjacent tensors into a canonical form.



Canonical form in ATRG 2/2

✔ **Reduced density matrix (RDM)** is simplified by the canonical form.
EVD/SVD for RDM gives us projectors that accomplish spacetime coarse-graining.



✔ **Adjacent tensors are compressed as a canonical MPS before we carry out spacetime coarse-graining.**

Grassmann TRG approach

Gu-Verstraete-Wen, arXiv:1004.2563

✔ Any TRG algorithm can be applied for fermions.
Fermionic path integral can be expressed as a tensor network generated by **Grassmann tensors**.

$$\mathcal{T}_{\eta_1\eta_2\eta_3\cdots} = \sum_{i_1,i_2,i_3,\cdots} T^{i_1i_2i_3\cdots}\eta_1^{i_1}\eta_2^{i_2}\eta_3^{i_3}\cdots$$

Gu, PRB88(2013)115139
Shimizu-Kuramashi, PRD90(2014)014508
Takeda-Yoshimura, PTEP2015(2015)043B01
Meurice, PoS LATTICE2018(2018)231
Bao's thesis, PhD, Uwaterloo
SA-Kadoh, JHEP10(2021)188

✔ A clear correspondence btw tensors and Grassmann tensors.

	Tensor	Grassmann tensor
index	integer	Grassmann number
contraction	$\Sigma_i \cdots$	$\int\int \mathrm{d}\bar{\eta}\mathrm{d}\eta \mathrm{e}^{-\bar{\eta}\eta}\cdots$

$$\mathrm{e}^{A\bar{\psi}_n\psi_{n+\mu}} = \left(\int\int \mathrm{d}\bar{\eta}_n\mathrm{d}\eta_n\mathrm{e}^{-\bar{\eta}_n\eta_n}\right)\exp\left[-\sqrt{A}\bar{\boldsymbol{\psi}}_{\boldsymbol{n}}\eta_n + \sqrt{A}\bar{\eta}_n\boldsymbol{\psi}_{\boldsymbol{n+\mu}}\right]$$

✔ A sample code of a novel GTRG is available on GitHub.

SA, JHEP11(2022)030

https://github.com/akiyama-es/Grassmann-BTRG

Bond-weighting method for the Grassmann TRG

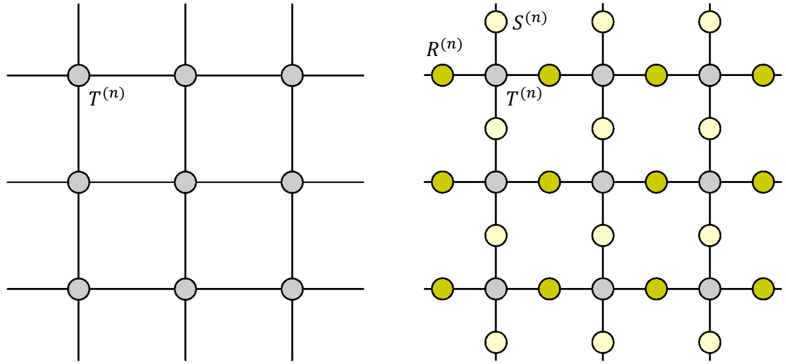
✔ Bond-weighting method is **a novel way to improve the accuracy of LN-TRG algorithms without increasing their computational costs.**

Adachi-Okubo-Todo, PRB105(2022)L060402

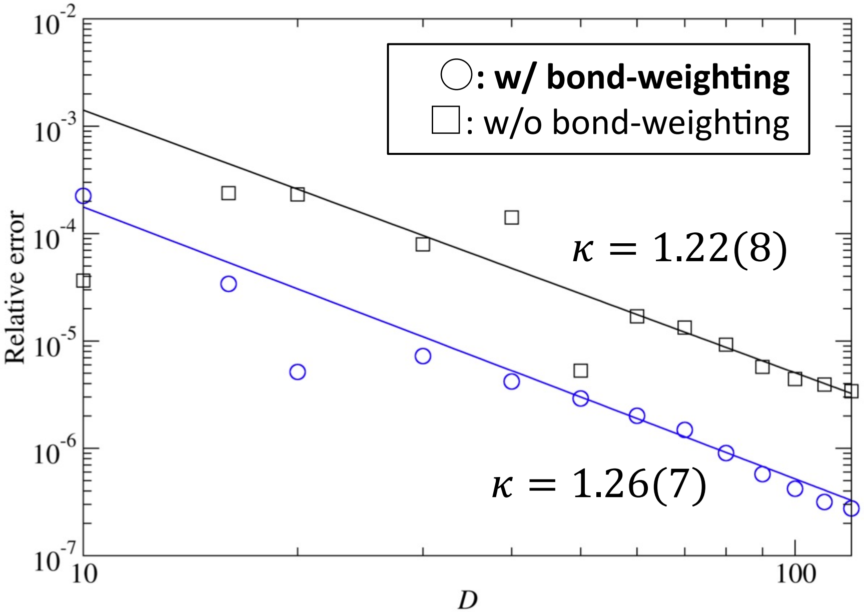
✔ Bond-weighting method works well also for lattice fermions. SA, JHEP11(2022)030

✔ **The sample code is available on GitHub.**

https://github.com/akiyama-es/Grassmann-BTRG



2d massless free Wilson fermion



$\delta f \sim D^{2\kappa}$ w/ $\kappa = 1.344$ when $c = 1$

Tagliacozzo+, PRB78(2008)024410
Pollmann+, PRL102(2009)255701

MPD for two- and three-flavor GNW model in 2D

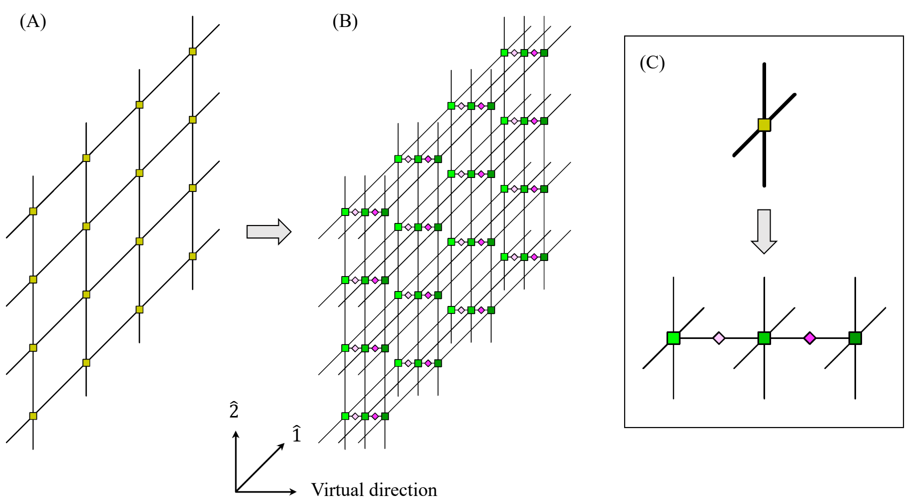
SA, arXiv:2304.01473

✔ **2D TN w/ $D = 4^{N_f}$ is equivalent to N_f-layer TN w/ $D = 4$.**

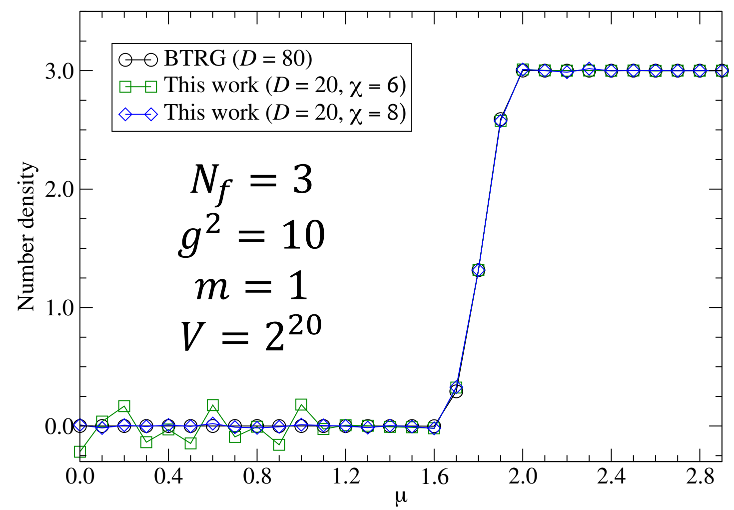
✔ Approximately contracting N_f-layer TN, one obtain the path integral.

✔ Is this kind of exact MPD available for 4D Wilson fermion w/ finite N_f?

MPD



Number density



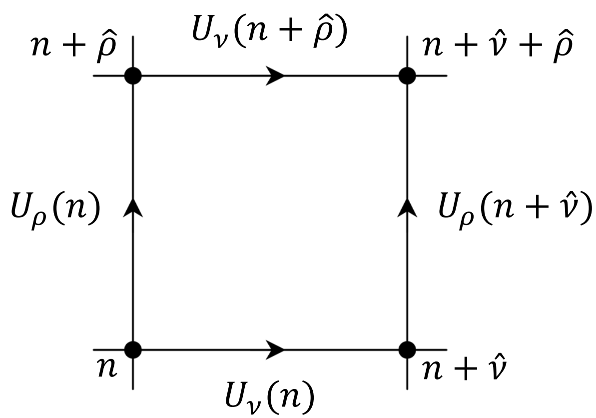
Chemical potential

TRG study of (3+1)D $\mathbb{Z}_2$ gauge-Higgs model on a lattice

$\mathbb{Z}_2$ gauge-Higgs model in the unitary gauge

✔ Action of the $(d+1)$-dimensional $\mathbb{Z}_2$ gauge-Higgs model

$$S = -\beta \sum_n \sum_{\nu>\rho} U_\nu(n) U_\rho(n+\hat{\nu}) U_\nu(n+\hat{\rho}) U_\rho(n)$$
$$-\eta \sum_n \sum_\nu \left[\mathrm{e}^{\mu\delta_{\nu,d+1}} \sigma(n) U_\nu(n) \sigma(n+\hat{\nu}) + \mathrm{e}^{-\mu\delta_{\nu,d+1}} \sigma(n) U_\nu(n-\hat{\nu}) \sigma(n-\hat{\nu}) \right]$$



$U_\nu(n) (\in \mathbb{Z}_2)$: link variable (gauge field)
$\sigma(n) (\in \mathbb{Z}_2)$: matter field

✔ **Choosing the unitary gauge, all the matter fields are eliminated**

$$\sigma(n) U_\nu(n) \sigma(n+\hat{\nu}) \mapsto U_\nu(n)$$

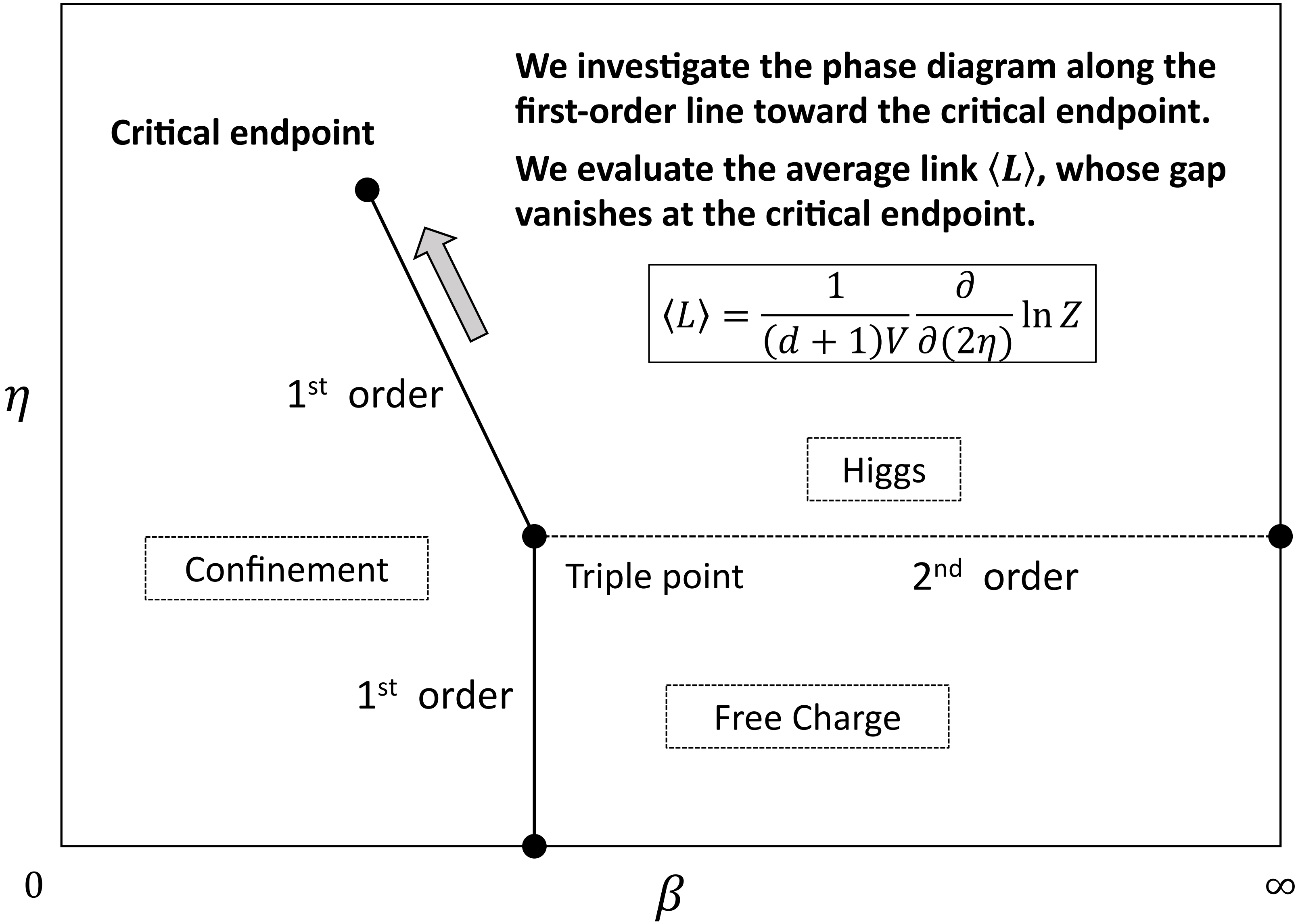
$$\boxed{S = -\beta \sum_n \sum_{\nu>\rho} U_\nu(n) U_\rho(n+\hat{\nu}) U_\nu(n+\hat{\rho}) U_\rho(n) - 2\eta \sum_n \sum_\nu \cosh\left(\mu\delta_{\nu,d+1}\right) U_\nu(n)}$$

Phase diagram of the (3+1)D model at $\mu = 0$

We investigate the phase diagram along the first-order line toward the critical endpoint.

We evaluate the average link $\langle L \rangle$, whose gap vanishes at the critical endpoint.

$$\langle L \rangle = \frac{1}{(d+1)V} \frac{\partial}{\partial(2\eta)} \ln Z$$



Motivation of studying $\mathbb{Z}_2$ gauge-Higgs model

✔ **The simplest lattice gauge theory coupling to a matter field**

A good target to see whether the TRG is efficient for the (3+1)D lattice gauge theory or not.

✔ **The model possesses the critical endpoint (CEP)**

QCD at finite temperature and density also has the CEP.
Can we use the TRG to specify the precise location of CEP?

✔ **We can consider the model at finite density**

We can investigate how the CEP moves by introducing the chemical potential.
Note that the model is free from the sign problem even at finite density.

Cf. TRG studies of gauge-Higgs models in 2D
Unmuth–Yockey+, PRD98(2018)094511
Bazavov+, PRD99(2019)114507
Butt+, PRD101(2020)094509

TN representation of LGT

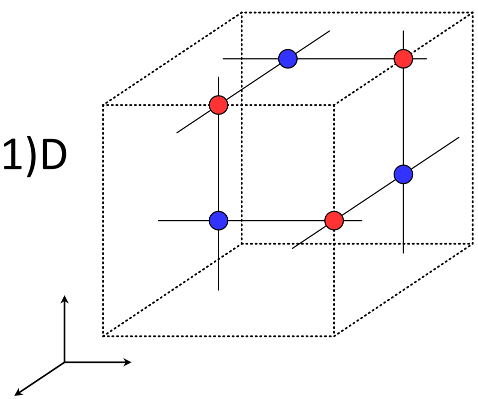
SA-Kuramashi, JHEP05(2022)102

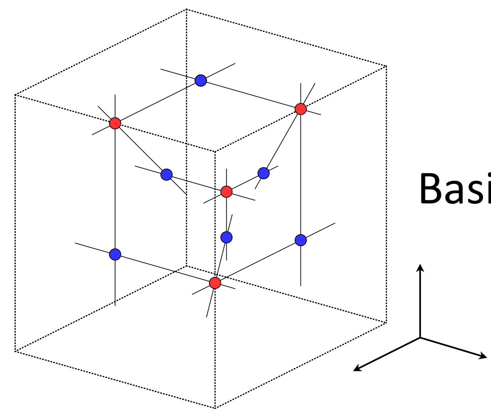
✔ We employ the HOSVD for the plaquette weight

$$\mathrm{e}^{\beta U_\nu(n) U_\rho(n+\hat{\nu}) U_\nu(n+\hat{\rho}) U_\rho(n)} = \sum_{a,b,c,d} V_{U_\nu(n)a} V_{U_\rho(n+\hat{\nu})b} V_{U_\nu(n+\hat{\rho})c} V_{U_\rho(n)d} B_{abcd}$$

✔ We follow the so-called asymmetric formulation, which allows us to have a uniform TN. Some contractions are necessary before the TRG is carried out.

Basic unit in (2+1)D





Basic unit in (3+1)D

Liu+, PRD88(2013)056005

Cf. Today's talk by Judah Unmuth-Yockey

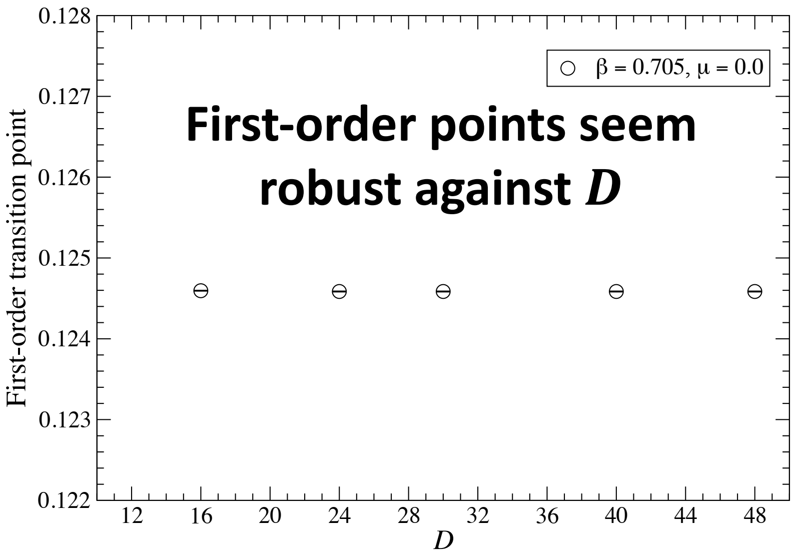
✔ The resulting TN is approximately contracted by the parallelized ATRG.

SA+, PoS(LATTICE2019)138

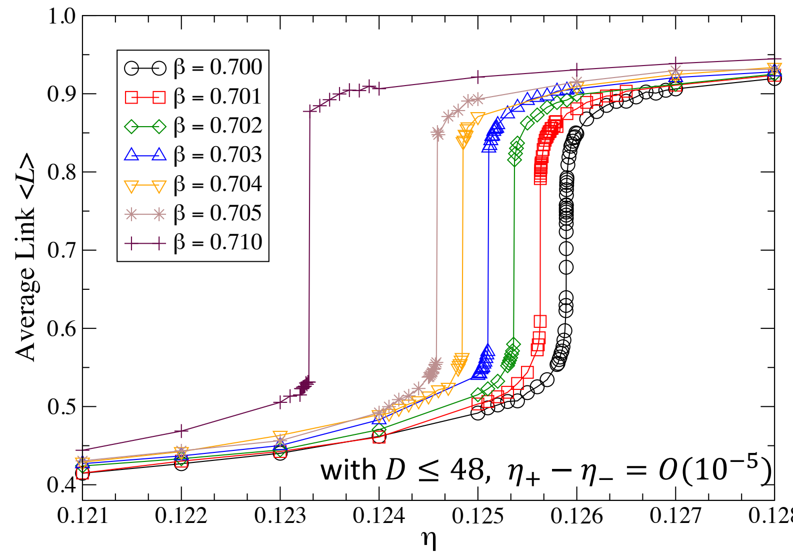
Study of the (2+1)D model at $\mu = 0$

SA-Kuramashi, JHEP05(2022)102

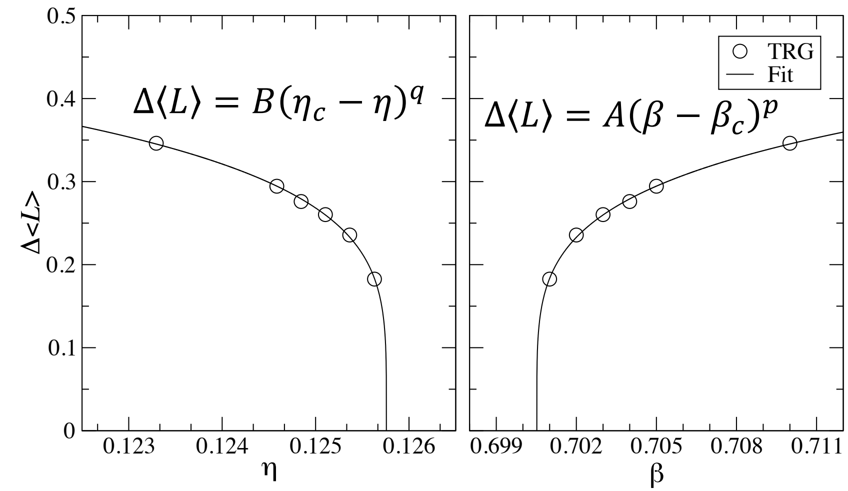
First-order points seem robust against *D*



Bond dimension

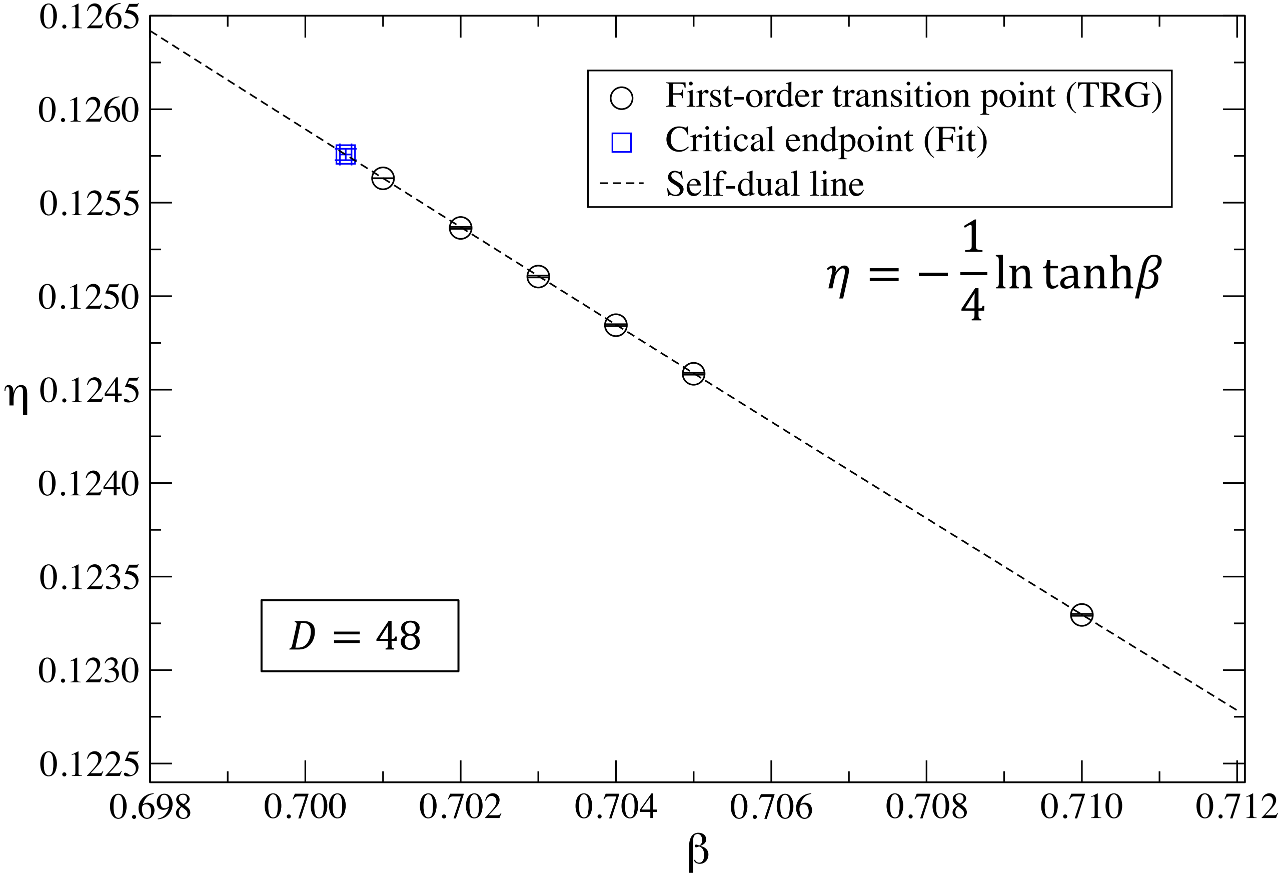


MC Somoza+, PRX11(2021)041008	$\beta_c \approx 0.701$
TRG this work	$\beta_c = 0.70051(7)$



Comparison with the self-dual line

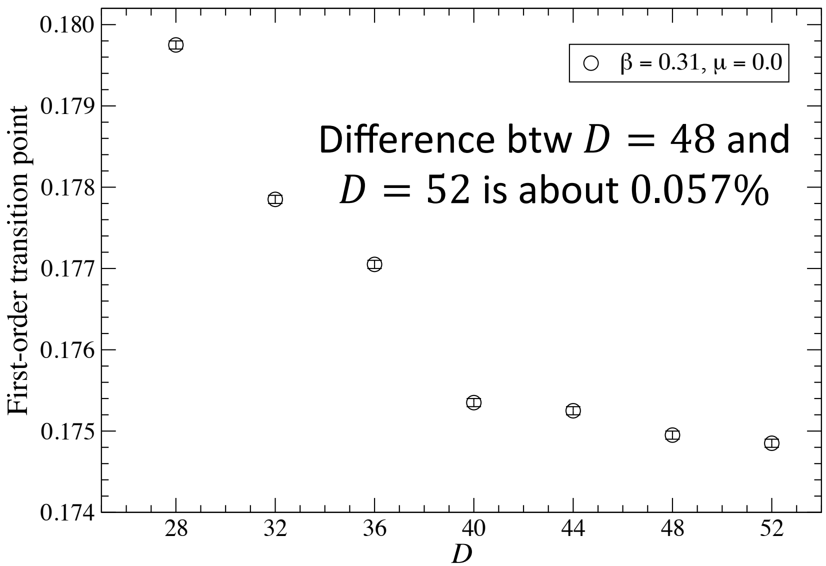
SA-Kuramashi, JHEP05(2022)102



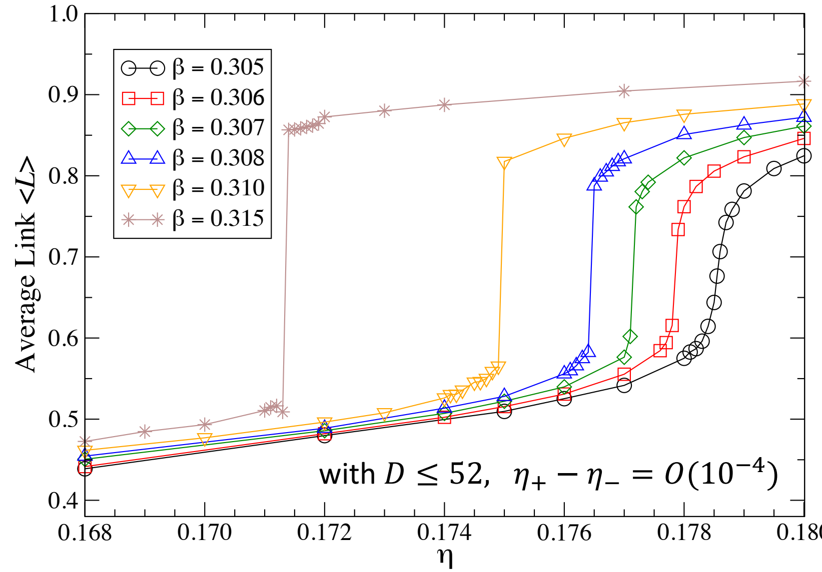
✔ **All transition points are well located on the self-dual line.**

(3+1)D model at vanishing density

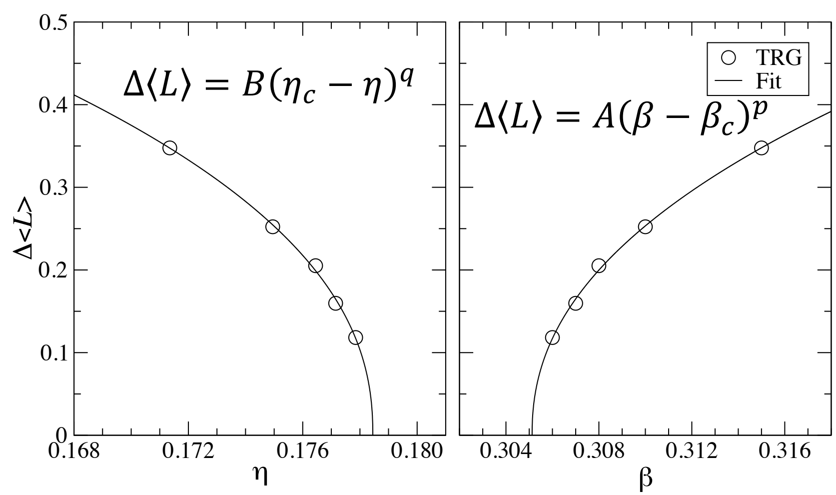
SA-Kuramashi, JHEP05(2022)102



Bond dimension

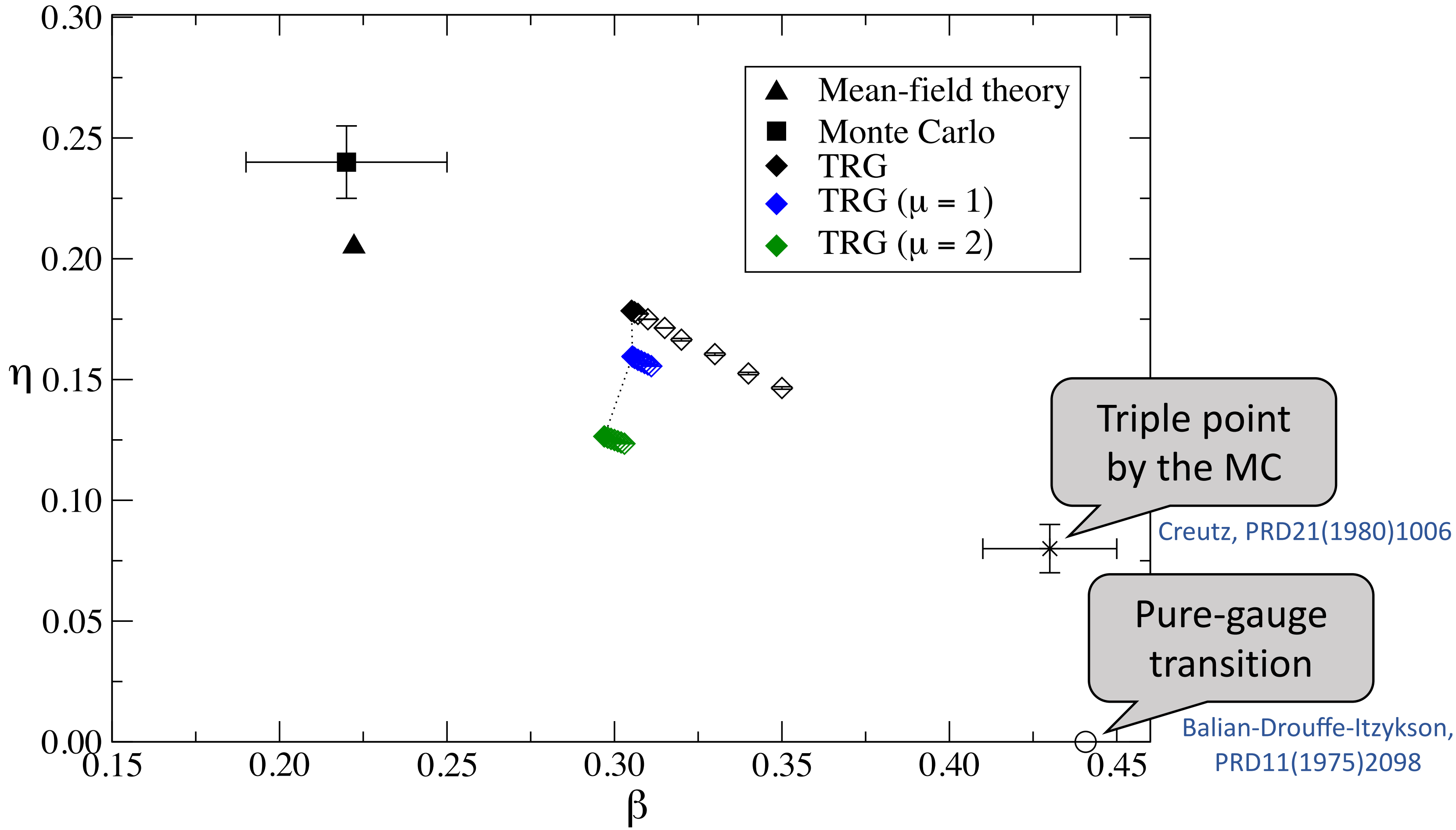


Mean-field Brezin-Drouffe, NPB200(1982)93	$(\beta_c, \eta_c) = (0.22, 0.205)$
MC on $V = 8^4$ Creutz, PRD21(1980)1006	$(\beta_c, \eta_c) = (0.22(3), 0.24(2))$
TRG w/ $D = 52$ this work	$(\beta_c, \eta_c) = (0.3051(2), 0.1784(2))$



Current status of the phase diagram near the CEP

SA-Kuramashi, JHEP05(2022)102



✔ It seems that TRG and MC share a similar first-order line at $\mu = 0$.

Summary

- ✔ TRG is a typical TN algorithm, which enables us to perform TN contraction approximately using the idea of RSRG.

- ✔ **TRG w/ parallel computation** has been a good way to investigate higher-dimensional QFT on a thermodynamic lattice.

- ✔ Although TRG is based on the Lagrangian formalism, several techniques are shared with TN methods based on the Hamiltonian approach.

- ✔ A sample code of BTRG for fermion is available on GitHub.

  https://github.com/akiyama-es/Grassmann-BTRG

  SA, JHEP11(2022)030

- ✔ The first application of TRG for (3+1)D LGT has been made.
  We have obtained the TRG estimate of **CEP in $\mathbb{Z}_2$ gauge-Higgs model at finite density**.

Future Perspective

✔ **A next interesting (challenging?) target can be the (3+1)D QED.**

- Variational approach based on the tree TN for the (3+1)D lattice QED ($L \leq 8$).

Magnifico+, Nature Commun. 12(2021)1

✔ **Is an exact MPD available for 4D interacting Wilson fermions?**

- Analytical considerations are necessary. Discussion welcome!

✔ **How can we approach $D \to \infty$?**

- This problem may be shared with PEPS or TTN.

Cf. Finite-entanglement scaling:
Tagliacozzo+, PRB78(2008)024410
Pollmann+, PRL102(2009)255701

✔ **Although TRG is based on Lagrangian formalism, some problems are shared with quantum computations based on Hamiltonian formalism.**

- How can we deal with higher-dimensional non-abelian gauge theories with TN?
  Cf. TRG approach for 3D pure SU(2) gauge theory Kuwahara-Tsuchiya, PTEP2022(2022)093B02
- TRG may give us insights from the viewpoint of classical computation (and vice versa).
- Which regimes seem to be difficult to study w/o quantum computation?