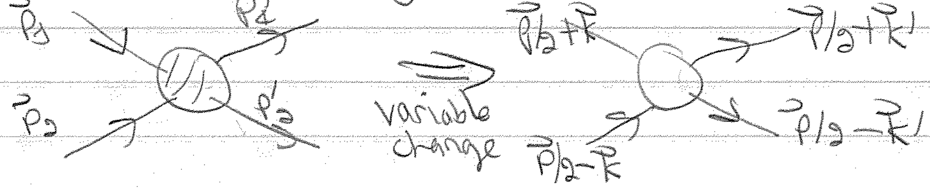
7/1/2013

m1b – Scattering Theory 1

- A more complete version of these notes is available on the TALENT website.

- Overview
  - Main source of info on NN force is NN scattering
  - You've seen at least the basics of scattering
    $\Rightarrow$ here: review and extend (also in exercises)

- Neglect V_{em} and n, p mass difference $\Rightarrow m \equiv \frac{1}{2}(m_n + m_p)$
  $\Rightarrow$ generic scattering of two-equal mass particles (nonrelativistic)
  - interacting with a short-ranged potential



$$\vec{r} = \vec{r}_1 - \vec{r}_2 \qquad \vec{R} = \tfrac{1}{2}(\vec{r}_1 + \vec{r}_2)$$

$$\vec{k} = \frac{\vec{p}_1 - \vec{p}_2}{2} \qquad \vec{P} = \vec{p}_1 + \vec{p}_2$$

} other conventions exist for relative and center-of-mass coordinates.

$$\hat{H} = \frac{\hat{p}_1^2}{2m} + \frac{\hat{p}_2^2}{2m} + \hat{V} \longrightarrow \hat{T}_{cm} + \hat{H}_{rel} = \underbrace{\frac{\hat{P}^2}{2M}}_{\text{com}} + \underbrace{\frac{\hat{k}^2}{2\mu} + \hat{V}}_{\text{"intrinsic" or "relative"}}$$

$\leftarrow$ key: independent of COM

$$\boxed{M = m_1 + m_2 = 2m} \; ; \; \boxed{\mu = \frac{m_1 m_2}{M} = \frac{m}{2}}$$

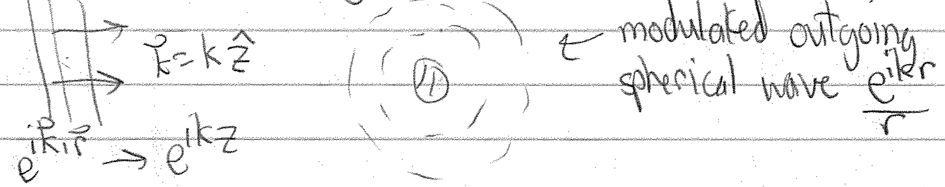
- So $|\Psi\rangle = |P\rangle|\Psi_{rel}\rangle$ $\leftarrow$ all the physics! $\quad$ Ignore $|P\rangle \Rightarrow$ or in com frame
  - $|P\rangle$: plane wave eigenstate of $\hat{P}$
  - $\leftarrow$ drop the rel

"on-shell" means $E_2 = \frac{p_2^2}{2m}$, $E_1 = \frac{p_1^2}{2m}$, etc. $\Rightarrow E_k = \frac{k^2}{2\mu} = \frac{k'^2}{2\mu}$ $\quad(\hbar = 1)$ ✗✗

- Elastic scattering $E_{in} = E_{out}$. Effective one-body problem.

7/1/2013

- Show F. Nunes scattering cartoon



$$\psi_E^{(+)}(\vec{r}) \xrightarrow[r\to\infty]{} \frac{1}{(2\pi)^{3/2}}\Big[\underset{\substack{\uparrow\\ \text{incoming}}}{e^{i\vec{k}\cdot\vec{r}}} + \underset{\substack{\nwarrow\\ \text{scattered}}}{f(k,\theta,\phi)\frac{e^{ikr}}{r}}\Big]$$

$$\frac{d\sigma}{d\Omega}(k,\theta,\phi) = \frac{\#\text{ scattering into } d\Omega \text{ per time}}{\#\text{ incident per area per time}} = \frac{k/\mu \cdot |f|^2/r^2 \cdot r^2}{k/\mu} \Bigg\}\text{ from probability current}$$

physics! $\Rightarrow \frac{d\sigma}{d\Omega} = |f(k,\theta,\varphi)|^2 \rightarrow |f(k,\theta)|^2$ (no φ dependence here from spin polarization)

Expand: $\psi(r,\theta) = \sum_{\ell=0}^{\infty} c_\ell \frac{u_\ell(r)}{r} P_\ell(\cos\theta)$

$$\Rightarrow -\frac{1}{2\mu}\frac{d^2u_\ell}{dr^2} + V(r)u_\ell + \frac{\ell(\ell+1)}{2\mu r^2}u_\ell = \frac{k^2}{2\mu}u_\ell \Rightarrow \frac{d^2u_\ell}{dr^2} - \left(\frac{\ell(\ell+1)}{r^2} + V(r) - k^2\right)u_\ell(r) = 0$$

solve to find scattering

Pick out θ dependence of f: $f(k,\theta) = \sum_{\ell=0}^{\infty}(2\ell+1)f_\ell(k)P_\ell(\cos\theta)$ [defines $f_\ell(k)$]
(central V here)

$$\text{incoming } e^{i\vec{k}\cdot\vec{r}} = e^{ikr\cos\theta} = \sum_{\ell=0}^{\infty}(2\ell+1)i^\ell j_\ell(kr)P_\ell(\cos\theta) \xrightarrow[r\to\infty]{} \sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\theta) \times \frac{\underset{\text{incoming spherical}}{(-1)^{\ell+1}e^{-ikr}} + \overset{\substack{\text{outgoing spherical}\\ \downarrow}}{e^{ikr}}}{2ikr}$$

outgoing scattering $f_\ell(k)\frac{e^{ikr}}{r}\cdot\frac{2ik}{2ik}$

$$\boxed{\psi_E^{(+)}(\vec{r}) \xrightarrow[r\to\infty]{} \frac{1}{(2\pi)^{3/2}}\sum_\ell (2\ell+1)P_\ell(\cos\theta)\frac{(-1)^{\ell+1}e^{-ikr} + S_\ell(k)e^{ikr}}{2ikr}}$$

$$\Rightarrow \underbrace{[1 + 2ikf_\ell(k)]}_{S_\ell(k)}\frac{e^{ikr}}{2ikr}$$

What is physical interpretation of the "1"?

partial wave S-matrix (warning: different normalizations)

Probability conservation: $|S_\ell(k)|^2 = 1 \Rightarrow S_\ell(k) = e^{2i\delta_\ell(k)} = \frac{e^{i\delta_\ell(k)}}{e^{-i\delta_\ell(k)}}$ (not a joke!)
(elastic) pure phase

- defines phase shift up to multiple of π

M1b-3

7/1/2013

Derive in exercises $f_\ell(k) = \frac{S_\ell(k)-1}{2ik} = \frac{e^{i\delta_\ell}\sin\delta_\ell}{k} = \frac{1}{k\cot\delta_\ell - ik}$

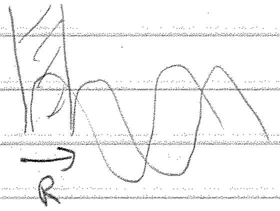
Units? $\hbar=1$, $\frac{1}{k}$ is length $\Rightarrow d\sigma \propto |f_\ell^2| \sim [L]^2$ ✓ we'll see again!

Combined! $\psi_E^{(+)}(\vec{r}) \xrightarrow[r\to\infty]{} \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(\cos\theta)\, i^\ell e^{i\delta_\ell} \frac{\sin(kr-\ell\frac{\pi}{2}+\delta_\ell)}{kr}$

and $u_\ell \propto \sin(kr - \ell\frac{\pi}{2} + \delta_\ell(k))$

2 min.: If I'm being careful, is the phase shift a function of energy or momentum? [on-shell $\Rightarrow$ either!]

- Ambiguity $\delta_\ell \to \delta_\ell + \pi$ or $\delta_\ell + 2\pi$ or ... $\Rightarrow$ physics is unchanged.
  - Levinson's theorem $\delta_\ell(k=0) = (\#\text{ bound states}) * \pi$ if $\delta_\ell(k)$ is continuous and $\delta_\ell(k\to\infty) = 0$ } explore numerically in exercise.
- Show pictures of phase shifts: repulsive pushed out, attractive pulled in



← hard sphere $u_0 \propto \sin(kr-kR) \Rightarrow \boxed{\delta_0(k) = -kR}$
S-wave

Think about numerical solution.

$u_0(r) \xrightarrow[r\to\infty]{} \sin(kr+\delta_0(r)) = \cos\delta_0 \sin kr + \sin\delta_0 \cos kr$

$\left[u_\ell(r) \to \cos\delta_\ell\, \hat{j}_\ell(kr) - \sin\delta_\ell\, \hat{n}_\ell(kr), \; \hat{j}_\ell(z) \equiv \frac{j_\ell(z)}{z}, \; \hat{n}_\ell(z) \equiv \frac{n_\ell(z)}{z}\right]$

integrate
S-eqn to
large enough
r (how large?)
for given k

Use $\frac{u_0(r_1)}{u_0(r_2)} \Rightarrow$ solve for $\tan\delta_\ell(k)$

or $\frac{u_0'(r_1)}{u_0(r_1)}$ (easy for square well, take $r_1 = R$]

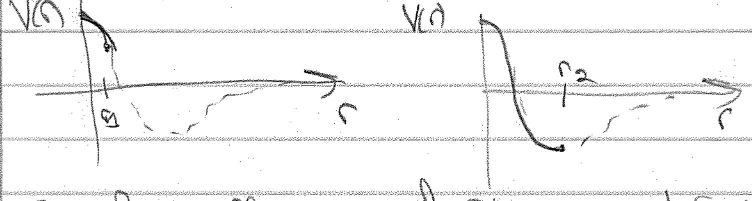
M1b-4

7/1/2013

Alternative: Variable Phase Approach (VPA)

given k: · imagine integrating out from the origin: pulled in or out according to V at that point ⇒ accumulate phase shift

· define $\delta(k,r)$ as phase shift at momentum k when potential cut at r



Class: $\delta(k,0) \overset{?}{=} 0$
$\delta(k,\infty) \overset{?}{=} \delta(k)$ (!)

Satisfies diff. eq: $\frac{d}{dr}\delta(k,r) = -\frac{1}{k}[2\mu V(r)] \sin^2[kr + \delta(k,r)]$

- nonlinear 1st order
- show Mathematica snippet ⇒ easy code! Play with notebook.
- $\sin^2[\,] \geq 0$ always ⇒ what does this say about phase when a potential is attractive or repulsive?
- derivation in notes [fill in details and generalize!] ⇒ Exercises

Non-uniqueness

- inverse scattering idea ⇒ given $\delta_l(k)$ for all k, find V(r) (or given $\delta_l(k)$ for all l at some k).
  - works if central and no bound states or bound state info given
- but unitary transformation → infinite "phase equivalent" potentials ⇒ same physics. Usually non-local!
- so idea that there is one true potential is misguided.

· Unitary transformations

- $|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle$ time evolution (also non-unitary $t \to -i\tau$)
- $U = e^{-i\vec{\alpha}\cdot\vec{G}}$ symmetry transformations
- unitary transformations of Hamiltonians (e.g. by RG methods)

7/1/2013

MLb-5

$$U^\dagger U = UU^\dagger = I \Rightarrow E_n = \langle \psi_n \underset{U^\dagger U}{|} \overset{=\hat H_0+\hat V}{\hat H} \underset{U^\dagger U}{|} \psi_n\rangle = \langle \tilde\psi_n | \underbrace{U\hat H U^\dagger}_{\tilde{\hat H}} | \underset{\nwarrow U|\psi_n\rangle}{\tilde\psi_n} \rangle$$

If U is short-ranged, then H and $\tilde H$ produce same phase shifts, energies

How do you transform $\hat O$? To preserve matrix elements $\hat O \to \tilde{\hat O} = U\hat O U^\dagger$.

What quantities are changed? [Exercise question]

- Local and non-local potentials

$1 = \int d^3r\, |\vec r\rangle\langle\vec r|$

and $\langle \vec r | \vec r\,'\rangle = \delta^3(\vec r - \vec r\,')$

  - Consider $\hat H = \frac{\hat k^2}{2\mu} + \hat V$ and $\langle\psi|\hat H|\psi\rangle$

Coordinate space: $\langle\psi|H|\psi\rangle = \int d^3r'\, d^3r\, \langle\psi|\vec r\,'\rangle\langle\vec r\,'|H|\vec r\rangle\langle\vec r|\psi\rangle$

See 1-D version on slides

$$\langle\vec r\,'|\frac{\hat k^2}{2\mu}|\vec r\rangle = \delta^3(\vec r - \vec r\,')\frac{-\hbar^2\nabla^2}{2\mu} \qquad \langle\vec r\,'|\hat V|\vec r\rangle = \begin{cases} V(\vec r)\,\delta^3(\vec r\,' - \vec r) & \text{if local} \\ V(\vec r\,', \vec r) & \text{otherwise} \end{cases}$$

not diagonal $\rightarrow$

$$\text{S-eqn. } \underline{-\frac{\hbar^2}{2\mu}\nabla^2\psi(\vec r) + V(\vec r)\psi(\vec r) = E\psi(\vec r)} \Rightarrow -\frac{\hbar^2}{2\mu}\nabla^2\psi(\vec r) + \int d^3r'\, V(\vec r, \vec r\,')\psi(\vec r\,') = E\psi(\vec r)$$

Momentum space: $\langle\psi|H|\psi\rangle = \int d^3k'\, d^3k\, \langle\psi|\vec k'\rangle\langle\vec k'|H|\vec k\rangle\langle\vec k|\psi\rangle$ $\leftarrow$ $1 = \int d^3k\, |\vec k\rangle\langle\vec k|$, $\langle\vec k|\vec k'\rangle = \delta^3(\vec k - \vec k')$

$$\Rightarrow \langle\vec k'|\frac{\hat k^2}{2\mu}|\vec k\rangle = \delta^3(\vec k - \vec k')\frac{\hbar^2 k^2}{2\mu} \qquad \langle\vec k'|\hat V|\vec k\rangle = \begin{cases} V(\vec k' - \vec k) & \text{if local} \\ V(\vec k', \vec k) & \text{otherwise} \end{cases}$$

$$\text{Yukawa} \Rightarrow \frac{e^{-m|\vec r|}}{4\pi|\vec r|} \xleftrightarrow[FT]{} \frac{1}{(\vec k - \vec k')^2 + m^2}$$

$\nwarrow$ momentum transfer (<u>not</u> relative momentum)

7/1/2013

Now partial wave expansion: (Taylor conventions)

$$\langle \vec{k}'|V|\vec{k}\rangle = \frac{2}{\pi}\sum_{l,m} V_l(k',k)\, Y^*_{lm}(\Omega_{k'})\, Y_{lm}(\Omega_k)$$

assuming central potential $\langle k' l' m'|V|k\, l\, m\rangle = \delta_{ll'}\delta_{mm'} V_l(k',k)$ ← hard to tell if local or nonlocal!

- tomorrow – mix different l's with <u>tensor</u>.

S-eqn ⇒ Lippmann-Schwinger equation for <u>T-matrix</u>

$$T^{(+)}(\vec{k}',\vec{k};E) = V(\vec{k}',\vec{k}) + \int d^3q\, \frac{V(\vec{k}',\vec{q})\, T(\vec{q},\vec{k};E)}{E - \frac{q^2}{m} + i\varepsilon}$$ (derive in exercises)

- Expand $\langle \vec{k}'|T^{(+)}(E)|\vec{k}\rangle$ like $\langle \vec{k}'|V|\vec{k}\rangle$

derive in exercises ⇒ $$T_l(k',k;E) = V_l(k',k) + \frac{2}{\pi}\int_0^\infty dq\, q^2\, \frac{V_l(k',q)\, T_l(q,k;E)}{E - E_q + i\varepsilon} \qquad \boxed{E_q = \frac{q^2}{m}}$$

- <u>any</u> $k', k; E$ works here
- but only <u>on-shell</u> related to scattering amplitude $f_l(k)$!

$$T_l(k,k;E=E_k) = -\frac{2\pi}{\mu} f_l(k)$$

- but if we put $k'=k$, $E=E_k$ on left, still need $T_l(q,k;E_k)$ for <u>all</u> $q \neq k$ on right ⇒ half-on shell.

- Operator form: $\hat{T}(z) = \hat{V} + \hat{V}\frac{1}{z-\hat{H}_0}\hat{T}(z)$

$$= \hat{V} + \hat{V}\frac{1}{z-\hat{H}_0}\hat{V} + \hat{V}\frac{1}{z-\hat{H}_0}\hat{V}\frac{1}{z-\hat{H}}\hat{V} + \ldots$$ ↙ Born series

- Take $\langle \vec{k}'|\ \ |\vec{k}\rangle$ matrix element and insert $1 = \int d^3q\, |\vec{q}\rangle\langle\vec{q}|$ to recover full LS equation or $1 = \frac{2}{\pi}\int q^2 dq\, |q\rangle\langle q|$ to get partial wave

- In exercises: <u>numerical</u> evaluation as matrix equation.

M1b-7

7/1/2013

Effective range expansion: 1st pass (we'll see it again!)

Schwinger: $k^{2l+1} \cot \delta_l(k)$ can be expanded in Taylor series in k^2

$\Rightarrow$ effective range expansion or ERE.

- The coefficients have names:

$$l=0 \quad k \cot \delta_0(k) = -\frac{1}{a_0} + \frac{1}{2} r_0 k^2 - P r_0^3 k^4 + \ldots$$

"shape parameter"

or a_s or a "scattering length"

or r_s or r_e "effective range"

- Try it for hard-sphere scattering (radius R) $\Rightarrow \delta_0(k) = -kR$

$$k \cot(-kR) \overset{\text{Taylor expansion}}{=} -\frac{1}{R} + \frac{1}{3} R k^2 + \ldots \Rightarrow a_0 = R, \; r_0 = \frac{2R}{3}$$

note sign

- More general:
  - $r_0 \sim R$, "range" of potential (for Yukawa?)
  - a_0 can be anything
    - if $a_0 \sim R$ then "natural"
    - if $|a_0| \gg R$ (unnatural), then interesting (e.g. neutrons, cold atoms)

- Associate sign and size of a_0 with behavior of scattering wave function as energy (or k) $\to 0$

$$\frac{\sin(kr + \delta_0(k))}{k} \xrightarrow[k \to 0]{} r - a_0 \quad \text{[show this)}$$

- See pictures: a_0 ranges from $-\infty$ to $+\infty$. Large near bound state at zero energy (or just miss)

- low-energy $l=0$, $f_0(k) = -\frac{1}{k \cot \delta_0 - ik} \doteq -\frac{1}{-1/a_0 - ik} \Rightarrow \sigma(k) = \frac{4\pi}{1/a_0^2 + k^2}$

natural: $\frac{d\sigma}{d\Omega} = a_0^2 \Rightarrow \sigma = 4\pi a_0^2$ $\quad |ka_0| \ll 1$

unnatural: $\frac{d\sigma}{d\Omega} \to \frac{1}{k^2} \Rightarrow \sigma = \frac{4\pi}{k^2}$ $\quad |ka_0| \gg 1$ $\quad$ "unitary limit"