Hadron Structure

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Lecture 1 - Recap

- Elastic scattering
  - Form factors
    - Surprises in their $Q^2$ dependence
    - Density distributions in a hadron
- Lattice techniques
  - Three-point functions via sequential source method
  - Extraction of matrix elements
Lecture 2 - all about Form Factors

- Extracting matrix elements from Lattice three-point functions
- Extracting form factors from matrix elements
- Lattice nucleon form factors
  - Compare with experiment
  - Investigation of systematic errors
  - Flavour dependence
- Lattice pion form factor
  - Twisted boundary conditions
- Other hadron form factors
Extracting Matrix Elements

• Recall hadronic form of the nucleon 3pt function

\[ G_\Gamma(t, \tau, \vec{p}, \vec{p}', O) = \sum_{s, s'} e^{-E_{p'}(t-\tau)} e^{-E_{p}\tau} \Gamma_{\beta\alpha} \langle \Omega \mid \chi_{\alpha}(0) \mid N(p', s') \rangle \langle N(p', s') \mid O(\vec{q}) \mid N(p, s) \rangle \langle Np, s \mid \bar{\chi}_{\beta}(0) \mid \Omega \rangle \]

• Need to remove time dependence and wave function amplitudes

\[ \text{Form a ratios with the nucleon 2pt function} \]

\[ G_2(t, \vec{p}) = \sum_s e^{-E_p t} \Gamma_{\beta\alpha} \langle \Omega \mid \chi_{\alpha} \mid N(p, s) \rangle \langle N(p, s) \mid \bar{\chi}_{\beta} \mid \Omega \rangle \]

• E.g.

\[ R(t, \tau; \vec{p}', \vec{p}; O) = \frac{G_\Gamma(t, \tau; \vec{p}', \vec{p}, O)}{G_2(t, \vec{p}')} \left[ \frac{G_2(\tau, \vec{p}')G_2(t, \vec{p}')G_2(t - \tau, \vec{p})}{G_2(\tau, \vec{p})G_2(t, \vec{p})G_2(t - \tau, \vec{p}')} \right]^{\frac{1}{2}} \]
Using the relation for spinors
\[ \bar{u}(\vec{p}, \sigma') \Gamma u(\vec{p}, \sigma) = \text{Tr} \Gamma (E \gamma_4 - i\vec{p} \cdot \vec{\gamma} + m) \frac{1}{2} \left( 1 - \gamma_5 \gamma_4 \frac{\vec{p} \cdot \vec{s}}{EM} + i\gamma_5 \frac{\vec{\gamma} \cdot \vec{s}}{m} \right) \delta_{\sigma \sigma'} \]

We can write the two point function as
\[ G_2(t, \vec{p}) = \sum_s \frac{\sqrt{Z_{\text{snk}}(\vec{p})} \sqrt{Z_{\text{src}}(\vec{p})}}{2E_{\vec{p}}} \text{Tr} \bar{u}(\vec{p}, s) \Gamma u(\vec{p}, s) [e^{-E_p t} + e^{-E_p' (T-t)}] + \nu\text{-spinor terms with opposite parity} \]

Use \( \Gamma_4 = \frac{1}{2} (1 + \gamma_4) \) to maximise overlap with positive parity forward propagating state
\[ G_2(t, \vec{p}) = \sqrt{Z_{\text{snk}}(\vec{p}) Z_{\text{src}}(\vec{p})} \left[ \left( \frac{E_p + m}{E_{\vec{p}}} \right) e^{-E_p t} + \left( \frac{E_{p'} + m'}{E_{\vec{p}'}^2} \right) e^{-E_{p'} (T-t)} \right] \]
Extracting Matrix Elements

• Similarly for the three-point function, if we express the nucleon matrix element under study as

\[ \langle N(p', s') | \mathcal{O}(\vec{q}) | N(p, s) \rangle = \bar{u}(p', s') \mathcal{J} u(p, s) \]

• E.g., for the EM current \( \mathcal{O} = J^\mu \)

\[ \mathcal{J} = \gamma^\mu F_1(Q^2) + i\sigma^{\mu\nu} \frac{q_\nu}{2M} F_2(Q^2) \]

• Then we have

\[ G_3(t, \tau; \vec{p}', \vec{p}; \Gamma, \mathcal{O}) = \sqrt{Z_{\text{snk}}(\vec{p}') Z_{\text{src}}(\vec{p})} F(\Gamma, \mathcal{J}) e^{-E_{\vec{p}'}(t-\tau)} e^{-E_{\vec{p}}\tau} \]

• where

\[ F(\Gamma, \mathcal{J}) = \frac{1}{4} \text{Tr} \left\{ \Gamma \left( \gamma_4 - i \frac{\vec{p}' \cdot \vec{\gamma}}{E_{\vec{p}'}} + \frac{m}{E_{\vec{p}'}} \right) \mathcal{J} \left( \gamma_4 - i \frac{\vec{p} \cdot \vec{\gamma}}{E_{\vec{p}}} + \frac{m}{E_{\vec{p}}} \right) \right\} \]
Example

- If we consider the particular case
  \[ \Gamma = \Gamma_{\text{unpol}} \equiv \frac{1}{2}(1 + \gamma_4), \quad \mathcal{O} = J^\mu, \quad \vec{p}' = \vec{p} \Rightarrow q = 0 \]

- then the contribution from \( F_2 \) to the matrix element drops out (proportional to \( q \))
  \[ \langle N(p', s')|J^\mu(0)|N(p, s)\rangle = \bar{u}(p', s')\gamma^\mu u(p, s)F_1(Q^2 = 0) + \bar{u}(p', s')i\frac{\sigma^{\mu\nu}q_\nu}{2M}u(p, s)F_2(Q^2 = 0) \]

- Euclideanisation
  \[ \gamma^M_0 = \gamma^E_4, \quad \gamma^M_i = -i\gamma^E_i \]
  \[ p^E_4 = i\rho^M_0 \equiv iE(\vec{p}), \quad p^E_i = -p^M_i \]

  \[ \langle N(p', s')|\bar{q}\gamma_\mu q|N(p, s)\rangle = \bar{u}(p', s')\gamma_\mu u(p, s)F_1(Q^2 = 0) + \bar{u}(p', s')i\frac{\sigma_\mu\nu q_\nu}{2M}u(p, s)F_2(Q^2 = 0) \]

Using the local vector current \( J^\mu = \bar{q}\gamma^\mu q \)
Example

Then the three-point function is now

\[ G_3(t, \tau; \vec{p}', \vec{p}; \Gamma, \mathcal{O}) = \sqrt{Z^{\text{nk}}(\vec{p}') Z^{\text{src}}(\vec{p})} F(\Gamma, \mathcal{J}) e^{-E_{\vec{p}'}(t-\tau)} e^{-E_{\vec{p}}\tau} \]

• with

\[ \mathcal{J} = \gamma^\mu F_1(Q^2) \]

• and

\[ F(\Gamma_{\text{unpol}}, \gamma_4) = \frac{1}{2E_{\vec{p}}E_{\vec{p}'}} [(E_{\vec{p}} + m)(E_{\vec{p}'} + m) + \vec{p}' \cdot \vec{p}] \]

\[ = 2 \]

\[ F(\Gamma_{\text{unpol}}, \gamma_i) = \frac{-i}{2E_{\vec{p}}E_{\vec{p}'}} [(E_{\vec{p}} + m)\vec{p}' + (E_{\vec{p}'} + m)\vec{p}] \]

\[ = 0 \]
Example

- So our ratio determines

\[
R(t, \tau; \vec{p}', \vec{p}; O) = \left( \frac{G_\Gamma(t, \tau; \vec{p}', \vec{p}, O)}{G_2(t, \vec{p}')} \right) \left[ \frac{G_2(\tau, \vec{p}')G_2(t, \vec{p}')G_2(t - \tau, \vec{p})}{G_2(\tau, \vec{p})G_2(t, \vec{p})G_2(t - \tau, \vec{p}')} \right]^{\frac{1}{2}} \\
= \sqrt{\frac{E_\vec{p}'E_{\vec{p}}}{(E_\vec{p} + m)(E_{\vec{p}} + m)}} F(\Gamma, J_0(q \bar{q})) \quad 0 \ll \tau \ll t \ll \frac{1}{2} T \\
= F_1(q^2 = 0) \quad \Gamma_{\text{unpol}} = \frac{1}{2}(1 + \gamma_4), \quad O = V_4 \equiv \gamma_4, \quad \vec{p}' = \vec{p} = 0
\]

\[
R(t = 16, \tau; \vec{p}, \vec{p}'; V_4)
\]
Other Useful Combinations

\[ R(t, \tau; \vec{0}, \vec{p}; V_4, \Gamma_4) = F_1(q^2) - \frac{E\vec{p} - M}{2M} F_2(q^2) = G_E(q^2) \]

\[ R(t, \tau; \vec{0}, \vec{p}; V_i, \Gamma_4) = -i \frac{q_i}{E + M} G_E(q^2) \]

\[ R(t, \tau; \vec{0}, \vec{p}; V_i, \Gamma_j) = -i \epsilon_{ijk} \frac{q_k}{E + M} G_M(q^2) \]

\[ \Gamma_4 \equiv \Gamma_{\text{unpol}} \]

\[ \Gamma_j = \frac{1}{2} (1 + \gamma_4) i \gamma_5 \gamma_j \]

• Certain combinations of parameters and kinematics give access to the form factors

• It is possible to have several choices giving access to the form factors at a fixed \( Q^2 \)

\[ \text{Overdetermined set of simultaneous equations that can be solved for } F_1, F_2 \text{ or } G_E, G_M \]
Typical Examples

More detailed look at lattice results to follow

\[ F_{1}^{u-d}(Q^2) \]

\[ F_{2}^{u-d}(Q^2) \]

QCDSF:
1106.3580
hep-lat/0608021
Some Recent Works

Nucleon

• Review: Ph. Hägler, 0912.5483
• QCDSF: 1106.3580
• ETMC: 1102.2208

Pion

• Mainz: 1109.0196
• PACS-CS: 1102.3652
• JLQCD/TWQCD: 0905.2465

[Not an exhaustive list]

• LHPC: 1001.3620
• RBC/UKQCD: 0904.2039
• CSSM: hep-lat/0604022
• ETMC: 0812.4042
• RBC/UKQCD: 0804.3971
• QCDSF: hep-lat/0608021
Electromagnetic Form Factors
Electromagnetic Form Factors

- Recall

\[ \frac{d\sigma}{d\Omega} = \left( \frac{d\sigma}{d\Omega} \right)_{\text{point}} |F(q^2)|^2 \]
If a nucleon was a point-like object with no internal structure, a probe would simply measure its e.g. charge for all $q^2$
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Electromagnetic Form Factors

- Recall

\[
\frac{d\sigma}{d\Omega} = \left( \frac{d\sigma}{d\Omega} \right)_{\text{point}} |F(q^2)|^2
\]
Example: Proton $F_1$ Form Factor
Example: Proton $F_1$ Form Factor

\[ r_i^2 = -6 \frac{dF_i(q^2)}{dq^2} \bigg|_{q^2=0} \]

charge radius
Electromagnetic Form Factors

\[ \langle p', s' | J^\mu (q) | p, s \rangle = \bar{u}(p', s') \left[ \gamma^\mu F_1(q^2) + i\sigma^{\mu\nu} \frac{q^\nu}{2m} F_2(q^2) \right] u(p, s) \]

Electric charge

\[ F_1(0) = Q \]
\[ F_2(0) = \kappa \]
\[ F_1(0) + F_2(0) = \mu \]

Anomalous magnetic moment

\[ Q_p = 1, \; Q^n = 0 \]
\[ \mu_p = 2.79\mu_N, \; \mu_n = -1.91\mu_N \]

Magnetic moment

Radii:

\[ r_i^2 = -6 \frac{dF_i(q^2)}{dq^2} \bigg|_{q^2=0} \]

\[ q^2 > 0 : \text{“Look inside” hadron} \]
Scaling of Form Factors

From dimensional counting [Brodsky & Farrar, 1973]

\[ F_1 \propto \frac{1}{Q^4} \quad \text{(dipole?)} \]

\[ F_2 \propto \frac{1}{Q^6} \quad \text{(tripole?)} \]

for \( Q^2 > \zeta_{pQCD} \)

\[ Q^2 \frac{F_2}{F_1} \propto \text{const} \]

\[ \frac{G_E}{G_M} \propto \text{const} \]
Scaling of Form Factors

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\[ Q^2 \frac{F_2}{F_1} \propto \text{const} \]

\[ Q \frac{F_2}{F_1} \propto \text{JLab} \]

\[ \frac{G_E}{G_M} \propto \text{const} \]
Scaling of Form Factors

From dimensional counting

$$F_1 \propto \frac{1}{Q^4} \quad \text{(dipole?)}$$

$$F_2 \propto \frac{1}{Q^6} \quad \text{(tripole?)}$$

for $Q^2 > \zeta_{PQCD}$

$$Q^2 \frac{F_2}{F_1} \propto \text{const}$$

$F(0) \frac{1}{(1 + Q^2/M^2)^p}$

$G_E/G_M \propto \text{const}$

\[ \frac{G_E}{G_M} \propto 1.0 \]

\[ \mu_p \frac{G^p}{G_M} \propto 0.0 \]

\[ \mu_p \frac{G^p}{G_M} \propto 0.5 \]

\[ \mu_p \frac{G^p}{G_M} \propto 1.0 \]

JLab

\[ 0 \]

\[ 0.25 \]

\[ 0.5 \]

\[ 0.75 \]

\[ 1 \]

\[ 1.25 \]

\[ 1.5 \]

\[ 1.75 \]

\[ 2 \]

\[ 2.25 \]

\[ 2.5 \]

\[ 2.75 \]

\[ 3 \]

\[ 3.25 \]

\[ 3.5 \]

\[ 3.75 \]

\[ 4 \]

\[ 4.25 \]

\[ 4.5 \]

\[ 4.75 \]

\[ 5 \]

\[ 5.25 \]

\[ 5.5 \]

\[ 5.75 \]

\[ 6 \]

(b)
**Q^2 Parameterization**

- Sachs form factors reasonably described by a dipole

\[
G_E^p(Q^2) = \frac{1}{(1 + Q^2/M_D^2)^2}
\]

\[
G_M^{p,n}(Q^2) = \frac{\mu^{p,n}}{(1 + Q^2/M_D^2)^2}
\]

- with

\[M_D \approx 0.71 \text{ GeV}\]

\[\mu^p = 2.79 \mu_N\]

\[\mu^n = -1.91 \mu_N\]

- But deviations seen, particularly at large Q^2
Q² Parameterisation


• Kelly proposed a simple parameterisation for the form factors

\[ G(Q^2) \propto \frac{\sum_{k=0}^{n} a_k \tau^k}{1 + \sum_{k=1}^{n+2} b_k \tau^k} \]

\[ \tau = Q^2 / 4M^2 \]

• with \( n=1 \) and \( a_0=1 \) for \( G_M^{p,n}(Q^2), \ G_E^p(Q^2) \)

[Recent work: Cloët & Miller, 1204.4422]
Form Factor Radii & Magnetic Moments

Search for non-analytic behaviour predicted by Chiral Perturbation Theory

Form factor radii: \[ r_i^2 = -6 \frac{dF_i(q^2)}{dq^2} \bigg|_{q^2=0} \]

Magnetic moment \( \mu \)/anomalous magnetic moment \( \kappa \)

\[ \mu = 1 + \kappa = G_m(0) \]

[see lectures by B. Tiburzi]
Form Factor Radii & Magnetic Moments

Search for non-analytic behaviour predicted by Chiral Perturbation Theory

Form factor radii: \( r_i^2 = -6 \frac{dF_i(q^2)}{dq^2} \bigg|_{q^2=0} \)

Magnetic moment \( \mu \)/anomalous magnetic moment \( \kappa \)

\[ \mu = 1 + \kappa = G_m(0) \]

[see lectures by B. Tiburzi]
Lattice Nucleon Form Factors
Systematics of a Lattice Calculation

• In the following slides, we will be looking at lattice results for the EM form factors of the proton which can be compared with the experimental results

• We need to be careful of systematic errors that could affect our results

  • Finite lattice spacing
  
  • Large quark masses
  
  • Finite volume
  
  • Contamination from excited states

• Will focus on recent results from QCDSF  
  
  PRD 84, 074507 (2011) [arXiv:1106.3580]
Comparison With Experiment

\[ F_{1}^{u-d}(Q^2) \]

- Isovector Dirac form factor
- Darker colours \( \rightarrow \) lighter masses
- Grey band \( \rightarrow \) parameterisation of experimental data
- Lattice results lie above experiment with smaller slope
Comparison With Experiment

\[ F_{2}^{u-d}(Q^2) \]

\[ m_\pi \geq 0.8 \text{GeV} \]

\[ 0.8 \text{GeV} \geq m_\pi \geq 0.4 \text{GeV} \]

- Isovector Pauli form factor
- Darker colours → lighter masses
- Grey band → parameterisation of experimental data
- Lattice results lie above experiment with smaller slope
**Systematic Errors**

**Lattice Spacing**

- Scan available datasets for bins with constant $m_{\pi}$, but with 3 or more different lattice spacings, $a$

- Plot results as a function of $a^2$

Grey band: parameterisation of experimental data

No visible dependence on $a$
Systematic Errors

Volume

• Scan datasets for bins with constant $m_\pi$ but with 2 or more spatial volumes, $L$

• Plot results as a function of $L$

Grey band: parameterisation of experimental data

- Small volume correction accounted for by exponential factor
  \[ a + b e^{(-m_\pi L)} \]

- $m_\pi = 0.4 \ldots 0.5 \text{GeV}$

- $m_\pi \approx 0.28 \text{GeV}$

\[ \langle r^2 \rangle \]
For small values of Euclidean time, effects from excited states may adversely affect the extraction of physical observable from the lattice, e.g.

\[ C_{2pt}(t) = A_0 e^{-M_0 t} + A_1 e^{-M_1 t} + \ldots \]

- Require distances between source \((t=0)\) - operator insertion \((\tau)\) - sink \((t_{snk}) \gg 1\)
- Simulate with multiple \(t_{snk}\)'s on a single dataset to test the validity of our original choice \(t_{snk}=13\)
Systematic Errors

• Systematics appear to be under control

  • Finite lattice spacing  √

  • Large quark masses

  • Finite volume  √

  • Contamination from excited states  √

• Remaining discrepancy must come from unphysical quark masses
• Isovector Dirac radius (squared)
• Isovector Pauli radius (squared)
• Isovector anomalous magnetic moment
• Dirac radius: different experimental values
Light Quark Mass Dependence

• Radii suppressed at large masses and small volumes

• Hint of sharp rise at small masses

• $r^2$ approaching experimental result

• $\kappa^u-d$ shows clear curvature at small masses

• Can the remaining discrepancy be due to the (still) unphysically large quark masses?

• Contact with ChPT?

• Popular expressions from *Phys. Rev. D71, 034508 (2005)* (SSE)

• But are they valid up to $m_\pi < 300$ MeV ?

• Check by: Varying unknown parameters over a “reasonable” range and extrapolate up from the chiral limit with the only constraint provided by the experimental point
• Rapidly decreasing isovector Dirac ms radius as pion mass increases

• Overlap with the lattice data points at $m_\pi \approx 250 \ldots 300$ MeV

• Similar observations for Pauli radius and anomalous magnetic moment

• Isoscalar $r_1$ indicates form not valid past physical pion mass

\[ \langle r^2 \rangle_{u-d} \text{[fm}^2 \rangle \]

\[ m_\pi \text{[GeV]} \]

\[ \beta=5.20 \]

\[ \beta=5.25 \]

\[ \beta=5.29 \]

\[ \beta=5.40 \]

\[ m_\pi \times L < 3.4 \]

\[ \text{PDG 2010} \]

\[ \text{Belushkin et al. '07} \]

\[ \text{Pohl et al. '10} \]
Flavour Distribution

- Individual flavour contributions not accessible directly in experiment

- Must be derived from a combination of proton and neutron form factors

  - (assuming charge symmetry $u^p = d^n$)

  $F^p = \frac{2}{3} F^p_u - \frac{1}{3} F^p_d$

  $F^n = -\frac{1}{3} F^p_u + \frac{2}{3} F^p_d$

- On the lattice we compute the individual quark contributions directly
d-quark contribution to $F_1(Q^2)$ falls off faster than the u-quark contribution

Effect is enhanced at lighter quark masses
In terms of charge radii, the d-quark in the proton has a larger charge radius than the u-quark.
Implications for Transverse Densities

Recall: \[ q(b_\perp^2) = \int d^2q_\perp \, e^{-ib_\perp \cdot q_\perp} F_1(q^2) \]

\[ r_{d,1,2} > r_{u,1,2} \]

Ph. Hägler (QCDSF) [PRL 98, 222001 (2007)]
Pion Form Factor

\[ \langle \pi(p') | J^\mu(\vec{q}) | \pi(p) \rangle = P^\mu F_\pi(q^2) \]

\[ q^2 = -Q^2 = (p' - p)^2 \]

\[ P^\mu = p'^\mu + p^\mu \]
Pion Form Factor

• Asymptotic normalisation known from $\pi \to \mu + \nu$ decay

$$F_\pi(Q^2 \to \infty) = \frac{16\pi\alpha_s(Q^2)f_\pi^2}{Q^2}$$

• Allows to study the transition from the soft to hard regimes

• Low $Q^2$: measured directly by scattering high energy pions from atomic electrons [CERN]

• High $Q^2$: quasi-elastic scattering off virtual pions [DESY & JLab]

Model dependence
bool registerAll()
{
    bool success = true;
    if (!registered)
    {
        success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("a0-a0"), mesA0A01SeqSrc);
        success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("a0-rho_x_1"), mesA0RhoX1SeqSrc);
        success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("a0-rho_y_1"), mesA0RhoY1SeqSrc);
        success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("a0-b1_z"), mesA0B1Z1SeqSrc);
        success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("a0-rho_z_1"), mesA0RhoZ1SeqSrc);
        success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("a0-b1_y"), mesA01B1Y1SeqSrc);
        success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("a0-b1_x"), mesA01B1X1SeqSrc);
        success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("a0-pion_2"), mesA01Pion2SeqSrc);
        success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("a0-a1_z"), mesA0A1Z1SeqSrc);
        success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("a0-rho_z_2"), mesA0RhoZ2SeqSrc);
        success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("a0-a1_y"), mesA0A1Y1SeqSrc);
        success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("a0-a1_x"), mesA0A1X1SeqSrc);
        success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("a0-pion_1"), mesA0Pion1SeqSrc);
        success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("pion_1-pion_1"), mesPion1Pion1SeqSrc);
        // keep for historical purposes
        success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("pion"), mesPion1Pion1SeqSrc);
    }
    return success;
}
Pion Form Factor

\[ Q^2 = -q^2 \]

**JLQCD:** arXiv:0810.2590 [hep-lat]

- \( m_{ud} = 0.050 \)
- fit: \( \rho \) pole + cubic
- VMD

\[ F_V(q^2) \]

**QCDSF, hep-lat/060821**
Pion Form Factor

**JLQCD:** arXiv:0810.2590 [hep-lat]

- $m_{ud} = 0.050$
- fit: $\rho$ pole + cubic
- VMD

Minimum lattice momentum:

\[ \vec{p} = \frac{2\pi}{L} \vec{n} \]

[affects determination of: \( \langle r_\pi^2 \rangle \)]
Discretised Momentum

• On a periodic lattice with spatial volume $L^3$, quark fields satisfy

$$
\psi(x + \bar{e}^i L) = \psi(x), \quad i = 1, 2, 3
$$

$$
\int d^4 p \ e^{-ip(x+\bar{e}^i L)} \tilde{\psi}(p) = \int d^4 p \ e^{-ipx} \tilde{\psi}(p), \quad i = 1, 2, 3
$$

• so we see that momenta are discretised in units of $p_i = \frac{2\pi}{L} n_i, \quad i = 1, 2, 3$

• For typical lattices, smallest non-zero momentum $\sim 400-500$ MeV

• Poor momentum resolution

• Can affect phenomenological observables e.g. form factors
Accessing small momenta: (partially) twisted boundary conditions

- On a periodic lattice with spatial volume $L^3$, quark fields satisfy

$$\psi(x + \vec{e}_i L) = \psi(x), \quad i = 1, 2, 3$$

$$\int d^4p e^{-ip(x+\vec{e}_i L)} \tilde{\psi}(p) = \int d^4p e^{-ipx} \tilde{\psi}(p), \quad i = 1, 2, 3$$

- so we see that momenta are discretised in units of

$$p_i = \frac{2\pi}{L} n_i, \quad i = 1, 2, 3$$

- Modify boundary conditions on the valence quarks

$$\psi(x + \vec{e}_i L) = e^{i\theta_i} \psi(x), \quad i = 1, 2, 3$$

- allows to tune the momenta continuously

$$p_i = \frac{2\pi}{L} n_i + \frac{\theta_i}{L}, \quad i = 1, 2, 3$$

- For a meson with quark flavours (1,2)

$$\vec{p} = \frac{2\pi}{L} \vec{n} + \frac{(\vec{\theta}_1 - \vec{\theta}_2)}{L}$$
Implementation

• Make a unitary Abelian transformation on the fields

\[ \psi(x) \longrightarrow \mathcal{U}(\theta, x)\tilde{\psi}(x) = e^{i\frac{\theta \cdot \vec{x}}{L}}\tilde{\psi}(x) \]

• Phase factor cancels in all terms of the lattice fermion action except the spatial hopping term

\[ \tilde{\psi}(x) \left[ e^{i\frac{a\theta}{L}} U_i(x)(1 - \gamma_i)\tilde{\psi}(x + \hat{i}) + e^{-i\frac{a\theta}{L}} U_i^\dagger(x - \hat{i})(1 + \gamma_i)\tilde{\psi}(x - \hat{i}) \right] \]

• In practice, compute quark propagator with gauge links

\[ \{U_i(x)\} \longrightarrow \{e^{i\frac{a\theta}{L}} U_i(x)\} \]

• Twisted boundary conditions for sea quarks requires generating new set of gauge fields for each twist

• only twist valence quarks \(\Longrightarrow\) partially twisted boundary conditions

• Introduces an additional finite size effect that is, however, exponentially suppressed
Additional Finite Volume Effects

$E_{\pi,\rho}^2 = m_{\pi,\rho}^2 + \left( \vec{p}_{\text{lat}} - \frac{\vec{\theta}_1 - \vec{\theta}_2}{L} \right)^2$

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$E_{\pi,\rho}^2 = m_{\pi,\rho}^2 + \left( \vec{p}_{\text{lat}} - \frac{\vec{\theta}_1 - \vec{\theta}_2}{L} \right)^2$

$m_{\pi}/m_{\rho} = 0.70$

$E_{\pi,\rho}^2 = m_{\pi,\rho}^2 + \left( \vec{p}_{\text{lat}} - \frac{\vec{\theta}_1 - \vec{\theta}_2}{L} \right)^2$

$E_{\pi,\rho}^2 = m_{\pi,\rho}^2 + \left( \vec{p}_{\text{lat}} - \frac{\vec{\theta}_1 - \vec{\theta}_2}{L} \right)^2$

$E_{\pi,\rho}^2 = m_{\pi,\rho}^2 + \left( \vec{p}_{\text{lat}} - \frac{\vec{\theta}_1 - \vec{\theta}_2}{L} \right)^2$
Implementation

[RBC/UKQCD, hep-lat/0705005]

- Use different (twisted) boundary conditions when computing the propagators either side of the current

- E.g. One possibility would be

\[ \vec{\theta}_{q_2} = \vec{\theta}_{q_3} = \vec{0}, \quad \vec{\theta}_{q_1} \neq \vec{0} \]
Pion charge radius

$\langle \pi \pi \rangle_{Q^2}$

$Q^2$ [GeV$^2$]

<table>
<thead>
<tr>
<th>maximum $Q^2$</th>
<th>linear</th>
<th>quadratic</th>
<th>cubic</th>
<th>pole</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.013 GeV$^2$</td>
<td>0.354(28)(11)</td>
<td>–</td>
<td>–</td>
<td>0.361(29)(12)</td>
</tr>
<tr>
<td>0.022 GeV$^2$</td>
<td>0.354(26)(11)</td>
<td>0.353(35)(11)</td>
<td>–</td>
<td>0.364(27)(12)</td>
</tr>
<tr>
<td>0.035 GeV$^2$</td>
<td>0.353(25)(11)</td>
<td>0.355(32)(11)</td>
<td>0.351(41)(11)</td>
<td>0.366(27)(12)</td>
</tr>
<tr>
<td>0.150 GeV$^2$</td>
<td>0.332(28)(11)</td>
<td>0.387(44)(13)</td>
<td>0.406(56)(13)</td>
<td>0.382(37)(12)</td>
</tr>
</tbody>
</table>

$m_\pi \approx 330$ MeV

$24^3 \times 64$

$a \approx 0.114$ fm
The fact that our result is in agreement with experiment, NLO ChPT fit to the three lowest solid blue curve. In addition we also represent the PDG wo $q^2$ dependence of $\langle \pi \rangle$ and pion form factors shows our SU(2) fit to the lattice data points for a pion with $K_{\pi}$ points for the physical pion.

Because our values of $H(2) = 0.65 \times (q^2)$ are very small, we apply NLO chiral perturbation theory (ChPT) to extract this LEC. The grey dashed curve on the right hand of Figure 2: Pion charge radius

$$m_\pi \approx 330 \text{MeV}$$

$$a \approx 0.114 \text{ fm}$$

$$Q_{\max}^2 = 0.035 \text{[GeV]}^2$$

$$[0.418(28)]$$

$$[0.452(11)]_{\text{exp}}$$
Pion form factor
(compared to experiment)

![Graph showing the pion form factor comparison to experimental data.]

- Experimental data NA7
- Lattice data for $m_\pi = 330$ MeV
- SU(2) NLO lattice-fit; $m_\pi = 330$ MeV
- SU(2) NLO lattice-fit; $m_\pi = 139.57$ MeV
- $1 + \frac{1}{6} \langle r^2 \rangle_{PDG} Q^2$

The fact that our result is in agreement with experiment, NLO ChPT fit to the three lowest using the black dashed line. Our best estimate for the pion ch...
Many choices of twist angles giving access to extremely small $Q^2$

Radii results increasing towards the experimental point at smaller quark masses
Other Hadron Form Factors

• Pion and nucleon form factors have received the most attention

• Small amount of work on form factors of other hadrons, e.g.
  
  • Hyperons
    CSSM: hep-lat/0604022, QCDSF: 1101.2806, H-W.Lin et al.: 0812.4456

  • Delta
    CSSM: 0902.4046, Alexandrou et al.: 0810.3976

  • Rho
    CSSM: hep-lat/0703014, QCDSF: PoS LAT2008, 051

  • $\gamma N \rightarrow \Delta$ transition
    Alexandrou et al.: 1011.3233

• Non-zero quadrupole moment $\rightarrow$ hadron deformation
  Review: [Alexandrou et al.: 1201.4511]