Hadron Structure

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Motivation

- We know the nucleon is not a point-like particle but in fact is composed of quarks and gluons.

- But how are these constituents distributed inside the nucleon?
  - E.g. The neutron has zero net charge, but does it have a +/- core?

- How do they combine to produce its experimentally observed properties?
  - For example:
    - “Spin crisis”: quarks carry on ~30% of the proton’s spin.
    - gluons? orbital angular momentum?

- Understanding how the nucleon is built from its quark and gluon constituents remains one the most important and challenging questions in modern nuclear physics.
Topics to cover

- Elastic scattering \rightarrow Electromagnetic form factors
- Neutron beta decay \rightarrow Nucleon axial charge
- Deep Inelastic Scattering \rightarrow Structure Functions and Parton Distribution Functions
- Generalised Parton Distribution Functions
- Hidden Flavour, e.g. strangeness content of the nucleon

*Focus on Nucleon and Pion*
Lecture 1

- A short history
- Elastic scattering
- Form Factors
  - Density distributions in the nucleon
- Lattice methods for computing hadronic matrix elements (3pt functions)
- A taste of some lattice results
Structure of the Proton

• Until 1932, proton was considered to be an elementary particle

• In 1933, Otto Stern measured the proton’s magnetic moment

\[ \mu_p \approx 2.5 \ (\text{today: } 2.7928456(11)) \ \frac{e}{2m_p} \]

• Deviates significantly from unity - the magnetic moment of point-like particle described by Dirac’s theory of relativistic fermions

\[ \mu_p = \mu_D \equiv \frac{e}{2m_p} \]

Proton is a composite particle

• The proton’s constituents were later “seen” in Deep-Inelastic Scattering experiments at SLAC (1968)
Nucleon Structure

• Nucleon structure is studied experimentally by electron-proton scattering

• Electron is a good probe because:
  
  • QED is a “well-understood” interaction

  • $\alpha_{em} = \frac{1}{137}$  

  • perturbation theory is valid

  • Electrons are charged and so easily accelerated

• Two types of e-p scattering:
  
  • Elastic scattering

  • Deep-Inelastic Scattering (DIS)
Elastic Scattering

- Final state nucleon remains intact, but with recoil
- Map out charge and density distributions inside the nucleon
- Dominated by single-photon exchange
- 4-momentum transfer \( q = k - k' = P' - P \)

\[
\frac{d\sigma}{d\Omega} = \left( \frac{d\sigma}{d\Omega} \right)_\text{point} |F(q^2)|^2
\]

Form Factor
Elastic Scattering

- When using a point target, \( F(q) = 1 \) and

\[
\left( \frac{d\sigma}{d\Omega} \right)_{\text{point}} = \frac{(Z\alpha)^2 E^2}{4k^2 \sin^4(\theta/2)} \left( 1 - \frac{k^2}{E^2} \sin^2(\theta/2) \right)
\]

- \( E, k \) - energy and momentum of electron

- \( \theta \) - scattering angle

\( q^2 = -4EE' \sin^2 \theta/2 \)

- But experimental cross-sections deviate from this description
Elastic Scattering

- Considering only one-photon exchange, justified because the fine structure constant is so small, the S-Matrix is then

\[ S = (2\pi)^4 \delta^4(k + P - P' - k') \overline{u}(k')( - ie \gamma^\mu) u(k) \frac{-i}{q^2} \langle P' | (ie) J^\mu | P \rangle \]

\[ = -i(2\pi)^4 \delta^4(k + P - P' - k') M \]

- The electromagnetic current is

\[ J^\mu = \sum_i e_i \overline{\psi}_i \gamma^\mu \psi_i \]

Sum over quark flavours with \( m_q \ll m_p \) (u,d,s)

- Can write cross-section in terms of invariant amplitude

\[ d\sigma = \frac{E'}{2EM^2} \frac{1}{1 + \frac{2E}{M} \sin^2 \frac{\theta}{2}} |M|^2 \frac{d\Omega}{(2\pi)^2} \]
Elastic Scattering

- The invariant amplitude squared is
  \[ |\mathcal{M}|^2 = \frac{e^4}{Q^4} \ell^{\mu\nu} W_{\mu\nu} \]

- Leptonic tensor
  \[ \ell^{\mu\nu} = \bar{u}(k') \gamma^\mu u(k) \bar{u}(k) \gamma^\nu u(k') \]

- Compute in QED

- Hadronic tensor
  \[ W^{\mu\nu} = \langle P | J^\nu | P' \rangle \langle P' | J^\mu | P \rangle \]

  - with matrix element between nucleon states defining two Lorentz-invariant form factors

  \[ \langle P' | J^\mu (\vec{q}) | P \rangle = \bar{u}(P') \left[ \gamma^\mu F_1(q^2) + i\sigma^{\mu\nu} \frac{q_\nu}{2m} F_2(q^2) \right] u(P) \]

  - Dirac
  - Pauli
Elastic Scattering

• Using the fact that both tensors are symmetric and conserved $q^\mu \ell_{\mu\nu} = q^\mu W_{\mu\nu} = 0$

• The elastic scattering cross-section in the lab frame becomes

$$\frac{d\sigma}{d\Omega} = \sigma_{\text{Mott}} \left[ \frac{G_E^2(Q^2) + \tau G_M^2(Q^2)}{1 + \tau} + 2\tau G_M^2(Q^2) \tan^2 \frac{\theta}{2} \right]$$

• where

$$G_E(Q^2) = F_1(Q^2) - \tau F_2(Q^2)$$

$$G_M(Q^2) = F_1(Q^2) + F_2(Q^2) \quad \tau = Q^2/(4M^2)$$

• are the Sachs electric and magnetic form factors

• Rewriting in terms of the virtual photon’s longitudinal polarisation

$$\frac{d\sigma}{d\Omega} = \sigma_{\text{Mott}} \frac{1}{1 + \tau} \left[ G_E^2(Q^2) + \frac{\tau \epsilon}{\epsilon} G_M^2(Q^2) \right]$$

$$\epsilon^{-1} = 1 + (1 + \tau)^2 \tan \theta/2 \quad \epsilon = 1 + 2(1 + \tau)^2 \tan \theta/2$$

• Need cross sections at fixed $Q^2$ but different scattering angle: Rosenbluth separation
Elastic Scattering - Rosenbluth

\[ \frac{d\sigma}{d\Omega} = \frac{\sigma_{\text{Mott}}}{1 + \tau} \left[ G_E^2(Q^2) + \frac{\tau}{\epsilon} G_M^2(Q^2) \right] \]

- At fixed $Q^2$, $G_E$ and $G_M$ determined from intercept and slope as a function of $\epsilon^{-1}$

- Drawback - reduced sensitivity to $G_E$ at large $Q^2$

- Both form factors reasonably well described by the dipole form

\[ G_E^p(Q^2) = \frac{G_M^p(Q^2)}{\mu_p} = \frac{1}{(1 + Q^2/M_D^2)^2} \]

- with $M_D^2 \approx 0.71 \text{ GeV}^2$
Elastic Scattering - Rosenbluth

\[
\frac{d\sigma}{d\Omega} = \frac{\sigma_{\text{Mott}}}{1 + \tau} \left[ G_E^2(Q^2) + \frac{\tau}{\epsilon} G_M^2(Q^2) \right]
\]

- For a neutron:
  - \( G_E \) small, so extraction near impossible
  - No neutron target so use deuterium and
    - subtract proton contribution
  - Model nuclear effects
Elastic Scattering - Polarisation Transfer

- Difficulties with unpolarised scattering → new techniques necessary

- Mid-‘90s brought

  - High luminosity, highly polarised electron beams
  
  - Polarised targets ($^1$H, $^2$H, $^3$He)

  - Large, efficient neutron detectors

- Polarisation transfer experiments provide access to the ratio $G_E/G_M$ directly from ratio of polarisation transverse and parallel to the momentum of the nucleon

  \[
  \frac{G_E}{G_M} = - \frac{P_t}{P_l} \frac{E + E'}{2M} \tan \frac{\theta}{2}
  \]

- Combine with previous accurate results for $G_M$ to also determine $G_E$
Elastic Scattering - Polarisation Transfer

- Precise results now available up to 8-9 GeV$^2$
- Does $G_E^p$ change sign?
- What is the origin of the linear fall-off?
Elastic Scattering - Polarisation Transfer
JLab, Hall A, PRC85 (2012) 045203

FIG. 17. (color online) Comparison of selected theoretical predictions to data for all four nucleon FFs at space-like $Q^2$. Theory curves are [15] (Diehl05), [18] (Eichmann11), [72] (Lomon06), [91] (Gross08) and [94] (Santopinto10). The data are from [5, 80, 81, 100–105] (cross section data, empty circles) and [1, 2, 25, 49–53, 106] (polarization data, filled circles), where the results of [2] have been replaced by the results of the present work (Table IV). The data are from [5, 80, 81, 100–102, 104, 105, 107–109]. The data are from [20, 110–121]. The data are from [21, 122–132].

$D/G = (1 + Q^2/\Lambda^2)^{-2}$, with $\Lambda^2 = 0.71$ GeV$^2$, is the standard dipole form factor.

Figure 17 summarizes the theoretical interpretation of the nucleon electromagnetic form factors, with representative examples from each of the classes of models discussed compared to the world data for all four nucleon electromagnetic form factors. Published results for $R = G_p^E/G_p^M$ were converted to $G_p^E$ values using the global fit of $G_p^E$ and $G_p^M$ from [43], updated to use the $R$ values of the present work, a change that does not noticeably affect $G_p^M$. Except at very low $Q^2$, the contribution of the uncertainty in $G_p^M$ to the resulting uncertainty in $G_p^E$ is negligible. At this juncture, it is worth recalling that the $G_p^E$ results extracted from cross section data are believed to be unreliable at high $Q^2$ due to incompletely understood TPEX corrections, which have not been applied to the data shown in Figures 14-17. Except for the DSE calculation of [18], all of the models shown describe existing data very well, which is to be expected given that the parameters of the models are fitted to reproduce the data. However, their predictions tend to diverge when extrapolated outside the $Q^2$ range of the data. That the DSE-based calculation of [18] fails to describe the data as well as the other calculations is not surprising, since it represents a more fundamental ab initio approach with virtually no adjustable parameters, but requires approximations that are not yet well-controlled. Significant progress in the quality of the predictions is nonetheless evident, as the data expose the weaknesses of different approximation schemes. Since the hard scattering mechanism leading to the asymptotic pQCD scaling relations is not expected to dominate the form factor behavior at presently accessible $Q^2$ values, phenomenological models and the ambitious ongoing efforts in lattice QCD and DSE calculations are of paramount importance to understanding the internal structure and dynamics of the nu-
Insights into Nucleon Structure

- $G_E \neq G_M \Rightarrow$ different charge and magnetisation distributions

- If $M \to \infty$ initial and final nucleons are fixed at the same location $Q^2 \ll M^2$

- Initial and final states have same internal state

- Fourier transformation of form factors are density distributions

- But $M$ is finite so need to consider nucleon recoil effects

- Initial and final states now sampled in different frames $\Rightarrow$ Lorentz contraction

- No model independent way to separate internal structure and recoil effects

- Work around: Breit frame or infinite momentum frame
Density Distributions

- Consider the Breit frame: $|P| = |P'|$

  - initial and final states have momenta with equal magnitude, hence similar Lorentz contraction

- $G_E(Q^2)$ can be interpreted as the Fourier transformation of the charge distribution

  $$G_E(Q^2) = \int e^{i\vec{q}\cdot\vec{x}} \rho(r) d^3r$$

  - expanding at small $Q^2$

    $$G_E(Q^2) = Q_e - \frac{1}{6} Q^2 \langle r^2 \rangle + \ldots$$

  - defines the charge radius of the nucleon

    $$\langle r^2 \rangle = -6 \left. \frac{dG_E(Q^2)}{dQ^2} \right|_{Q^2=0}$$
Size of the Proton

- $>5\sigma$ discrepancy between muonic hydrogen and e-p scattering

- $r_p=0.84184(67)\text{ fm}$  \[\text{[Nature 466, 213 (2010)]}\]

- $r_p=0.875(8)(6)\text{ fm}$  \[\text{[arXiv:1102.0318]}\]
Transverse Spatial Distributions

- Model independent relation between form factors and transverse spatial distributions occurs in the infinite momentum frame.

- Quark (charge) distribution in the transverse plane:

\[ q(b_{\perp}^2) = \int d^2 q_{\perp} e^{-i\vec{b}_{\perp} \cdot \vec{q}_{\perp}} F_1(q^2) \]

**Distance of (active) quark to the centre of momentum in a fast moving nucleon**

**Provide information on the size and internal charge densities**
Electromagnetic Form Factors

- Can some of these questions be answered by a calculation from QCD?
- Form factors are nonperturbative quantities

\[ \langle p', s' | J^\mu (\vec{q}) | p, s \rangle = \bar{u}(p', s') \left[ \gamma^\mu F_1(q^2) + i\sigma^{\mu\nu} \frac{q_\nu}{2m} F_2(q^2) \right] u(p, s) \]
Calculating Matrix Elements

\[ \langle H' | \mathcal{O} | H \rangle \]

\( H, H' : \pi, K p, n, \ldots \)

\( \mathcal{O} : V_\mu, A_\mu, \ldots \)
Calculating Matrix Elements

Spin-0
\[ \langle \pi(p')|J^\mu(\vec{q})|\pi(p)\rangle = P^\mu F_\pi(q^2) \]

Spin-1/2
\[ \langle N(p', s')|J^\mu(\vec{q})|N(p, s)\rangle = \bar{u}(p', s') \left[ \gamma^\mu F_1(q^2) + i\sigma^\mu\nu \frac{q\nu}{2m} F_2(q^2) \right] u(p, s) \]

Spin-1
\[ \langle \rho(p', s')|J^\mu(\vec{q})|\rho(p, s)\rangle = \]
\[ - (\epsilon'^* \cdot \epsilon) P^\mu G_1(Q^2) - [(\epsilon'^* \cdot q)\epsilon^\mu - (\epsilon \cdot q)\epsilon'^{\ast\mu}] G_2(Q^2) + (\epsilon \cdot q)(\epsilon'^* \cdot q) \frac{P^\mu}{(2m^2)^2} G_3(Q^2) \]

Spin-3/2
\[ \langle \Delta(p', s')|J^\mu(\vec{q})|\Delta(p, s)\rangle = \]
\[ \bar{u}_\alpha(p', s') \left\{ - g^{\alpha\beta} \left[ \gamma^\mu a_1(Q^2) + \frac{P^\mu}{2M_\Delta} a_2(Q^2) \right] - \frac{q^\alpha q^\beta}{(2M_\Delta)^2} \left[ \gamma^\mu c_1(Q^2) + d \frac{P^\mu}{2M_\Delta} c_2(Q^2) \right] \right\} u_\beta(p, s) \]
Lattice 3pt Functions

\[ \langle \Omega | T (\overline{\chi}_\beta(0)) | \Omega \rangle \]

- Create a state (with quantum numbers of the proton) at time \( t=0 \)
Lattice 3pt Functions

\[ \langle \Omega | T (\mathcal{O}(\vec{x}_1, \tau) \bar{\chi}_\beta(0)) | \Omega \rangle \]

- Create a state (with quantum numbers of the proton) at time \( t=0 \)
- Insert an operator, \( \mathcal{O} \), at some time \( \tau \)
Lattice 3pt Functions

\[ \langle \Omega | T (\chi_\alpha (\vec{x}_2, t) \mathcal{O} (\vec{x}_1, \tau) \bar{\chi}_\beta (0)) | \Omega \rangle \]

• Create a state (with quantum numbers of the proton) at time \( t=0 \)

• Insert an operator, \( \mathcal{O} \), at some time \( \tau \)

• Annihilate state at final time \( t \)
Lattice 3pt Functions

\[ G(t, \tau, p, p') = \sum_{\vec{x}_2, \vec{x}_1} e^{-ip'(\vec{x}_2 - \vec{x}_1)} e^{-ip\cdot\vec{x}_1} \Gamma_{\beta\alpha} \langle \Omega | T (\chi_{\alpha}(\vec{x}_2, t) \mathcal{O}(\vec{x}_1, \tau) \bar{\chi}_\beta(0)) | \Omega \rangle \]

- Create a state (with quantum numbers of the proton) at time \( t=0 \)
- Insert an operator, \( \mathcal{O} \), at some time \( \tau \)
- Annihilate state at final time \( t \)
Lattice 3pt Functions

\[ G(t, \tau, p, p') = \sum_{\vec{x}_2, \vec{x}_1} e^{-ip' \cdot (\vec{x}_2 - \vec{x}_1)} e^{-i\vec{p} \cdot \vec{x}_1} \Gamma_{\beta \alpha} \langle \Omega | T \left( \chi_{\alpha}(\vec{x}_2, t) \mathcal{O}(\vec{x}_1, \tau) \bar{\chi}_{\beta}(0) \right) | \Omega \rangle \]

- Insert complete set of states
  \[ I = \sum_{B', p', s'} |B', p', s'\rangle \langle B', p', s'| \quad I = \sum_{B, p, s} |B, p, s\rangle \langle B, p, s| \]

- Make use of translational invariance
  \[ \chi(\vec{x}, t) = e^{\hat{H} t} e^{-i\hat{P} \cdot \vec{x}} \chi(0) e^{i\hat{P} \cdot \vec{x}} e^{-\hat{H} t} \]

\[ G(t, \tau, \vec{p}, \vec{p}') = \sum_{B, B'} \sum_{s, s'} e^{-E_{B'}(\vec{p}')(t-\tau)} e^{-E_B(\vec{p}) \tau} \Gamma_{\beta \alpha} \]
\[ \times \langle \Omega | \chi_{\alpha}(0) | B', p', s'\rangle \langle B', p', s' | \mathcal{O}(\vec{q}) | B, p, s \rangle \langle B, p, s | \bar{\chi}_{\beta}(0) | \Omega \rangle \]

- Evolve to large Euclidean times to isolate ground state \( 0 \ll \tau \ll t \)

\[ G(t, \tau, \vec{p}, \vec{p}') = \sum_{s, s'} e^{-E_{\vec{p}'}(t-\tau)} e^{-E_{\vec{p}} \tau} \Gamma_{\beta \alpha} \langle \Omega | \chi_{\alpha}(0) | N(p', s') \rangle \langle N(p', s') | \mathcal{O}(\vec{q}) | N(p, s) \rangle \langle N, p, s | \bar{\chi}_{\beta}(0) | \Omega \rangle \]
Lattice 3pt Functions

• Consider a pion 3pt function

\[ G(t, \tau, p, p') = \sum_{\vec{x}_2, \vec{x}_1} e^{-i\vec{p} \cdot (\vec{x}_2 - \vec{x}_1)} e^{-i\vec{p} \cdot \vec{x}_1} \langle \Omega | T (\chi(\vec{x}_2, t) \mathcal{O}(\vec{x}_1, \tau) \chi^\dagger(0)) | \Omega \rangle \]

• With interpolating operator \( \chi(x) = \bar{d}(x)\gamma_5 u(x) \)

• And insert the local operator (quark bi-linear) \( \bar{q}(x)\mathcal{O}q(x) \)

\[ -\bar{d}(x_2)\gamma_5 u(x_2)\bar{u}(x_1)\mathcal{O}u(x_1)\bar{u}(0)\gamma_5 d(0) \]

\( \mathcal{O} \): Combination of \( \gamma \) matrices and derivatives

\( \bar{u} \)-quark
Consider a pion 3pt function

\[ G(t, \tau, p, p') = \sum_{\vec{x}_2, \vec{x}_1} e^{-i\vec{p}' \cdot (\vec{x}_2 - \vec{x}_1)} e^{-i\vec{p} \cdot \vec{x}_1} \langle \Omega | T \left( \chi(\vec{x}_2, t) \mathcal{O}(\vec{x}_1, \tau) \chi^\dagger(0) \right) | \Omega \rangle \]

With interpolating operator \( \chi(x) = \bar{d}(x)\gamma_5 u(x) \)

And insert the local operator (quark bi-linear) \( \bar{q}(x)\mathcal{O}q(x) \)

\[ -\bar{d}(x_2)\gamma_5 u(x_2)\bar{u}(x_1)\mathcal{O}u(x_1)\bar{u}(0)\gamma_5 d(0) \]

\( \mathcal{O} \): Combination of \( \gamma \) matrices and derivatives

\( \bar{u} \)-quark
Lattice 3pt Functions

\[-d_\beta^a(x_2)\gamma_5\beta\gamma_u\gamma^a(x_2)\bar{u}^b_\rho(x_1)\Gamma_{\rho\delta}u^b_\delta(x_1)\bar{u}^c_\xi(0)\gamma_5\xi\alpha d^c_\alpha(0)\]

• all possible Wick contractions
Lattice 3pt Functions

- all possible Wick contractions

- connected

\[-\bar{d}_\beta^a(x_2)\gamma_{5\beta\gamma}u_\gamma^a(x_2)\bar{u}_\rho^b(x_1)\Gamma_{\rho\delta}u_\delta^b(x_1)\bar{u}_\xi^c(0)\gamma_{5\xi\alpha}d_\alpha^c(0)\]
Lattice 3pt Functions

- all possible Wick contractions

- connected

\[ -\bar{d}_{\beta}^a(x_2)\gamma_5\beta\gamma u_{\gamma}^a(x_2)\bar{u}_{\rho}^b(x_1)\Gamma_{\rho\delta}u_{\delta}^b(x_1)\bar{u}_{\xi}^c(0)\gamma_5\xi\alpha d_{\alpha}^c(0) \]

- disconnected

\[ S_{d\alpha\beta}^{ca}(0, x_2)\gamma_5\beta\gamma S_{u\gamma\rho}^{ab}(x_2, x_1)\Gamma_{\rho\delta}S_{u\delta\xi}^{bc}(x_1, 0)\gamma_5\xi\alpha \]

\[ -S_{d\alpha\beta}^{ca}(0, x_2)\gamma_5\beta\gamma S_{u\gamma\xi}^{ac}(x_2, 0)\gamma_5\xi\alpha S_{u\delta\rho}^{bb}(x_1, x_1)\Gamma_{\rho\delta} \]
Lattice 3pt Functions

all possible Wick contractions

connected

\[ \text{Tr} \left[ S_d(0, x_2) \gamma_5 S_u(x_2, x_1) \Gamma S_u(x_1, 0) \gamma_5 \right] \]

disconnected

\[ \text{Tr} \left[ - S_d(0, x_2) \gamma_5 S_u(x_2, 0) \gamma_5 \right] \text{Tr} \left[ S_u(x_1, x_1) \Gamma \right] \]
Lattice 3pt Functions

\[-\bar{d}_\beta(x_2) \gamma_{5\beta\gamma} u_\gamma(x_2) \bar{u}_\rho(x_1) \Gamma_{\rho\delta} u^\delta(x_1) \bar{u}_\xi(0) \gamma_{5\xi\alpha} d^\alpha(0)\]

• all possible Wick contractions

• connected

\[
\text{Tr} \left[ S_{d}^\dagger(x_2, 0) S_{u}(x_2, x_1) \Gamma S_{u}(x_1, 0) \right]
\]

• disconnected

\[
\text{Tr} \left[ - S_{d}^\dagger(x_2, 0) S_{u}(x_2, 0) \right] \text{Tr} \left[ S_{u}(x_1, x_1) \Gamma \right]
\]

• all-to-all propagators
Lattice 3pt Functions

\[ G_{\Gamma}(t, \tau; p', \vec{p}) = \sum_{\vec{x}_2, \vec{x}_1} e^{-i\vec{p}' \cdot (\vec{x}_2 - \vec{x}_1)} e^{-i\vec{p} \cdot \vec{x}_1} \Gamma_{\beta\alpha} \langle \Omega | T [ \chi_{\alpha}(t, \vec{x}_2) \mathcal{O}(\tau, \vec{x}_1) \bar{\chi}_{\beta}(0) ] | \Omega \rangle \]

- Use the following interpolating operator to create a proton

\[ \chi_{\alpha}(x) = \epsilon^{abc} \left( u^T a(x) \ C \gamma_5 \ d^b(x) \right) u^c(x) \]

- And insert the local operator (quark bi-linear)

\[ \bar{q}(x) \mathcal{O} q(x) \]

\( \mathcal{O} \): Combination of \( \gamma \) matrices and derivatives

- Perform all possible (connected) Wick contractions

\[ \epsilon^{abc} \epsilon^{a'b'c'} \left( u^T a(x_2) \ C \gamma_5 \ d^b(x_2) \right) u^c(x_2) \bar{u}(x_1) \mathcal{O} u(x_1) \bar{u}^c(0) \left( \bar{d}^{b'}(0) C \gamma_5 \bar{u}^T a'(0) \right) \]
• Use the following interpolating operator to create a proton

\[ \chi_\alpha(x) = \epsilon^{abc} \left( u^T a(x) \, C \gamma_5 \, d^b(x) \right) u_c^\alpha(x) \]

• And insert the local operator (quark bi-linear) \( \bar{q}(x) \mathcal{O} q(x) \) \( \mathcal{O} \): Combination of \( \gamma \) matrices and derivatives

• Perform all possible (connected) Wick contractions
Lattice 3pt Functions

\[ G_{\Gamma}(t, \tau; \vec{p}', \vec{p}) = \sum_{\vec{x}_2, \vec{x}_1} e^{-i\vec{p}' \cdot (\vec{x}_2 - \vec{x}_1)} e^{-i\vec{p} \cdot \vec{x}_1} \Gamma_{\beta\alpha} \langle \Omega | T \left[ \chi_{\alpha}(t, \vec{x}_2) \mathcal{O}(\tau, \vec{x}_1) \bar{\chi}_{\beta}(0) \right] | \Omega \rangle \]

- Use the following interpolating operator to create a proton

\[ \chi_{\alpha}(x) = \epsilon^{abc} \left( u^{T_a}(x) \ C \gamma_5 \ d^b(x) \right) u^c(x) \]

- And insert the local operator (quark bi-linear) \( \bar{q}(x) \mathcal{O} q(x) \)

\( \mathcal{O} \): Combination of \( \gamma \) matrices and derivatives

- Perform all possible (connected) Wick contractions

\[ \epsilon^{abc} \epsilon^{a'b'c'} \left( u^{T_a}(x_2) \ C \gamma_5 \ d^b(x_2) \right) u^c(x_2) \bar{u}(x_1) \mathcal{O} u(x_1) \bar{u}^{c'}(0) \left( \bar{d}^{b'}(0) C \gamma_5 \bar{u}^{T_{a'}}(0) \right) \]
Lattice 3pt Functions

\[
G_{\Gamma}(t, \tau; \vec{p}', \vec{p}) = \sum_{\vec{x}_2, \vec{x}_1} e^{-i\vec{p}' \cdot (\vec{x}_2 - \vec{x}_1)} e^{-i\vec{p} \cdot \vec{x}_1} \Gamma_{\beta\alpha} \langle \Omega | T \left[ \chi_{\alpha}(t, \vec{x}_2) \mathcal{O}(\tau, \vec{x}_1) \bar{\chi}_{\beta}(0) \right] | \Omega \rangle
\]

- Use the following interpolating operator to create a proton

\[
\chi_{\alpha}(x) = \epsilon^{abc} \left( u^{Ta}(x) \ C\gamma_5 \ d^b(x) \right) u^c_{\alpha}(x)
\]

- And insert the local operator (quark bi-linear) \( \bar{q}(x)\mathcal{O}q(x) \)

- Perform all possible (connected) Wick contractions

\[
\epsilon^{abc} \epsilon^{a'b'c'} \left( u^{Ta}(x_2) \ C\gamma_5 \ d^b(x_2) \right) u^c_{\alpha}(x_2) \bar{u}(x_1)\mathcal{O}u(x_1)\bar{u}^{c'}(0) \left( \bar{d}^{b'}(0)C\gamma_5\bar{u}^{Ta'}(0) \right)
\]

\( \mathcal{O} \): Combination of \( \gamma \) matrices and derivatives

u-quark (4 terms)
Lattice 3pt Functions

\[ G_\Gamma(t, \tau; \vec{p}', \vec{p}) = \sum_{\vec{x}_2, \vec{x}_1} e^{-i\vec{p}' \cdot (\vec{x}_2 - \vec{x}_1)} e^{-i\vec{p} \cdot \vec{x}_1} \Gamma_\alpha \langle \Omega | T [\chi_\alpha(t, \vec{x}_2) \mathcal{O}(\tau, \vec{x}_1) \bar{\chi}_\beta(0)] | \Omega \rangle \]

- Use the following interpolating operator to create a proton

\[ \chi_\alpha(x) = \epsilon^{abc} (u^{Ta}(x) C \gamma_5 d^b(x)) u^c_\alpha(x) \]

- And insert the local operator (quark bi-linear)

\[ \bar{q}(x) \mathcal{O} q(x) \quad \mathcal{O}: \text{Combination of } \gamma \text{ matrices and derivatives} \]

- Perform all possible (connected) Wick contractions

\[ \epsilon^{a'b'c'} (u^{Ta}(x_2) C \gamma_5 d^b(x_2)) u^c_\alpha(x_2) \bar{d}(x_1) \mathcal{O} d(x_1) \bar{u}^c(0) \left( \bar{d}^{b'}(0) C \gamma_5 \bar{u}^{Ta'}(0) \right) \]

\( d \)-quark (2 terms)
Lattice 3pt Functions

\[ G_{\Gamma}(t, \tau; \vec{p}', \vec{p}) = \sum_{\vec{x}_2, \vec{x}_1} e^{-i\vec{p}' \cdot (\vec{x}_2 - \vec{x}_1)} e^{-i\vec{p} \cdot \vec{x}_1} \Gamma_{\beta\alpha} \langle \Omega | T \left[ \chi_{\alpha}(t, \vec{x}_2) \mathcal{O}(\tau, \vec{x}_1) \bar{\chi}_{\beta}(0) \right] | \Omega \rangle \]

- Use the following interpolating operator to create a proton
  \[ \chi_{\alpha}(x) = \epsilon^{abc} (u^{Ta}(x) C \gamma_5 d^b(x)) u^c_{\alpha}(x) \]
- And insert the local operator (quark bi-linear): \[ \bar{q}(x) \mathcal{O} q(x) \]
  \( \mathcal{O} \): Combination of \( \gamma \) matrices and derivatives
- Perform all possible (connected) Wick contractions

\[ \epsilon^{abc} \epsilon^{a'b'c'} (u^{Ta}(x_2) C \gamma_5 d^b(x_2)) u^c_{\alpha}(x_2) \bar{d}(x_1) \mathcal{O} d(x_1) \bar{u}^c(0) \left( \bar{d}^{b'}(0) C \gamma_5 \bar{u}^{Ta'}(0) \right) \]

- d-quark (2 terms)
Lattice 3pt Functions

• Pictorially:
  
  • u-quark
  
  • d-quark
  
  • s-quark

• quark-line disconnected contributions drop out in isovector quantities ($u-d$) if isospin is exact ($m_u=m_d$)
Lattice 3pt Functions at the quark level

\[ C_\Gamma(t, \tau; \vec{p}', \vec{p}) = \sum_{\vec{x}_1} e^{i \vec{q} \cdot \vec{x}_1} \left\langle \text{Tr} \left[ \Sigma_\Gamma(\vec{0}, 0; \vec{p}', t) \mathcal{O}(\vec{x}_1, \tau) G(\vec{x}_1, 0) \right] \right\rangle_{\{U\}} \]
**Lattice 3pt Functions at the quark level**

\[
C_\Gamma(t, \tau; \vec{p}', \vec{p}) = \sum_{\vec{x}_1} e^{i\vec{q} \cdot \vec{x}_1} \left\langle \text{Tr} \left[ \Sigma_\Gamma(\vec{0}, 0; \vec{p}', t) \mathcal{O}(\vec{x}_1, \tau) G(\vec{x}_1, 0) \right] \right\rangle_{\{U\}}
\]

\[
\Sigma_\Gamma(\vec{0}, 0; \vec{x}_1; \vec{p}', t) = \sum_{\vec{x}_2} S_\Gamma(\vec{x}_2, t; \vec{0}, 0; \vec{p}') G(\vec{x}_2, t; \vec{x}_1)
\]

**Diagram:**

- **u-quark**
- **d-quark**
Lattice 3pt Functions at the quark level

\[ C_{\Gamma}(t, \tau; \vec{p}', \vec{p}) = \sum_{\vec{x}_1} e^{i\vec{q} \cdot \vec{x}_1} \left\langle \text{Tr} \left[ \Sigma_{\Gamma}(\vec{0}, 0; \vec{p}', t) \mathcal{O}(\vec{x}_1, \tau) G(\vec{x}_1, 0) \right] \right\rangle \{U\} \]

\[ \Sigma_{\Gamma}(\vec{0}, 0; \vec{x}_1; \vec{p}', t) = \sum_{\vec{x}_2} S_{\Gamma}(\vec{x}_2, t; \vec{0}, \vec{0}; \vec{p}') G(\vec{x}_2, t; \vec{x}_1) \]

Exercise: Prove \[ \tilde{G} = C\gamma_5 G^T \gamma_5 C \]

\[
S_{\Gamma}^{u; a' a} (\vec{x}_2, t; \vec{0}, 0; \vec{p}') = e^{-i\vec{p}' \cdot \vec{x}_2} \epsilon^{abc} \epsilon^{a'b'c'} \times \\
\left[ \tilde{G}^{d; bb'} (\vec{x}_2, t; \vec{0}, 0) G^{u; cc'} (\vec{x}_2, t; \vec{0}, 0) \Gamma + \text{Tr}_D [\tilde{G}^{d; bb'} (\vec{x}_2, t; \vec{0}, 0) G^{u; cc'} (\vec{x}_2, t; \vec{0}, 0)] \Gamma \\
+ \Gamma G^{u; bb'} (\vec{x}_2, t; \vec{0}, 0) \tilde{G}^{d; cc'} (\vec{x}, t; \vec{0}, 0) + \text{Tr}_D [\Gamma G^{u; bb'} (\vec{x}_2, t; \vec{0}, 0)] \tilde{G}^{d; cc'} (\vec{x}_2, t; \vec{0}, 0) \right]
\]

\[
S_{\Gamma}^{d; a' a} (\vec{x}_2, t; \vec{0}, 0; \vec{p}') = e^{-i\vec{p}' \cdot \vec{x}_2} \epsilon^{abc} \epsilon^{a'b'c'} \times \\
\left[ \tilde{G}^{u; bb'} (\vec{x}_2, t; \vec{0}, 0) \tilde{G}^{r; cc'} (\vec{x}_2, t; \vec{0}, 0) + \text{Tr}_D [\Gamma G^{u; bb'} (\vec{x}_2, t; \vec{0}, 0)] \tilde{G}^{r; cc'} (\vec{x}_2, t; \vec{0}, 0) \right]
\]
Lattice 3pt Functions

• \( \Sigma_\Gamma(\vec{0}, 0; \vec{x}_1; \vec{p}', t) = \sum_{\vec{x}_2} S_\Gamma(\vec{x}_2, t; \vec{0}, 0; \vec{p}') G(\vec{x}_2, t; \vec{x}_1) \) can be computed from the linear system of equations

\[
\sum_v M(v', v) \gamma_5 \Sigma^\dagger_\Gamma(\vec{0}, 0; v; \vec{p}', t) = \gamma_5 S^\dagger_\Gamma(\vec{v}, t; \vec{0}, 0; \vec{p}') \delta_{v', t}
\]

Fermion matrix

• so \( \Sigma_\Gamma(\vec{0}, 0; \vec{x}_1; \vec{p}', t) \) is a sequential propagator based on a source \( S_\Gamma(\vec{x}_2, t; \vec{0}, 0; \vec{p}') \) constructed from two ordinary propagators at time \( t \)
Sequential Source Technique

- First compute ordinary propagators $G(x, 0)$
Sequential Source Technique

- Construct sources

\[ S_{\Gamma}^{u; a'} a(\vec{x}_2, t; \vec{0}, 0; \vec{p}') \] or \[ S_{\Gamma}^{d; a'} a(\vec{x}_2, t; \vec{0}, 0; \vec{p}') \]
Sequential Source Technique

- Compute sequential propagators

\[
\Sigma_{\Gamma}(\vec{0}, 0; \vec{x}_1; \vec{p}', t) = \sum_{\vec{x}_2} S_{\Gamma}(\vec{x}_2, t; \vec{0}, 0; \vec{p}') S(\vec{x}_2, t; \vec{x}_1)
\]

- via the second inversion

\[
\sum_v M(v', v) \gamma_5 \Sigma^\dagger_{\Gamma}(\vec{0}, 0; v; \vec{p}', t) = \gamma_5 S^\dagger_{\Gamma}(\vec{v}, t; \vec{0}, 0; \vec{p}') \delta_{v_0, t}
\]
Sequential Source Technique

\[ \Sigma_{\Gamma}(\vec{0}, 0; \vec{p}', t) \mathcal{O}(\vec{x}_1, \tau) \]
Sequential Source Technique

- Tie everything together with an ordinary propagator

\[ C_{\Gamma}(t, \tau; \vec{p}', \vec{p}) = \sum_{\vec{x}_1} e^{i\vec{q} \cdot \vec{x}_1} \left\langle \text{Tr} \left[ \Sigma_{\Gamma}(\vec{0}, 0; \vec{p}', t) \mathcal{O}(\vec{x}_1, \tau) G(\vec{x}_1, 0) \right] \right\rangle \{U\} \]
Sequential Source Technique

• **Advantages:** Free choice of
  - Momentum transfer
  - Operator (vector/axial/tensor)
  - Ideal for Form Factors, Structure Functions, GPDs

• **Disadvantages:** Separate 3-pt inversion for each
  - Quark flavour
  - Hadron eg. p, Σ, Δ, π, N → γΔ
  - Polarisation
  - Sink momentum
Sequential Source Technique

- Alternative method involves computing a sequential propagator “through the operator”
Sequential Source Technique

**Advantages:** Free choice of

- Quark flavour
- Hadron e.g. $p, \Sigma, \Delta, \pi, N \to \gamma\Delta$
- Polarisation
- Sink momentum
- Ideal for studying flavour dependence in a hadron multiplet

**Disadvantages:** Separate 3-pt inversion for each

- Momentum transfer
- Operator (vector/axial/tensor)
\[
S^u; a' a (\vec{x}_2, t; \vec{0}, 0; \vec{p}') = e^{-i\vec{p}' \cdot \vec{x}} \epsilon^{abc} \epsilon^{a'b'c'} \times
\]

\[
\begin{align*}
& \left[ \tilde{G}^{d; \bar{b}b'} (\vec{x}_2, t; \vec{0}, 0) G^{u; cc'} (\vec{x}_2, t; \vec{0}, 0) \Gamma + \text{Tr}_D[\tilde{G}^{d; \bar{b}b'} (\vec{x}_2, t; \vec{0}, 0) G^{u; cc'} (\vec{x}_2, t; \vec{0}, 0)] \Gamma \\
& + \Gamma G^{u; \bar{b}b'} (\vec{x}_2, t; \vec{0}, 0) \tilde{G}^{d; cc'} (\vec{x}, t; \vec{0}, 0) + \text{Tr}_D[\Gamma G^{u; \bar{b}b'} (\vec{x}_2, t; \vec{0}, 0) \tilde{G}^{d; cc'} (\vec{x}_2, t; \vec{0}, 0)] \right]
\end{align*}
\]

/* \"bar u 0 u" insertion in NR proton, ie. 
*(u Cg5 d) u */
/* Some generic T */

// Use precomputed Cg5
q1_tmp = quark_propagators[0] * Cg5;
q2_tmp = Cg5 * quark_propagators[1];
di_quark = quarkContract24(q1_tmp, q2_tmp);

// First term
src_prop_tmp = T * di_quark;

// Now the second term
src_prop_tmp += traceSpin(di_quark) * T;

// The third term...
q1_tmp = q2_tmp * Cg5;
q2_tmp = quark_propagators[0] * T;

src_prop_tmp -= quarkContract13(q1_tmp, q2_tmp) + transposeSpin(quarkContract12(q2_tmp, q1_tmp));

END_CODE();

return projectBaryon(src_prop_tmp, forward_headers);

Lattice 3pt Functions in Chroma

\[
S^u; a' a (\vec{x}_2, t; \vec{0}, 0; \vec{p}') = e^{-i\vec{p}' \cdot \vec{x}} \epsilon^{abc} \epsilon^{a'b'c'} \times
\]

\[
\begin{align*}
& \left[ \tilde{G}^{d; \bar{b}b'} (\vec{x}_2, t; \vec{0}, 0) G^{u; cc'} (\vec{x}_2, t; \vec{0}, 0) \Gamma + \text{Tr}_D[\tilde{G}^{d; \bar{b}b'} (\vec{x}_2, t; \vec{0}, 0) G^{u; cc'} (\vec{x}_2, t; \vec{0}, 0)] \Gamma \\
& + \Gamma G^{u; \bar{b}b'} (\vec{x}_2, t; \vec{0}, 0) \tilde{G}^{d; cc'} (\vec{x}, t; \vec{0}, 0) + \text{Tr}_D[\Gamma G^{u; \bar{b}b'} (\vec{x}_2, t; \vec{0}, 0) \tilde{G}^{d; cc'} (\vec{x}_2, t; \vec{0}, 0)] \right]
\end{align*}
\]
success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("NUCL-U"), barNuclNuclU);
success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("NUCL-D"), barNuclNuclD);
success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("NUCL-U UNPOL"), barNuclUUnpol);
success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("NUCL-D UNPOL"), barNuclDUnpol);
success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("NUCL-U POL"), barNuclUPol);
success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("NUCL-D POL"), barNuclDPol);
success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("NUCL-U UNPOL_NONREL"), barNuclUUnpolNR);
success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("NUCL-D UNPOL_NONREL"), barNuclDUnpolNR);
success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("NUCL-U POL_NONREL"), barNuclUPolNR);
success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("NUCL-D POL_NONREL"), barNuclDPolNR);
success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("NUCL-U_MIXED_NONREL"), barNuclUMixedNR);
success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("NUCL-D_MIXED_NONREL"), barNuclDMixedNR);
success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("NUCL-U_MIXED_NONREL_NEGPAR"), barNuclUMixedNRnegPar);
success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("NUCL-D_MIXED_NONREL_NEGPAR"), barNuclDMixedNRnegPar);
success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("XI_D_MIXED_NONREL"), barXiDMixedNR);
success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("DELTA-DELTA_U"), barDeltaDeltaU);
success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("DELTA-DELTA_D"), barDeltaDeltaD);
success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("XI_D_MIXED_NONREL_NEGPAR"), barXiDMixedNRnegPar);
success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("XI_D_MIXED_NONREL_NEGPAR"), barXiDMixedNRnegPar);
success &= Chroma::TheWilsonHadronSeqSourceFactory::Instance().registerObject(string("XI_D_MIXED_NONREL_NEGPAR"), barXiDMixedNRnegPar);
Chroma xml for Sequential Source

\[
\begin{align*}
\text{\textless elem\textgreater}
\text{\textless annotation\textgreater}; \text{NUCL\textunderscore U\textunderscore UNPOL seqsource</annotation>}
\text{\textless Name\textgreater} SEQSOURCE</Name>
\text{\textless Frequency\textgreater} 1</Frequency>
\text{\textless Param\textgreater}
\text{\textless version\textgreater} 1</version>
\text{\textless seq_src\textgreater} NUCL\textunderscore U\textunderscore UNPOL</seq_src>
\text{\textless t_sink\textgreater} 13</t_sink>
\text{\textless sink\_mom\textgreater} 0 0 0</sink\_mom>
\text{\textless Param\textgreater}
\text{\textless version\textgreater} 2</version>
\text{\textless Sink\textgreater}
\text{\textless version\textgreater} 2</version>
\text{\textless SinkType\textgreater} SHELL\_SINK</SinkType>
\text{\textless j\_decay\textgreater} 3</j\_decay>
\text{\textless SmearingParam\textgreater}
\text{\textless wvf\_kind\textgreater} GAUGE\_INV\_GAUSSIAN</wvf\_kind>
\text{\textless wvf\_param\textgreater} 2.0</wvf\_param>
\text{\textless wvf\_IntPar\textgreater} 5</wvf\_IntPar>
\text{\textless no\_smear\_dir\textgreater} 3</no\_smear\_dir>
\text{\textless SmearingParam\textgreater}
\text{\textless Sink\textgreater}
\text{\textless PropSink\textgreater}
\text{\textless version\textgreater} 5</version>
\text{\textless Sink\textgreater}
\text{\textless SinkType\textgreater} SHELL\_SINK</SinkType>
\text{\textless j\_decay\textgreater} 3</j\_decay>
\text{\textless SmearingParam\textgreater}
\text{\textless wvf\_kind\textgreater} GAUGE\_INV\_GAUSSIAN</wvf\_kind>
\text{\textless wvf\_param\textgreater} 2.0</wvf\_param>
\text{\textless wvf\_IntPar\textgreater} 5</wvf\_IntPar>
\text{\textless no\_smear\_dir\textgreater} 3</no\_smear\_dir>
\text{\textless SmearingParam\textgreater}
\text{\textless Sink\textgreater}
\text{\textless PropSink\textgreater}
\text{\textless version\textgreater} 5</version>
\text{\textless NamedObject\textgreater}
\text{\textless gauge\_id\textgreater} gauge</gauge\_id>
\text{\textless prop\_ids\textgreater}
\text{\textless elem\textgreater} sh\_prop\_1</elem>
\text{\textless elem\textgreater} sh\_prop\_1</elem>
\text{\textless NamedObject\textgreater}
\end{align*}
\]
Exercise

• Using the following interpolating operator

\[ \chi^{\Delta^+}(x) = \frac{1}{\sqrt{3}} \epsilon^{abc} \left[ 2(u^T(x)C\gamma+ d^b(x))u^c(x) + (u^T(x)C\gamma+ u^b(x))d^c(x) \right] \]

perform the appropriate Wick contractions and write down the \( \Delta^+ \) 3pt function and compare your result to the source implemented in Chroma

• Work out the sequential sources required for \( \gamma^N \rightarrow \Delta \)

\[ \langle \Omega | \Delta^+(x_2) j^\mu(x_1) N(0) | \Omega \rangle \]
Extracting matrix elements

- Recall hadronic form of the nucleon 3pt function

\[
G(t, \tau, \vec{p}, \vec{p}') = \sum_{s,s'} e^{-E_{\vec{p}'}(t-\tau)} e^{-E_{\vec{p}}} \Gamma_{\beta \alpha} \langle \Omega | \chi_{\alpha}(0) | N(p', s') \rangle \langle N(p', s') | \mathcal{O}(\vec{q}) | N(p, s) \rangle \langle N(p, s) | \chi_{\beta}(0) | \Omega \rangle
\]

- Need to remove time dependence and wave function amplitudes

\[
\text{Form a ratios with the nucleon 2pt function}
\]

\[
G_2(t, \vec{p}) = \sum_s e^{-E_p t} \Gamma_{\beta \alpha} \langle \Omega | \chi_{\alpha} | N(p, s) \rangle \langle N(p, s) | \chi_{\beta} | \Omega \rangle
\]

- E.g.

\[
R(t, \tau; \vec{p}', \vec{p}; \mathcal{O}) = \frac{G(\Gamma)(t, \tau; \vec{p}', \vec{p}, \mathcal{O})}{G_2(t, \vec{p}')} \left[ \frac{G_2(t, \vec{p}') G_2(t, \vec{p}) G_2(t - \tau, \vec{p})}{G_2(\tau, \vec{p}) G_2(t, \vec{p}) G_2(t - \tau, \vec{p}')} \right]^\frac{1}{2}
\]
Extracting matrix elements

• Using the relation for spinors
  \[ \bar{u}(\vec{p}, \sigma') \Gamma u(\vec{p}, \sigma) = \text{Tr} \Gamma (E_4 - i \vec{p} \cdot \vec{\gamma} + m) \frac{1}{2} \left( 1 - \gamma_5 \gamma_4 \frac{\vec{p} \cdot \vec{s}}{EM} + i \gamma_5 \frac{\vec{\gamma} \cdot \vec{s}}{m} \right) \delta_{\sigma \sigma'} \]

• We can write the two point function as
  \[ G_2(t, \vec{p}) = \sum_s \sqrt{Z_{\text{snk}}(\vec{p})} \sqrt{Z_{\text{src}}(\vec{p})} \frac{1}{2E_\vec{p}} \text{Tr} \bar{u}(\vec{p}, s) \Gamma u(\vec{p}, s) \left[ e^{-E_p t} + e^{-E_p'(T-t)} \right] \]
  + \nu\text{-spinor terms with opposite parity}

• Use \( \Gamma_4 = \frac{1}{2} (1 + \gamma_4) \) to maximise overlap with positive parity forward propagating state
  \[ G_2(t, \vec{p}) = \sqrt{Z_{\text{snk}}(\vec{p})} Z_{\text{src}}(\vec{p}) \left[ \left( \frac{E_\vec{p} + m}{E_\vec{p}} \right) e^{-E_p t} + \left( \frac{E_p' + m'}{E_p'} \right) e^{-E_p'(T-t)} \right] \]

• Similarly for the three-point function
  \[ G_3(t, \tau; \vec{p}', \vec{p}; \Gamma, \mathcal{O}) = \sqrt{Z_{\text{snk}}(\vec{p}') Z_{\text{src}}(\vec{p})} F(\Gamma, \mathcal{F}) e^{-E_{\vec{p}'}(t-\tau)} e^{-E_{\vec{p}} \tau} \]

• where
  \[ F(\Gamma, \mathcal{J}) = \frac{1}{4} \text{Tr} \left( \gamma_4 - i \frac{\vec{p}'}{E_{\vec{p}'}} \cdot \vec{\gamma} + \frac{m}{E_{\vec{p}'}} \right) \mathcal{J} \left( \gamma_4 - i \frac{\vec{p} \cdot \vec{\gamma}}{E_{\vec{p}}} + \frac{m}{E_{\vec{p}}} \right) \]

• and
  \[ \langle N(p', s') | \mathcal{O}(\vec{q}) | N(p, s) \rangle = \bar{u}(p', s') \mathcal{J} u(p, s) \]
Example

- So our ratio determines

\[
R(t, \tau; \vec{p}', \vec{p}; \mathcal{O}) = \frac{G_1(t, \tau; \vec{p}', \vec{p}, \mathcal{O})}{G_2(t, \vec{p}')} \left[ \frac{G_2(\tau, \vec{p}')G_2(t, \vec{p}')G_2(t - \tau, \vec{p})}{G_2(\tau, \vec{p})G_2(t, \vec{p})G_2(t - \tau, \vec{p}')} \right]^{\frac{1}{2}}
\]

\[
= \sqrt{\frac{E_{\vec{p}'}E_{\vec{p}}}{(E_{\vec{p}} + m)(E_{\vec{p}} + m)}} F(\Gamma, \mathcal{J}_\mathcal{O}(\vec{q})) \quad 0 \ll \tau \ll t \ll \frac{1}{2} T
\]

\[
= F_1(q^2 = 0)
\]

\[
\Gamma_{\text{unpol}} = \frac{1}{2}(1 + \gamma_4), \quad \mathcal{O} = V_4 \equiv \gamma_4, \quad \vec{p}' = \vec{p} = 0
\]
Other Useful Combinations

Exercise: Prove them!

\[ R(t, \tau; \vec{0}, \vec{p}; V_4, \Gamma_4) = F_1(q^2) - \frac{E\vec{p} - M}{2M} F_2(q^2) = G_E(q^2) \]

\[ R(t, \tau; \vec{0}, \vec{p}; V_i, \Gamma_4) = -i \frac{q_i}{E + M} G_E(q^2) \]

\[ R(t, \tau; \vec{0}, \vec{p}; V_i, \Gamma_j) = -i \epsilon_{ijk} \frac{q_k}{E + M} G_M(q^2) \]

\[ \Gamma_j = \frac{1}{2} (1 + \gamma_4) i \gamma_5 \gamma_j \]

- Certain combinations of parameters and kinematics give access to the form factors

- It is possible to have several choices giving access to the form factors at a fixed \( Q^2 \)

Overdetermined set of simultaneous equations that can be solved for \( F_1, F_2 \) or \( G_E, G_M \)
Typical Examples

More detailed look at lattice results for form factors tomorrow

\begin{align*}
F_{1}^{u-d}(Q^2) \\
F_{2}^{u-d}(Q^2) \\
F_{\pi}(Q^2)
\end{align*}

QCDSF: 1106.3580
hep-lat/0608021
Some Recent Works

[Not an exhaustive list]

Nucleon

• Review: Ph. Hägler, 0912.5483

• QCDSF: 1106.3580

• ETMC: 1102.2208

• LHPC: 1001.3620

• RBC/UKQCD: 0904.2039

• CSSM: hep-lat/0604022

Pion

• Mainz: 1109.0196

• PACS-CS: 1102.3652

• JLQCD/TWQCD: 0905.2465

• ETMC: 0812.4042

• RBC/UKQCD: 0804.3971

• QCDSF: hep-lat/0608021