Lecture 2: two-body

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Lecture content

- Two particles in finite volume: Lüscher method
- Scattering
- Resonances
- Two-particle bound states
Theory
Maiani-Testa no-go theorem

- Scattering and decays are real-time processes

- How can Euclidean space (imaginary time) calculations address generic Minkowski space correlations?

- Maiani & Testa [91]: Euclidean correlators with initial/final states at kinematic thresholds allow access to physical information (matrix elements, weak decays)

- In infinite volume away from kinematic thresholds, scattering continuum masks the physically interesting information

- Example: $K \rightarrow \pi \pi$ weak decay

- Consider

$$C(t_1, t_2) = \langle \mathcal{O}_K(t_1) \mathcal{O}_{\text{weak}}(0) \mathcal{O}_{\pi \pi}(t_2) \rangle$$

Take large $|t_1, t_2|$ to get single state

$$\Rightarrow \langle K | \mathcal{O}_{\text{weak}} | \pi(\hat{p}) \pi(-\hat{p}) \rangle + \ldots$$

[see also Michael 89]
Maiani-Testa no-go theorem

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  \[
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  \]
  Take large \(|t_1, t_2|\) to get single state
  \[
  \not\to \langle K| O_{\text{weak}}|\pi(p)\pi(-p)\rangle + \ldots
  \]

[see also Michael 89]
Maiani-Testa no-go theorem

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- How can Euclidean space (imaginary time) calculations address generic Minkowski space correlations?
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- Example: $K \rightarrow \pi \pi$ weak decay

\[ C(t_1, t_2) = \langle O_K(t_1) O_{weak}(0) O_{\pi\pi}(t_2) \rangle \]

Take large $|t_1, t_2|$ to get single state

$\rightarrow \langle K | O_{weak} | \pi(\hat{p}) \pi(-\hat{p}) \rangle + \ldots$

[see also Michael 89]
Two particles in a box

- Long realised that forcing particles to be in a finite volume shifts their energy in a way that depends on their interactions

- Uhlenbeck 1930’s; Bogoliubov 1940’s; Lee, Huang, Yang 1950’s, ...

- Lüscher (1986,1991) demonstrated that this is also true in QFT up to inelastic thresholds (see also Hamber, Marinari, Parisi & Rebbi)

- Energy eigenvalues of discrete scattering states well defined – no issue in Minkowski-Euclidean connection

- Bypasses Maiani-Testa NGT
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Two particles in a box

\[ p = \frac{2\pi}{L} n, \quad n_i \in \mathbb{Z} \]

\[ E_n(L) = 2 \sqrt{m^2 + \frac{4\pi^2}{L^2} |n|^2} \]
Two particles in a box

- Consider first the non-interacting two particle system with particles of mass $m$ in a box of dimensions $L^3$ with zero CoM momentum.

- Particles constrained to have momenta

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- Modified by interactions in a calculable way
Consider simple effective field theory of a scalar particle interacting via contact interactions

\[ \mathcal{L} = \partial \phi \partial \phi + M\phi^2 + C_0 \phi^4 + C_2 \phi D^2 \phi^2 + \ldots \]

Scattering amplitude given by bubble sum

\[ \sum_i C_{2i} p^{2i} \]

In infinite volume

\[ \mathcal{A} \equiv \frac{4\pi/M}{p \cot \delta(p) - ip} = \frac{\sum_n C_{2n} p^{2n}}{1 - I_0(p) \sum_n C_{2n} p^{2n}} \]

Making non-relativistic approx (doing relativistically is not much harder) and using power divergent subtraction regularisation scheme [Kaplan, Savage, Wise]

\[ I_0^{PDS}(p) = \mu^{4-d} \int \frac{d^{d-1}k}{E - |k|^2/M + i \epsilon} = -\frac{M}{4\pi} (\mu + ip) \]
In finite volume, integral restricts to allowed mode sum

\[ \mathcal{A}(L) = \frac{\sum_n C_{2n} p^{2n}}{1 - I_0^L(p) \sum_n C_{2n} p^{2n}} \quad p \cot \delta(p) = \frac{4\pi}{M} \frac{1}{\sum_n C_{2n} p^{2n}} + \mu \]

\[ 0 = \mathcal{A}^{-1}(L) = p \cot \delta(p) - \frac{M\mu}{4\pi} - I_0^L(p) \]

Define PDS regulated sum as

\[ I_0^L(p) \equiv \frac{1}{L^3} \sum_{\text{PDS}} \frac{1}{E - |k|^2/M} \]

\[ = \frac{1}{L^3} \sum_{k < \Lambda} \frac{1}{E - |k|^2/M} + \int_{\Lambda} \frac{d^3k/(2\pi)^3}{|k|^2/M} - \int_{\text{PDS}} \frac{d^3k/(2\pi)^3}{|k|^2/M} \]

\[ = \frac{M}{4\pi} \left[ -\frac{1}{\pi L} \sum_{n \Lambda} \frac{1}{|n|^2 - \frac{L^2 EM}{4\pi^2} - \frac{4\Lambda}{L} - \mu} \right] \]
Finite volume energies

- Energies satisfy eigenvalue equation (Lüscher’s method)

\[ p \cot \delta(p) - \frac{1}{\pi L} S \left( \frac{L^2 p^2}{4\pi^2} \right) = 0 \]

where

\[ S(x) = \sum_{n}^{\Lambda} \frac{1}{|n|^2 - x} - 4\pi\Lambda \quad \text{[3D zeta function]} \]

- Result valid for momenta up to inelastic threshold

- Valid up to exponentially small corrections

- Eg: lowest energy level (zero rel. mom.)

\[ \Delta E_0 = \frac{4\pi a}{ML^3} \left[ 1 + \hat{c}_1 \frac{a}{L} + \hat{c}_2 \left( \frac{a}{L} \right)^2 + \ldots \right] \]

- Calculation of energy levels on the lattice determines scattering parameters

HW: 1. derive the energy shift \( \Delta E_0 \) from the Lüscher formula above assuming small \( a/L \).

2. Calculate the coefficient \( c_1 \).
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known Lüscher coefficients

- Calculation of energy levels on the lattice determines scattering parameters

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Two particle energies

Intersections correspond to eigen-energies of states in lattice volume
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Asymptotic expansions

- Ground state energy shift
  \[ \Delta E_0 = \frac{4\pi a}{ML^3} \left[ 1 + c_1 \frac{a}{L} + c_2 \left( \frac{a}{L} \right)^2 + \ldots \right] \]

- First excited state energy shift
  \[ \Delta E_1 = \frac{4\pi}{ML^2} - \frac{12}{ML^2} \tan \delta_0 \left[ 1 + c'_1 \tan \delta_0 + c'_2 \tan^2 \delta_0 \right] + \ldots \text{ where } \delta_0 = \delta(p_{E_1}) \]

- Each new level extracted or new volume used adds information on the phase shift at a different energy

- Bound states can also be described
  \[ \Delta E_{-1} = -\frac{\gamma^2}{M} \left[ 1 + \frac{12}{\gamma_0 L(1 - \gamma r_3)} e^{-\gamma L} \right] + \ldots \]

- Expansions also for L/a << 1 [Beane et al.]
Bound states at finite volume

• Lüscher eigenvalue equation also includes solutions with $p^2 < 0$ (bound state?)

• Two particle scattering amplitude in infinite volume

$$A \equiv \frac{4\pi/M}{p \cot \delta(p) - i \, p}$$

bound state at $p^2 = -\gamma^2$ when $\cot \delta(i\gamma) = i$

• Binding energy $E_B$ related to binding momentum as $\gamma = \sqrt{2ME_B}$

• Scattering amplitude in finite volume (another way of expanding Lüscher eqn)

$$\cot \delta(i\kappa) = i - i \sum_{\tilde{m} \neq 0} \frac{e^{-|\tilde{m}| \kappa L}}{|\tilde{m}| \kappa L}$$

$\kappa \xrightarrow{L \to \infty} \gamma$

• Multiple volumes required to show a negatively shifted state is bound
Boosted systems

- Boost of the two body system CoM relative to boundary conditions (lab) changes the effective shape of the box as seen by the interacting system

- First studied by Rummukainen & Gottlieb [95]; Further study by Kim, Sachrajda & Sharpe [05]; Kim Christ & Sachrajda [05];

- Generalised to other frames [Feng, Renner & Jansen]

- Allows access to phase shift at different momenta

- Effects on bound states investigated [Bour et al; Davoudi & Savage]

- Can be used to cancel leading exponential FV corrections to binding energies
Asymmetric boxes

Asymmetry box of geometry $\eta_1 L \times \eta_2 L \times L$

Eigenvalue equation modified

$$S(\tilde{p}^2) \rightarrow S(\tilde{p}^2, \eta_1, \eta_2) = \frac{1}{\eta_1 \eta_2} \sum_{\tilde{n}} \frac{1}{|\tilde{n}|^2 - \tilde{p}^2 - 4\pi \Lambda_n}$$

where $\tilde{n} = \left( \frac{n_1}{\eta_1}, \frac{n_2}{\eta_2}, n_3 \right)$

Asymptotic expansion

$$\Delta E_0 = \frac{4\pi a}{\eta_1 \eta_2 M L^3} \left[ 1 + c_1(\eta_1, 2) \frac{a}{L} + c_2(\eta_1, 2) \left( \frac{a}{L} \right)^2 + \ldots \right]$$

Geometric coefficients

$$c_1(\eta_1, 2) = \frac{1}{\pi} \left( \frac{1}{\eta_1 \eta_2} \sum_{\tilde{n} \neq 0} \frac{1}{|\tilde{n}|^2 - 4\pi \Lambda_n} \right)$$

Asymmetries exist where sub-leading FV effects are suppressed: $c_i(\eta_1, \eta_2) = 0$
Resonances at finite volume

- Pion is light so very few stable hadrons in the real world – often the lightest particle of a given set of conserved quantum numbers and not much else
  - Ex: $\rho(770)$ decays to $\pi\pi$ and to $\pi\pi\pi\pi$; $\Delta(1232)$ decays to $N\pi$
- Ex: $I=1$ $\pi\pi$ phase shift

- Extensive experimental efforts to understand excited spectrum of hadrons
- At finite volume, spectrum is discrete: how do resonances manifest?
- Spectrum gets large modifications as a function of volume near resonance energy – embodies in solutions of Lüscher eigenvalue equation
- See excellent lectures of Jo Dudek at HUGS 2012 for details [http://www.jlab.org/hugs/program.html]
Resonances at finite volume

- Consider simple spin model in 3+1 D [Rummukainen & Gottlieb hep-lat/9509088]

\[ S = -\kappa_{\phi} \sum_{x;\mu} \phi_x \phi_{x+\mu} - \kappa_{\rho} \sum_{x;\mu} \rho_x \rho_{x+\mu} + g \sum_{x;\mu} \rho_x \phi_x \phi_{x+\mu} \]

- Left hand case \( g=0 \) (no resonance); right hand case \( g=0.021 \) (\( \rho \) appears as resonance in \( \phi\phi \) channel)

Monte Carlo simulations

Field variables \( \phi \) and \( \rho \) are restricted to \( \{-1, 1\} \). Choosing the coupling constants suitably \( \rho \) appears as a resonance in \( s\)-wave \( \phi\phi \) scattering. We used 3 sets of couplings (A, B, C), given in the table below, together with the final results:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \kappa_{\phi} )</td>
<td>0.0742</td>
<td>0.07325</td>
</tr>
<tr>
<td>( \kappa_{\rho} )</td>
<td>0.0708</td>
<td>0.0718</td>
</tr>
<tr>
<td>( g )</td>
<td>0</td>
<td>0.008</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th># of lattice sizes</th>
<th>9</th>
<th>10</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_{\phi} )</td>
<td>0.1856(4)</td>
<td>0.1996(5)</td>
<td>0.3081(4)</td>
</tr>
<tr>
<td>( m_{\rho} )</td>
<td>0.5049(5)</td>
<td>0.5306(13)</td>
<td>0.8206(11)</td>
</tr>
<tr>
<td>( \Gamma )</td>
<td>0</td>
<td>0.0044(2)</td>
<td>0.0178(7)</td>
</tr>
<tr>
<td>( g_{R} )</td>
<td>0</td>
<td>0.598(14)</td>
<td>1.49(3)</td>
</tr>
<tr>
<td>( \lambda_{R} )</td>
<td>28.1(1.1)</td>
<td>36.8(1.3)</td>
<td>48.3(2.0)</td>
</tr>
</tbody>
</table>

In case A \( g=0 \) and \( \rho \) is stable. In cases B and C \( g \) increases causing an increase in the resonance width \( \Gamma \). \( g_{R} \) and \( \lambda_{R} \) are renormalized \( \phi\phi\rho \) - and \( \phi^4 \) couplings, respectively.

Gattringer and Lang used similar action to study resonance scattering in 1+1 dimensions [Nucl.Phys.B 391(1993)].
Inelastic scattering

- In most physical scattering processes, inelastic contributions are important. Ex: $I=0 \, \pi \pi$
- Derivation of Lüscher method breaks down at inelastic thresholds.
- Various attempts to get around this:
  - Treat the system purely quantum mechanically [He, Liu et al.]
  - Effective field theory at finite volume [Bernard et al.; Döring et al.; Briceno&Davoudi; Hansen&Sharpe]
  - Introduces some level of systematic uncertainty (high order effects, ...)
- Active area of research
Bethe-Salpeter wave functions

- An alternate way of learning about scattering is based on determination of (Nambu-)Bethe-Salpeter wavefunction [Lüscher; Lin et al.; Aoki et al.; HALQCD]

\[
\psi_k(x - y) = \langle 0 | T[\phi(x)\phi(y)]|\phi(k)\phi(-k)\rangle_{\text{in}}
\]
chosen interpolating operator probes content of state

- Satisfies Schrödinger equation for non-local BS kernel \(U(x,y)\)

\[
\frac{1}{2\mu} \left[ \nabla^2 + |k|^2 \right] \psi_k(x) = \int d^3y U(x,y) \psi_k(y) \tag{\*}
\]

- Provided \(U(x,y)=0\) for large \(|x-y|\) asymptotic behaviour of partial waves given by

\[
\psi^\ell_k(x) \rightarrow A_{\ell} \frac{\sin(|k|x) - \ell \pi/2 + \delta_{\ell}(k)}{|k|x}
\]

\(\ell=0,1,\ldots\)
can be used to determine phase shift

- HALQCD method: invert (\*) to determine a potential by approximating

\[
U(x,y) = V(x,\nabla)\delta^3(x-y) = \delta^3(x-y) \left[V(x) + \mathcal{O}(\nabla^2)\right]
\]
giving

\[
V(x) = \frac{1}{2\mu} \left[ \nabla^2 + |k|^2 \right] \frac{\psi_k(x)}{\psi_k(|bf\cdot x|)}
\]

**Convergence of velocity expansion**

If the higher order terms are large, LO potentials determined from NBS wave functions at different energy become different. (cf. LOC of ChPT).

Numerical check in quenched QCD

\[m_\pi \approx 0.53\text{ GeV}\]

K. Murano, N. Ishii, S. Aoki, T. Hatsuda

PTP 125 (2011)1225.

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PTP 125 (2011)1225.
Bethe-Salpeter wave functions

- BS wavefunctions can be determined from LQCD correlation functions

\[
C(r, t) = \langle 0 | T[\phi(x + r, t)\phi(x, t)]\mathcal{J}^\dagger(0)|0\rangle \\
= \sum_n \langle 0 | T[\phi(x + r, t)\phi(x, t)]|\phi\phi^{(n)}\rangle \langle \phi\phi^{(n)}|\mathcal{J}^\dagger(0)|0\rangle \\
= \sum_n Z_n e^{-E_n t} \psi_{k_n}(r) \\
\xrightarrow{t \to \infty} Z_0 e^{-E_0 t} \psi_{k_0}(r)
\]

- Various extensions considered – see recent review [Aoki et al. 1206.5088]

- Potentials obtained from this method are:
  - Energy dependent (weakly in some cases – see [Murano et al, 1103.0619])
  - Only guaranteed to reproduce phase shift at \( E_0 \)
  - Sink dependent – not an issue as observable is phase shift at measured energy
  - Extraction of phase shift from lattice potential introduces model dependence as a functional form must be fit to finite lattice data – probably a mild problem

Eigen-energies contain scattering information
Numerical Investigations
Two body scattering studies

- **Meson-meson**
  - \(\pi - \pi\) (I=2, 1, 0): [CP-PACS; NPLQCD; Feng et al; HadSpec; Fu; ... many others]
  - \(\pi-K\) (I=1/2, 3/2): [NPLQCD; Z Fu; Nagata et al; PACS-CS; Lang et al.]
  - \(K-K\) (I=1): [NPLQCD; Z Fu]

- **Meson-baryon**
  - Five simple octet baryon – octet meson channels studied [NPLQCD]
  - \(J/\psi\)-nucleon [Liu et al; Kawanai & Sasaki]

- **Baryon-baryon**
  - Various octet baryon – octet baryon scattering [HALQCD, NPLQCD]
  - Omega (sss)–Omega scattering [Buchoff, Luu & Wasem]

- A rapidly growing field
Example: \( I=2 \, \pi\pi \)

- \( I=2 \, \pi-\pi \) "easy" as no disconnected contractions
- Measure multiple energy levels of two pions in a box for multiple volumes and with multiple \( P_{CM} \)

![Graph showing energy levels and wave functions for \( I=2 \) pions](image-url)

- \( L / b_s = 32 \), \( P_{cm} = 0 \)

@ \( m_\pi = 390 \) MeV
Example: $l=2 \pi\pi$

- $l=2 \pi-\pi$ “easy” as no disconnected contractions
- Measure multiple energy levels of two pions in a box for multiple volumes and with multiple $P_{cm}$

\[ P_{cm}=0, n=1 \]
\[ P_{cm}=1, n=0 \]
\[ P_{cm}=\sqrt{2}, n=0 / P_{cm}=0, n=1 \]
\[ P_{cm}=1, n=0 \]
\[ P_{cm}=0, n=0 \]

Dashed lines are non-interacting energy levels

$b_t, E$

$L / b_t = 16\ 20\ 24\ 32$

@ $m_\pi = 390$ MeV
Example: \( l = 2 \) \( \pi \pi \)

- Input into Lüscher eigenvalue equation
- Allows phase shift to be extracted at multiple energies
Combine with chiral perturbation theory (low-momentum interactions turn off in the chiral limit) to interpolate to physical pion mass.
Meson-baryon scattering

- Also constrained in the chiral limit
- Study of channels with no annihilation using DWF on MILC lattices

\[ m_\pi = 350 \text{ MeV} \]

\[ \Delta E / m_\pi \]

\[ \cot \delta / m_\pi \]

\[ \tilde{a}_{\pi^+ \Sigma^+}, SU(2) \]

\[ \tilde{a}_{\pi^+ \Xi^0}, SU(2) \]

[Torok et al. 0907.1913]
Studies by L. Liu; Kawanai & Sasaki

Interactions are purely gluon/sea quark effects

Phenomenological studies suggest $J/\Psi$ might bind in a nucleus through attraction of multiple nucleons

Interactions are small

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Interactions are small
NN phase shifts

- Potential extracted for BS wave function method at $E \sim 0$, use to evaluate phase shift
- NB: energy dependence of potential neglected

\[ \delta(1S0) \]

\[ \bar{\delta}(3S1) \]

\[ m_{\pi} = 700 \text{ MeV} \quad m_{\pi} = 570 \text{ MeV} \quad m_{\pi} = 411 \text{ MeV} \]

[Stapp's convention is employed for the scattering phases and mixing parameter.]

[N Ishii, Chiral Dynamics 2012]
Resonance studies: rho

- Map out phase shift to identify a resonance
- Having multiple P\(_{\text{CM}}\) frames is crucial
- Phase shift can be well determined at heavy quark masses

- Properties of resonance requires modelling—eg fit with modified Breit-Wigner
- Much current work addressing best way to get the most information
Delta baryon studied by QCDSF-Bonn-Jülich collaboration [Meißner QNP12] – a more challenging case [see Schierholz talk tomorrow at INT program]

Multiple volumes at light quark masses
H-dibaryon

- First QCD bound state observed in LQCD

- Jaffe [1977]: chromo-magnetic interaction between quarks
  \[ \langle H_m \rangle \sim \frac{1}{4} N(N - 10) + \frac{1}{3} S(S + 1) + \frac{1}{2} C_c^2 + C_f^2 \]
  most attractive for spin, colour, flavour singlet

- H-dibaryon (uuddss) J=I=0, s=-2 most stable
  \[ \Psi_H = \frac{1}{\sqrt{8}} \left( \Lambda\Lambda + \sqrt{3} \Sigma\Sigma + 2 \Xi N \right) \]

- Bound in a many hadronic models

- Experimental searches
  - Emulsion expts, heavy-ion, stopped kaons
  - No conclusive evidence for or against

\[ \text{KEK-ps (2007)} \]
\[ K^- {}^{12}\text{C} \rightarrow K^+ \Lambda\Lambda X \]
H dibaryon in QCD

- Extract energy eigenstates from large Euclidean time behaviour of two-point correlators

\[
C_{\Lambda}(t) = \sum_x \langle 0 | \chi(\mathbf{x}, t) \chi(0) | 0 \rangle \xrightarrow{t \to \infty} Z_{\Lambda} e^{-M_{\Lambda} t}
\]

\[
C_{\Lambda\Lambda}(t) = \sum_x \langle 0 | \phi(\mathbf{x}, t) \phi(0) | 0 \rangle \xrightarrow{t \to \infty} Z_{\Lambda\Lambda} e^{-E_{\Lambda\Lambda} t}
\]

- Correlator ratio allows direct access to energy shift

\[
R(t) = \frac{C_{\Lambda\Lambda}(t)}{C_{\Lambda}(t)^2} \xrightarrow{t \to \infty} \tilde{Z} e^{-\Delta E_{\Lambda\Lambda} t}
\]
Simple extrapolations

- After volume extrapolation, H bound at unphysical quark masses
- Quark mass extrapolation is uncertain and unconstrained
  \[ B_{H}^{\text{quad}} = +11.5 \pm 2.8 \pm 6.0 \text{ MeV} \]
  \[ B_{H}^{\text{lin}} = +4.9 \pm 4.0 \pm 8.3 \text{ MeV} \]
- Other extrapolations, see [Shanahan, Thomas & Young PRL. 107 (2011) 092004, Haidenbauer & Meissner 1109.3590]
- Suggests H is weakly bound or just unbound
- More study required at light masses

* 230 MeV point preliminary (one volume)
Deuteron and Dineutron

- Deuteron, di-neutron also investigated
- NPLQCD
- PACS-CS
- More work needed at lighter masses
- Also s=-4 \Xi\Xi dibaryon is found to be bound

FIG. 9: Same as Fig. 3 for $^1S_0$ channel.

FIG. 10: $m_\pi^2$ dependence of $\Delta E(3S_1)$ [GeV].

FIG. 11: Same as Fig. 10 for $1S_0$ channel.
Deuteron and Dineutron

- Deuteron, di-neutron also investigated
- NPLQCD
- PACS-CS
- More work needed at lighter masses
- Also $s=-4$ $\Xi\Xi$ dibaryon is found to be bound