Collective Neutrino Oscillations as a many-body problem

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Supernova neutrinos

- \( M_{\text{progenitor}} \geq 8 M_\odot \Rightarrow \Delta E \sim 10^{59} \text{ MeV} \)
- 99% of this energy is carried away by neutrinos and antineutrinos with \( 10 \leq E_\nu \leq 30 \text{ MeV} \)
  \( \Rightarrow 10^{58} \) neutrinos!

A neutrino many-body system!

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Collective Neutrino Oscillations as a many-body problem
Balantekin and Fuller, arXiv:1303.3874 [nucl-th]
Neutrino many-body system

This is the only many-body system driven by the weak interactions:

Table: Many-body systems

<table>
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<tr>
<th>Nuclei</th>
<th>Strong</th>
<th>at most $\sim 250$ particles</th>
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<tr>
<td>Condensed matter</td>
<td>E&amp;M</td>
<td>at most $N_A$ particles</td>
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<td>Neutron Star</td>
<td>Gravity + Strong</td>
<td>?</td>
</tr>
<tr>
<td>$\nu$’s in SN</td>
<td>Weak</td>
<td>$\sim 10^{58}$ particles</td>
</tr>
</tbody>
</table>

Astrophysical extremes allow us to test physics that cannot be tested elsewhere!
Neutrino Mixing

Mass and Flavor States

\[ a_1(p) = \cos \theta \ a_e(p) - \sin \theta \ a_x(p) \]
\[ a_2(p) = \sin \theta \ a_e(p) + \cos \theta \ a_x(p) \]

Neutrino Flavor Isospin Operators

\[ \hat{J}_p^+ = a_e^\dagger(p) a_x(p), \quad \hat{J}_p^- = a_x^\dagger(p) a_e(p), \]
\[ \hat{J}_p^0 = \frac{1}{2} \left( a_e^\dagger(p) a_e(p) - a_x^\dagger(p) a_x(p) \right) \]
\[ [\hat{J}_p^+, \hat{J}_q^-] = 2\delta_{pq} \hat{J}_p^0, \quad [\hat{J}_p^0, \hat{J}_q^\pm] = \pm \delta_{pq} \hat{J}_p^\pm, \]
Neutrino Hamiltonian

**Vacuum Oscillation Term**

\[ \hat{H}_{\nu}^{(1)} = \sum_p \left( \frac{m_1^2}{2p} a_1^\dagger(p) a_1(p) + \frac{m_2^2}{2p} a_2^\dagger(p) a_2(p) \right) + \hat{I}(\ldots). \]

\[ \hat{H}_{\nu}^{(1)} = \sum_p \frac{\delta m^2}{2p} \hat{B} \cdot \vec{J}_p, \quad \hat{B} = (\sin 2\theta, 0, -\cos 2\theta) \]

**One-Body Hamiltonian including interactions with the electron background**

\[ \hat{H}_\nu = \sum_p \left( \frac{\delta m^2}{2p} \hat{B} \cdot \vec{J}_p - \sqrt{2} G_F N_e J_p^0 \right) \]
Neutrino Hamiltonian

\[ \hat{H}_{\nu\nu} = \frac{\sqrt{2} G_F}{V} \sum_{p,q} (1 - \cos \vartheta_{pq}) \vec{J}_p \cdot \vec{J}_q \]

Note: due to the \((1 - \cos \vartheta)\) term there is no interaction between neutrinos moving in the same direction.
The total neutrino Hamiltonian

\[ \hat{H}_{\text{total}} = H_{\nu} + H_{\nu\nu} = \left( \sum_{p} \frac{\delta m^2}{2p} \hat{\mathbf{B}} \cdot \mathbf{\hat{J}}_p - \sqrt{2} G_F N_e J_0^p \right) \]
\[ + \frac{\sqrt{2} G_F}{V} \sum_{p,q} \left( 1 - \cos \vartheta_{pq} \right) \mathbf{\hat{J}}_p \cdot \mathbf{\hat{J}}_q \]

Dasgupta, Duan, Fogli, Friedland, Fuller, Lisi, Lunardini, McKellar, Mirizzi, Qian, Pantaleone, Pastor, Pehlivan, Raffelt, Sawyer, Sigl, Smirnov, Balantekin, ...
Antineutrinos and three flavors

Including antineutrinos

\[ H = H_{\nu} + H_{\bar{\nu}} + H_{\nu\nu} + H_{\nu\bar{\nu}} + H_{\bar{\nu}\bar{\nu}} \]

Requires introduction of a second set of SU(2) algebras!

Including three flavors

Requires introduction of SU(3) algebras.

Both extensions are straightforward, but tedious!
Neutrino Hamiltonian

Neutrino Hamiltonian with $\nu - \nu$ interactions

$$\hat{H}_{\text{total}} = \sum_p \frac{\delta m^2}{2p} \vec{B} \cdot \vec{J}_p + \frac{\sqrt{2} G_F}{V} \sum_{p,q} \left(1 - \cos \vartheta_{pq}\right) \vec{J}_p \cdot \vec{J}_q$$

Single-angle approximation $\Rightarrow$

$$\hat{H}_{\text{total}} = \sum_p \frac{\delta m^2}{2p} \vec{B} \cdot \vec{J}_p + \frac{\sqrt{2} G_F}{V} \langle(1 - \cos \vartheta_{pq})\rangle \sum_{p \neq q} \vec{J}_p \cdot \vec{J}_q$$

Defining $\mu = \frac{\sqrt{2} G_F}{V} \langle(1 - \cos \vartheta_{pq})\rangle$, and $\omega_p = \frac{\delta m^2}{2p}$ one can write

$$\hat{H}_{\text{total}} = \sum_p \omega_p \vec{B} \cdot \vec{J}_p + \mu \sum_{p \neq q} \vec{J}_p \cdot \vec{J}_q$$
BCS Hamiltonian

Hamiltonian in Quasi-spin basis

\[ \hat{H}_{\text{BCS}} = \sum_k 2\epsilon_k \hat{t}_k^0 - |G| \hat{T}^+ \hat{T}. \]

Quasi-spin operators:

\[ \hat{t}_k^+ = c_{k\uparrow}^\dagger c_{k\downarrow}^\dagger, \quad \hat{t}_k^- = c_{k\downarrow} c_{k\uparrow}, \quad \hat{t}_k^0 = \frac{1}{2} \left( c_{k\uparrow}^\dagger c_{k\uparrow} + c_{k\downarrow}^\dagger c_{k\downarrow} - 1 \right) \]

\[ [\hat{t}_k^+, \hat{t}_l^-] = 2\delta_{kl} \hat{t}_k^0, \quad [\hat{t}_k^0, \hat{t}_l^\pm] = \pm \delta_{kl} \hat{t}_k^\pm. \]

Richardson gave a solution of this problem. Hence there exist invariants of motion.
### Gaudin algebra

\[
[S^+(\lambda), S^-(\mu)] = 2 \frac{S^0(\lambda) - S^0(\mu)}{\lambda - \mu}
\]

\[
[S^0(\lambda), S^{\pm}(\mu)] = \pm \frac{S^{\pm}(\lambda) - S^{\pm}(\mu)}{\lambda - \mu}
\]

\[
[S^0(\lambda), S^0(\mu)] = [S^{\pm}(\lambda), S^{\pm}(\mu)] = 0
\]

\(\lambda\) is an arbitrary complex parameter. The operators

\[
X(\lambda) = S^0(\lambda)S^0(\lambda) + \frac{1}{2}S^+(\lambda)S^-(\lambda) + \frac{1}{2}S^-(\lambda)S^+(\lambda)
\]

satisfy \([X(\lambda), X(\mu)] = 0, \ \lambda \neq \mu\).
Lowest weight vector is chosen to satisfy
\[ S^-(\lambda)|0\rangle = 0, \quad \text{and} \quad S^0(\lambda)|0\rangle = W(\lambda)|0\rangle, \]
\[ \Rightarrow X(\lambda)|0\rangle = \left[ W(\lambda)^2 - \frac{\partial W(\lambda)}{\partial \lambda} \right]|0\rangle. \]

Excited states are given by
\[ |\xi > \equiv |\xi_1, \xi_2, \ldots, \xi_n > \equiv S^+(\xi_1)S^+(\xi_2)\ldots S^+(\xi_n)|0 >. \]

The complex numbers \( \xi_1, \xi_2, \ldots, \xi_n \) satisfy the Bethe Ansatz equations:
\[ W(\xi_\alpha) = \sum_{\beta=1}^{n} \frac{1}{\xi_\alpha - \xi_\beta} \quad \text{for} \quad \alpha = 1, 2, \ldots, n. \]

Corresponding eigenvalue of \( X(\lambda) \) is
\[ E_n(\lambda) = \left[ W(\lambda)^2 - W'(\lambda) \right] - 2 \sum_{\alpha=1}^{n} \frac{W(\lambda) - W(\xi_\alpha)}{\lambda - \xi_\alpha}. \]
Neutrino representation of the Gaudin algebra

\[ S^0(\lambda) = A + \sum_k \frac{\hat{J}^0_k}{\omega_k - \lambda} \quad S^\pm(\lambda) = \sum_k \frac{\hat{J}^\pm_k}{\omega_k - \lambda} \]

\( \omega_k \) and \( A \) are arbitrary constants. We choose the mass basis for the operators, \( \omega_k = \delta m^2 / 2k \) and \( A = -1/2\mu. \)

\[ X(\lambda) = \sum_p \frac{\mathbf{J}_p^2}{(\omega_p - \lambda)^2} + A^2 + Y(\lambda) \]

\[ Y(\lambda) = \sum_{p,q,p \neq q} \frac{\mathbf{J}_p \cdot \mathbf{J}_q}{(\omega_p - \lambda)(\omega_q - \lambda)} + 2A \sum_p \frac{\mathbf{J}_p^0}{(\omega_p - \lambda)}. \]

\[ [X(\lambda), Y(\nu)] = 0 = [Y(\lambda), Y(\nu)] \]
\[
\frac{h_p}{\mu} \equiv \lim_{\lambda \to \omega_p} (\lambda - \omega_p) Y(\lambda) = 2 \sum_{q \neq p, q \neq p} \frac{J_p \cdot J_q}{\omega_p - \omega_q} + \frac{1}{\mu} J^0_p
\]

\[
\frac{H}{\mu} = \sum_p \omega_p \frac{h_p}{\mu} = 2 \sum_{q \neq p, q \neq p} J_p \cdot J_q + \frac{1}{\mu} \sum_p \omega_p J^0_p.
\]

\[
[H, h_p] = 0
\]

Above equations are written in the mass basis, but the
transformation to flavor basis is easy

\[
H_{\text{flavor}} = T H_{\text{mass}} T^{-1} \quad T = e^{\theta (a_1^\dagger a_2 - a_2^\dagger a_1)}
\]
The duality between $\nu - \nu$ and BCS Hamiltonians

The $\nu - \nu$ Hamiltonian

$$\hat{H} = \sum_p \frac{\delta m^2}{2p} \vec{B} \cdot \vec{J}_p + \frac{\sqrt{2} G_F}{V} \vec{J} \cdot \vec{J}$$

The BCS Hamiltonian

$$\hat{H}_{\text{BCS}} = \sum_k 2\epsilon_k \hat{t}_k^0 - |G| \hat{T}^+ \hat{T}$$

Same symmetries leading to Analogous (dual) dynamics!

Bethe ansatz equations

\[ \sum_p \frac{-j_p}{\omega_p - \xi_\alpha} = \frac{1}{2\mu} + \sum_{\beta=1}^{N} \frac{1}{\xi_\alpha - \xi_\beta}. \]
Adiabatic solution of the exact many-body problem

Pehlivan et al., AIP Conf. Proc. 1743, 040007 (2016)
When $[\hat{O}_1, \hat{O}_2] \sim 0$. Approximate the operator product as

$$\hat{O}_1 \hat{O}_2 \sim \hat{O}_1 \langle \hat{O}_2 \rangle + \langle \hat{O}_1 \rangle \hat{O}_2 - \langle \hat{O}_1 \rangle \langle \hat{O}_2 \rangle,$$

where the expectation values should be calculated with respect to a state $|\Psi\rangle$ which satisfies the condition $\langle \hat{O}_1 \hat{O}_2 \rangle = \langle \hat{O}_1 \rangle \langle \hat{O}_2 \rangle$. This reduces the two-body problem to a one-body problem:

$$a^\dagger a^\dagger a a \Rightarrow \langle a^\dagger a \rangle a^\dagger a + \langle a^\dagger a^\dagger \rangle a a + \text{h.c.}$$
Mean Field Approximations

\[ \hat{H} \sim \sum_p \omega_p \hat{B} \cdot \vec{J}_p + \vec{P} \cdot \vec{J} \]

Polarization vector: \( \vec{P}_{p,s} = 2 \mu \langle \vec{J}_{p,s} \rangle \). Use SU(2) coherent states for the expectation value.
Mean-neutrino field

Polarization vectors

\[ \hat{H} \sim \sum_p \omega_p \hat{B} \cdot \vec{J}_p + \vec{P} \cdot \vec{J} \]

\[ \vec{P}_{p,s} = 2\langle \vec{J}_{p,s} \rangle \]

Eqs. of motion:

\[ \frac{d}{d\tau} \vec{J}_p = -i[\vec{J}_p, \hat{H}^{RPA}] = (\omega_p \hat{B} + \vec{P}) \times \vec{J}_p \]

Mean Field Consistency requirement \( \Rightarrow \) \[ \frac{d}{d\tau} \vec{P}_p = (\omega_p \hat{B} + \vec{P}) \times \vec{P}_p \]

Invariants

\[ I_p = 2\langle \hat{h}_p \rangle = \hat{B} \cdot \vec{P}_p + \sum_{q(\neq p)} \frac{\vec{P}_p \cdot \vec{P}_q}{\omega_p - \omega_q} \Rightarrow \frac{d}{d\tau} I_p = 0 \]

Raffelt; Pehlivan, Balantekin, Kajino, Yoshida
Mean-neutrino field

Possible mean fields

Neutrino-Neutrino Interaction:

\[ \bar{\Psi}_{\nu L} \gamma^\mu \psi_{\nu L} \bar{\Psi}_{\nu L} \gamma_\mu \psi_{\nu L} \Rightarrow \bar{\Psi}_{\nu L} \gamma^\mu \psi_{\nu L} \langle \bar{\Psi}_{\nu L} \gamma_\mu \psi_{\nu L} \rangle + \cdots \]

Antineutrino-Antineutrino Interaction:

\[ \bar{\Psi}_{\bar{\nu} R} \gamma^\mu \psi_{\bar{\nu} R} \bar{\Psi}_{\bar{\nu} R} \gamma_\mu \psi_{\bar{\nu} R} \Rightarrow \bar{\Psi}_{\bar{\nu} R} \gamma^\mu \psi_{\bar{\nu} R} \langle \bar{\Psi}_{\bar{\nu} R} \gamma_\mu \psi_{\bar{\nu} R} \rangle + \cdots \]

Neutrino-Antineutrino Interaction:

\[ \bar{\Psi}_{\nu L} \gamma^\mu \psi_{\nu L} \bar{\Psi}_{\bar{\nu} R} \gamma_\mu \psi_{\bar{\nu} R} \Rightarrow \bar{\Psi}_{\nu L} \gamma^\mu \psi_{\nu L} \langle \bar{\Psi}_{\bar{\nu} R} \gamma_\mu \psi_{\bar{\nu} R} \rangle + \cdots \]

Balantekin and Pehlivan, JPG 34, 1783 (2007)
Mean-neutrino field

Possible mean fields

Neutrino-Antineutrino can also have an additional mean field:

\[ \overline{\psi}_L \gamma^\mu \psi_L \overline{\psi}_R \gamma_\mu \psi_R \Rightarrow \overline{\psi}_L \gamma^\mu \langle \psi_L \overline{\psi}_R \gamma_\mu \rangle \psi_R + \cdots \]

However note that

\[ \langle \psi_L \overline{\psi}_R \gamma_\mu \rangle \propto m_\nu \]

(negligible is the medium isotropic)

Fuller et al., Volpe
CP-violating phases in collective oscillations

Neutrinos: \[ T_{ij}(p, \vec{p}) = a_i^\dagger(\vec{p})a_j(\vec{p}) \]

Antineutrinos: \[ T_{ij}(-p, \vec{p}) = -b_j^\dagger(\vec{p})b_i(\vec{p}) \]

\[ H_{\nu\nu} = \frac{G_F}{\sqrt{2}} \sqrt{V} \sum_{i,j=1}^{3} \sum_{E, \vec{p}, E', \vec{p}'} \sum_{\alpha, \alpha'} \left( 1 - \cos \theta_{\vec{p}\vec{p}'} \right) T_{\alpha_i\alpha_j}(E, \vec{p}) T_{\alpha_j\alpha_i}(E', \vec{p}') \]

\[ H_\nu + H_{\nu\nu} = S_\tau^\dagger (H_\nu + H_{\nu\nu}) S_\tau \]

with \( \delta \neq 0 \)

\[ S_\tau = e^{-i\delta(T_{\tau\tau} + \bar{T}_{\tau\tau})} \]
Including Spin-Flavor Precession

\[ H_\nu + H_{\nu\nu} + H_{\text{SFP}}(\mu) = S^\dagger_\tau H_\nu + H_{\nu\nu} + H_{\text{SFP}}(\mu^{\text{eff}}) S_\tau \]

with \( \delta \neq 0 \)

\[ \mu^{\text{eff}} = S\mu S = \begin{pmatrix} 0 & \mu_{12} & \mu_{13}e^{i\delta} \\ -\mu_{12} & 0 & \mu_{23}e^{i\delta} \\ -\mu_{13}e^{i\delta} & -\mu_{23}e^{i\delta} & 0 \end{pmatrix} \]

In the Early Universe weak and magnetic cross sections have a very different energy dependence. This has potentially many interesting ramifications for decoupling in the BBN epoch.

Majorana magnetic moment in the Early Universe

\[
\rho_{\text{relativistic}} = \frac{\pi^2}{15} T^4 \left[ 1 + \frac{7}{8} N_{\text{eff}} \left( \frac{4}{11} \right)^{4/3} \right]
\]

Planck:
\[ N_{\text{eff}} = 3.30 \pm 0.27 \quad \Rightarrow \quad \mu_{\text{Majorana}} \leq 6 \times 10^{-10} \mu_B \]
How can we make further progress?

- We examined the many-neutrino gas both from the exact many-body perspective and an effective one-body description following introduction of a mean field. In the limit of the single angle approximation, both pictures possess constants of motion.

- At least in the single angle approximation, we can solve the full many-body problem in the adiabatic limit for a few simple cases. To go beyond those special cases we need to solve the Bethe ansatz equations.

- The condensed-matter community has developed many advanced tools to solve those equations. Bringing those two communities together to exchange ideas would significantly help SN neutrino physics community.