LESS KNOWN ASPECTS
of NUCLEAR PAIRING

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Topics for Discussion

- Exact solution of nuclear pairing
- Properties of the ground state
- Excited states
- New approximations
- Chaos and pairing
- Phase transitions
- Extending to continuum
- Ternary correlations
Pairing Hamiltonian

- Pairing on degenerate time-conjugate orbitals
- Pair operators $P = (a a) (J=0, T=1)$
- Number of unpaired fermions is seniority $s$
- Unpaired fermions are untouched by $H$

$$H = \sum_1^{\epsilon_1} N_1 - \sum_{12} G_{12} P_1^\dagger P_2$$

Diagram showing energy levels $\epsilon_1$, $\epsilon_2$ with seniority $s_1 = 0$ and $s_2 = 1$.
Approaching the solution of pairing problem

- **Approximate**
  - BCS - HFB theory
    + corrections + RPA
  - Iterative techniques

- **Exact solution**
  - Richardson solution (special choices of \( G \))
  - Algebraic methods
  - Direct diagonalization + quasispin symmetry
SHORTCOMINGS of BCS

PARTICLE NUMBER NONCONSERVATION

\[ |0\rangle = \prod_{\lambda \text{(doublets)}} \{ u_\lambda - v_\lambda P_\lambda^\dagger \} |\text{vac}\rangle \]

Projection correction: gauge freedom

\[ v \Rightarrow ve^{i\phi} \]

(losing the variational character of the solution...)

NO SOLUTION for WEAK PAIRING

Simple model: \[ G_{\lambda\lambda'} = -G \text{ const near Fermi-surface} \]

\[ \Delta_\lambda = G \sum_{\lambda'} \frac{\Delta_{\lambda'}}{2E_{\lambda'}} \]

Nontrivial solution

\[ 1 = G \sum_{\lambda} \frac{1}{2E_\lambda} \]

Critical point, \( G = G_c, \Delta = 0 \)

\[ 1 = G_c \sum_{\lambda} \frac{1}{2|\epsilon_\lambda'|} \]

Macroscopic superconductor:

\[ 1 = G \nu \int_0^{\hbar \omega} \frac{d\epsilon}{\sqrt{\epsilon^2 + \Delta^2}} \Rightarrow \Delta \approx 2\hbar \omega \exp \left( -\frac{1}{G \nu} \right) \]
Quasispin and exact solution of pairing problem (EP)

For each single $j$-level

- Operators $P^+_j$, $P_j$ and $N_j$ form a SU(2) group
- Quasispin $L^2_j$ is a constant of motion,
  seniority $s_j = (2j+1) - 2L_j$

- Each $s_j$ is conserved but $N_j$ is not

Practically easy:

Example: $^{116}$Sn: 601,080,390 m-scheme states
  272,828 $J=0$ states
  110 $s=0$ states

- Generalization to isovector pairing, $R_5$ group
Drawbacks of BCS

- Particle number non-conservation
- Sharp phase transition
- No correlations beyond phase transition
- BCS fails for weak pairing
- Excited states and pair vibrations
Pairing phase transition

- BCS has a sharp phase transition

Diagram showing
- BCS solution
- Exact solution

Pairing correlations

 Weak pairing region

$G_{critical}$

Pairing strength
Occupation numbers and spectroscopic factors

- Occupation numbers
  \[ n_j = \langle N | a_j^\dagger a_j | N \rangle \]

- Spectroscopic factors
  \[ v_j = \langle N - 1 | a_j | N \rangle \]
  \[ u_j = \langle N + 1 | a_j^\dagger | N \rangle \]

- Two-body spectroscopic factors
  \[ P_j(N) = \langle N - 2 | P_j | N \rangle \]
  \[ P_j^\dagger(N) = P_j(N + 2) = \langle N + 2 | P_j^\dagger | N \rangle \]

BCS: \[ n_j = v_j^2 = 1 - u_j^2, \quad P_j(\overline{N}) = \sqrt{n_j(1 - n_j)} = u_j v_j \]
BCS works well: $^{114}_{\text{Sn}}$

**Exact calculation and BCS**

**Separation energy:**
$S(N) = E(N-1) - E(N)$

<table>
<thead>
<tr>
<th></th>
<th>$g_{7/2}$</th>
<th>$d_{5/2}$</th>
<th>$d_{3/2}$</th>
<th>$s_{1/2}$</th>
<th>$h_{11/2}$</th>
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<tr>
<td>$N_j$</td>
<td>6.96</td>
<td>4.46</td>
<td>0.627</td>
<td>0.356</td>
<td>1.60</td>
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<tr>
<td>$N_i$</td>
<td>6.71</td>
<td>4.14</td>
<td>0.726</td>
<td>0.507</td>
<td>1.91</td>
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<td>$n_j$</td>
<td>0.870</td>
<td>0.744</td>
<td>0.157</td>
<td>0.178</td>
<td>0.133</td>
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<tr>
<td>$1-u^2_j$</td>
<td>0.872</td>
<td>0.748</td>
<td>0.162</td>
<td>0.183</td>
<td>0.137</td>
</tr>
<tr>
<td>$\nu^2_j$</td>
<td>0.865</td>
<td>0.736</td>
<td>0.155</td>
<td>0.177</td>
<td>0.131</td>
</tr>
<tr>
<td>$n_i$</td>
<td>0.839</td>
<td>0.690</td>
<td>0.181</td>
<td>0.254</td>
<td>0.159</td>
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<tr>
<td>$S(N+1)$</td>
<td>2.80</td>
<td>3.13</td>
<td>3.14</td>
<td>3.39</td>
<td>3.29</td>
</tr>
<tr>
<td>$S(N+1)$</td>
<td>2.89</td>
<td>3.21</td>
<td>3.11</td>
<td>3.21</td>
<td>3.26</td>
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<tr>
<td>$S(N)$</td>
<td>6.86</td>
<td>6.55</td>
<td>7.25</td>
<td>6.98</td>
<td>7.12</td>
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<tr>
<td>$S(N)$</td>
<td>6.89</td>
<td>6.64</td>
<td>7.20</td>
<td>7.03</td>
<td>7.06</td>
</tr>
<tr>
<td>$P(N+2)$</td>
<td>0.680</td>
<td>0.779</td>
<td>0.617</td>
<td>0.514</td>
<td>1.03</td>
</tr>
<tr>
<td>$P(N)$</td>
<td>0.810</td>
<td>0.930</td>
<td>0.524</td>
<td>0.396</td>
<td>0.845</td>
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<tr>
<td>$P$</td>
<td>0.734</td>
<td>0.801</td>
<td>0.545</td>
<td>0.435</td>
<td>0.896</td>
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</table>
Ladder-system with constant $G$

occupation numbers
single nucleon transitions

two nucleon emission

Ladder with 12 levels, $N=12$
Ca isotopes

Monopole contribution: (FPD6)

$$\sum_j \overline{V}_{jj} \frac{N_j(N_j-1)}{2} + \sum_{j \neq j'} \overline{V}_{jj'}, N_j N_{j'}$$

$$\overline{V}_{jj} = \frac{1}{j(2j+1)} \sum_{L \neq 0} (2L+1) V_j (jj jj), \quad \overline{V}_{jj'} = \frac{1}{(2j+1)(2j'+1)} \sum_{L \neq 0} (2L+1) V_j (jj'; jj')$$ (j \neq j')
Hartree-Fock + EP

- Mean field (HF)
- Other interactions
- Pairing
- Occupation numbers

- We use Skyrme HF (SKX$^{(1)}$)
- Pairing matrix elements from G-matrix calculations $^{(2)}$

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Pair vibrations in realistic cases

- $J=0$ pair-vibration states - "worsened" copies of the ground state.
- Less affected by other interactions.
- Energy above $2\Delta$.
\( {^{116} \text{Sn}} \) example

- Neutron system on \( h_{1/2}, d_{3/2}, s_{1/2}, g_{7/2}, d_{5/2} \)
- Realistic interactions Nijm-I G-matrix\(^\dagger\) (pairing part \( J=0, T=1 \))
- Exact solution determines all 601080390 m.b. states found in 420 representations, seniority=0 has 110 spin 0 states

Moment of inertia  \( E = \frac{\hbar^2 J(J+1)}{2I} \)

EP I=32 MeV\(^{-1}\), Rigid body I=35 MeV\(^{-1}\), Experiment I=22 MeV\(^{-1}\)


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NEW ENTROPY

Response to noise - random parameter(s) $\lambda$:

$$|\alpha; \lambda\rangle = \sum_{k} C_{k}^{\alpha}(\lambda)|k\rangle$$

Density matrix of an **individual** state

$$(\rho_{\alpha})_{kk'} = \langle C_{k}^{\alpha}(\lambda)C_{k'}^{\alpha*}(\lambda)\rangle_{av} \text{ over } \lambda$$

Invariant correlational entropy (ICE)

$$S_{\alpha} = -\text{Trace}(\rho_{\alpha} \ln \rho_{\alpha})$$

**Sensitive to structural changes of wave functions**
This secular equation is also valid for the normal modes generated by pairing with no condensate present when we have only the trivial BCS solutions $X_i = 0$ and $\Delta_i = 0$, the vibrational amplitude $z_i$ vanishes, and the single-particle energies are equal to $|\epsilon_i|$. With the extension to higher orders of the same mapping procedure to the collective variables as in Eq. (11), one can study the anharmonic effects. However, if the anharmonicity is indeed important, as for example in the region of very low frequencies and correspondingly large amplitude collective motion, it is simpler (at least in the pure pairing problem) to switch to the exact solution.

B. Example: two-level system with off-diagonal pairing

As an example of the analytically solved RPA, we consider an interesting particular case of two levels with capacitites $\Omega_{1,2}$ and single-particle energies $\epsilon_{1,2}$ when the pairing interaction has only the off-diagonal amplitude $\gamma_{12} = V_{12} = g$ (in this case the sign of $g$ does not matter). With the effective interaction parameters $\lambda_{1,2} = g \Omega_{1,2}/4$, the set of the BCS gap equations (17) takes the forms

\[
\Delta_1 = \frac{\lambda_1 + \lambda_2}{\epsilon_1 + \epsilon_2} \Delta_2, \quad \Delta_2 = \frac{\lambda_1 + \lambda_2}{\epsilon_1 + \epsilon_2} \Delta_1,
\]

which leads to the exact solutions

\[
\Delta_1 = \frac{\lambda_1 + \lambda_2}{\epsilon_1 + \epsilon_2} \Delta_2, \quad \Delta_2 = \frac{\lambda_1 + \lambda_2}{\epsilon_1 + \epsilon_2} \Delta_1.
\]

The corresponding quasiparticle energies are given by

\[
\epsilon_1^2 = \lambda_1 \epsilon_1, \quad \epsilon_2^2 = \lambda_2 \epsilon_2,
\]

with the useful identity

\[
\epsilon_1 \epsilon_2 = \lambda_1 \lambda_2
\]

being valid. The BCS solution collapses, $\Delta_{1,2} \rightarrow 0$, at the critical coupling strength determined by $\lambda_1 \lambda_2 = \epsilon_1^2 \epsilon_2^2$, or at

\[
g^2 \approx g_c \approx \frac{16 \epsilon_1^2 \epsilon_2^2}{\Omega_1 \Omega_2}.
\]

The secular equation [see Eqs. (21) and (23)], with the help of identity (26), gives, along with the spurious mode $\omega^2 = 0$, the physical root

\[
\omega^2 = 4(\epsilon_1^2 + \epsilon_2^2 + 2 \epsilon_1 \epsilon_2) = 4(\epsilon_1^2 + \epsilon_2^2 + \Delta_1^2 + \Delta_2^2).
\]

C. Exact solution versus BCS+RPA

The behavior of energies and entropy in the BCS phase transition region is illustrated in Fig. 3 for a two-level model, and in Fig. 4 for the realistic case. First we consider the two-level model, defined in Sec. IV B; see Fig. 3. In the case of half occupancy and two levels of equal capacity, $N = \Omega_1 = \Omega_2$, with energies $\pm \epsilon$ symmetric with respect to the chemical potential $\mu = 0$, the RPA predicts $\omega^2 = 8 \Delta^2$, see Eq. (28). Note that in the case of many interacting levels with $V_{ij} = \text{const}$ the spectrum of normal modes starts with the lower value $\omega^2 = 4 \Delta^2$ (the threshold of pair breaking).

Panel (b) in the middle shows the energies of the lowest pair vibration state (thick solid line) and the lowest state with one broken pair (thick dashed line) as a function of the pairing strength $g$. The excitation energy of the pair vibration is compared with the RPA prediction shown by the thin dotted line. The BCS phase transition occurs, in the units corresponding to $\epsilon_{1,2} = 1$, at $g = g_c = 0.25$, where we see the breakdown of the RPA and the vanishing collective frequency. Below this point the RPA is constructed on the background of the normal Fermi distribution, while, above $g = g_c$, the RPA is built on the pairing condensate. The limits $g \rightarrow 0$ and $g \rightarrow \infty$ are described well within the RPA, while near the phase transition large fluctuations make the BCS + RPA description unreliable. The failure of the BCS is well discussed in the literature and studied using simple models [27]. In accordance with this instability, a sharp rise of en-
Scaling of other matrix elements

48 Ca
ground state
(fp shell model)

Occupation of f7/2
Phase diagram $^{29}\text{Mg}$
(invariant entropy)
Fragmentation of single-particle strength

$1k_{17/2}^{209\, \text{Pb}}$

($1 \text{phonon} + 2 \text{phonons}$)
Doubling phase transition

$k = 2.0$
**sd-SHELL MODEL $^{24}$Mg**

(a) degenerate $\epsilon_{s.p.}$, 63 random m.e.
$J_0 = 0, T_0 = 0$ 59.1%; overlap 2%

(b) realistic $\epsilon_{s.p.}$, 63 random m.e.
$J_0 = 0, T_0 = 0$ 49.3%; overlap 5.3%

(c) realistic $\epsilon_{s.p.}$ and 6 pairing m.e., 57 random m.e.
$J_0 = 0, T_0 = 0$ 67.8%; overlap 10.6%

(d) degenerate $\epsilon_{s.p.}$, 6 random pairing m.e.
$J_0 = 0, T_0 = 0$ 92.2%; overlap 5.2%

Many spins 1/2: $J_0 = 0, T_0 = 0$ 99%

Quantum glass $J_0 \sim \sqrt{N}$, $H = \sum_{12} J_{12}(s_1 \cdot s_2)$
single-j level model that the ground-state wave function carries very little effect of pairing correlations. However, this overlap is still greater than one would expect in the case of extreme chaoticty when the components $C_\alpha$ of the wave function are uniformly distributed over the unit sphere in space of the corresponding dimension $N$ and $|\langle \Psi | \rangle|^2 = 1/N$ which gives rise to the so-called $N$-scaling [17,18]. In our case the dimension for the $J = 0, T = 0$ states is $N = 325$ which would give the average chaotic overlap factor 0.3%.

![Graph](image)

**FIG. 2.** Distribution of $B(E2)_R/B(E2)_W$ values from the first $2^+0$ state to the $0^+0$ g.s. for models (a-d), where $B(E2)_R$ are the values obtained from the random interactions and $B(E2)_W$ is the value from the $W$ interaction.

The overlap is even greater in other models. The maximum of 11% is reached in model (c) because of the combined action of two effects. First, the presence of realistic pairing lowers the energy of a state with paired particles. On the other hand, basis states with large seniority (the number of unpaired particles) are now effectively removed from contributing considerably to the ground-state wave functions. This makes the effective dimension $N$ smaller than the nominal one. This phenomenon was clearly seen for a simple $N = 3$ single-j case in Ref. [11]. The stabilizing presence of the mean-field orbitals, model (b), also increases the overlap with the realistic ground-state wave function.

In Fig. 1 it is seen that models (a) and (d) have overlaps which are strongly peaked at small values. Model (c) leads to a more smooth and uniform overlap distribution. As discussed for the average values, this means that the realistic mean-field orbitals (given by their single-particle energies) have a strong influence on the overlap. The overlap is more enhanced by realistic pairing (c), but it is still far from unity.

The predominantly chaotic nature of the low-lying states is confirmed by the weakness of multipole-multipole correlations produced by the random interactions. As an illustration, results for the quadrupole transition probabilities from the lowest $2^+$ state to the ground $0^+$ state are shown at the end of Table I and in Fig. 2. Typically, the $B(E2)$ value is by more than an order of magnitude weaker than that obtained with the realistic interactions. The distribution of the $B(E2)$ values for model (a) is close to the Porter-Thomas as one expected for matrix elements of one-body operators between two complicated states [17,19]; the $2^+$ state is even less ordered than the ground state. A trace of collective strength appears in the model (c). In this respect one can recall that low-lying collective vibrations, in contrast to high-lying giant resonances that are less sensitive to the residual interactions, emerge only in a superfluid Fermi-system. In a normal Fermi-system, the low-lying vibrations are not shifted outside the particle-hole continuum and have only a single-particle strength [20]. It means that again we see the pronounced pairing effects only if the residual interaction explicitly contains the pairing part.

In conclusion, with the aid of random rotationally- and isospin-invariant two-body interactions in the $sd$ shell model we have studied the main features of the structure of the ground and low-lying eigenstates. We confirm the strong enhancement of the probability of the quantum numbers $J^T = 0^+0$ for the ground states. However, the resulting ground-state wave functions have only a weak overlap with the realistic ground states that depends on the specific model of randomness. No considerable pairing effects generated by the random interactions were observed. The quadrupole transitions between the lowest $2^+$ states and the ground states also do not reveal significant collectivity. We can claim that the apparent regular geometric pattern of the low-lying spectra coexists, in the case of the random two-body Hamiltonian, with mainly incoherent ("chaotic") structure of the eigenfunctions. Small hints of coherent components in the wave functions generated presumably by the off-diagonal pairing matrix elements and observed also in the earlier studies [11] require a more detailed analysis.

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Overlap

\[ x = \left| \langle \text{g.s. paired} | \text{g.s. random int.} \rangle \right|^2 \]

\[ (s = 0) \]

\[ N = 6 \]

\[ j = \frac{15}{2} \]

\[ N = 4 \]

\[ \frac{dP(x)}{dx} \]

\[ \frac{dP(x)}{dx} \]

\[ J_{gs} = 0, \text{ and } J_{1} = 2 \]

\[ \text{prediction} \]
FIG. 1: Single-\(j\) system \(j = 15/2\) is considered with \(N = 4\) and \(N = 6\) particles, upper and lower panels, respectively. Dynamics of these systems is driven by two-body random interactions, so that each matrix elements \(V_{ij}\) has a gaussian distribution centered at zero and width 1. 100000 random realizations are considered in both cases. From all these realizations only those that result in the \(J=0\) ground state and \(J=2\) first excited state were selected. The number of such cases is 6650 (i.e. 6.7\%) for \(N = 4\), and 9240 (9.2\%) for \(N = 6\). The histogram shows the distribution of the ratio \(A = Q^2 / B(E2)\) for all selected cases. Here \(Q\) is the quadrupole moment of the first excited \(J = 2\) state and \(B(E2)\) is a transition strength for decay of this excited state.
Is there chaotic in pairing

Level Spacing Distribution

Spectral Rigidity $\Delta_3$

- Pairing vibrations
- GOE
- Poisson

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TERNARY CORRELATIONS

Remnants of three-body forces??

\[ H^{(3)} = \sum_{1231'2'3'} g(123; 1'2'3') a_1 a_1' a_2 a_2' a_3 a_3' \]

Possible collective effects

- Monopole correction to pairing
  \[ (\text{Pair})_0 (a^1 a)_0 (\text{Pair})_0 \]
- Cubic quadrupole anharmonicity
  \[ [(\text{Phonon})_2 (\text{Phonon})_2 (\text{Phonon})_2]_0 \]
- Shape fluctuations and giant dipole resonance
  \[ [(\text{Phonon})_1 (\text{Phonon})_2 (\text{Phonon})_1]_0 \]
- Renormalization of octupole mode
  \[ [(\text{Phonon})_3 (\text{Phonon})_2 (\text{Phonon})_3]_0 \]
- .......

1
MONOPOLE CORRECTION TO PAIRING

\[ H = H_0 + H_P + H', \quad H_P = -\sum_{12} G_{12}P_1^TP_2, \quad H' = -\sum_{12,3m} g_{12,3}T_{1,3m}^TT_{2,3m}, \quad T_{1,3m} = \alpha_{3m}P_1 \]

Degenerate case:

\[ H = \epsilon N - GP^TP + gP^NP \]

Renormalization, \( G \Rightarrow G - g(N - 2) \),

Odd-even mass difference

\[ \Delta = E_1(N + 1) - \frac{1}{2} [E_0(N) + E_0(N + 2)] = \frac{1}{4} [G\Omega - gN(\Omega - 2)] \]

Xe isotopes: 126-136

\( G \approx 0.5 \text{ MeV}, \quad g \approx 26 \text{ keV} \)

BCS-type solution:

\[ \Delta'_1 = \sum_{2} \sum_{3} \frac{\Delta'_3}{4E'_2} \left( G_{12} - \sum_{3} g_{12,3}\langle n_3 \rangle \right) \]

\[ \langle n_1 \rangle = \Omega_1 \nu_1^2 \]

\[ \epsilon'_1 = \epsilon_1 + \sum_{23} g_{23,1}\Omega_2\Omega_3 \frac{\Delta'_2\Delta'_3}{16E'_2E'_3} \]

\[ \epsilon'_1 = \epsilon_1 + \sum_{12,3} g_{31,2}\Omega_1\Omega_3 \frac{\Delta'_1\Delta'_3}{16E'_1E'_3} \]
CUBIC ANHARMONICITY

General form:

For identical phonons - suppressed by Furry theorem - particle-hole symmetry ~ (v^2 - v^3)

Large contribution to the width of Isovector Giant Resonances (Quadrupole and Dipole);

- microscopic mechanism of shape fluctuations

\[ \lambda_2 = \lambda_3 = 2^+ \quad \text{IVGQR} \]

\[ \lambda_2 = \lambda_3 = 1^- \quad \text{IVGDR} \]
SOFT MODE DYNAMICS

Adiabatic adjustment to low-lying quadrupole mode


New experiment W.F. Mueller et al. MSU (2005)

\[ E(3^-) = E_0 - \frac{\text{const}}{E(2^+)} \]

\[ B(E3 \uparrow) = K \frac{Z^2 A^{1/3}}{E(3^-)} \frac{82 - N}{12} \]

Xe isotopes

\( h_{1/2} \) occupation
PARITY and TIME-REVERSAL VIOLATION

From Schiff moment to atomic EDM

\[ S = \frac{1}{10} \sum_a e_a r_a \left[ r_a^2 - \frac{5}{3} \langle r_{ch}^2 \rangle \right]. \]

Expectation value

\[ \langle S \rangle = \frac{\langle (S \cdot J) \rangle}{J(J+1)} \]

violates \( P \)- and \( T \)-invariance.

With static quadrupole and octupole deformation

\[ S_{\text{intr}} = S_{\text{intr}} n \]

\[ \langle n_s \rangle = 2\alpha \frac{KM}{J(J+1)} \]

\[ \alpha = \frac{\langle J^+ | W(PT) | J^- \rangle}{E_+ - E_-} \]

\[ S_{\text{intr}} \propto \beta_2 \beta_3 \]

\[ S \sim S_{\text{intr}} \frac{2\alpha^2}{J+1} \ll \beta_2 \beta_3^2 \]

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Possible enhancement in spherical nuclei with soft modes??

$W_{12}$: Coupling through continuum

\[ G = \frac{1}{E - H} \quad \text{Intrinsic propagator} \]

\[ g = \frac{1}{E - \hbar \omega} \quad \Rightarrow \quad g = \frac{1}{G} \cdot \frac{1}{G} \sum_{12} g \]

\[ \Sigma_{12}(E) \sim \sum_{\text{channels}} \int (\prod_{\text{c}} d^3p_c) \quad \frac{(c \rightarrow 1) (2 \rightarrow c)}{E - E_p + i\hbar} \]

\[ = \text{Re} \Sigma_{12} + i \text{Im} \Sigma_{12} \]

- principal value
  - (virtual processes)
  - (virtual processes)
  - (virtual processes)

- closed and open channels

- $H \Rightarrow H'$

\[ i \left(-\frac{1}{2} W_{12}\right) \]

\[ W_{12} = \sum_c A_1^c A_2^{c*} \]

(open)

Channel variables are eliminated
EFFECTIVE HAMILTONIAN

\[ \mathcal{H}(E) = H - \frac{i}{2} W(E) \text{ - non-Hermitian} \]

\[ W_{12} = \sum_{c; \text{open}(E)} A_1^c A_2^c \]

**Internal representation:** \( H \rightarrow \epsilon_n, \)

\[ \mathcal{H} = \begin{pmatrix}
\epsilon_1 - (i/2)A_1^2 & -(i/2)A_1A_2 & -(i/2)A_1A_3 \\
-(i/2)A_1A_2 & \epsilon_2 - (i/2)A_2^2 & -(i/2)A_2A_3 \\
-(i/2)A_1A_3 & -(i/2)A_2A_3 & \epsilon_3 - (i/2)A_3^2
\end{pmatrix} \]

Weak coupling, \( \kappa \ll 1 \) – isolated resonances

\[ \mathcal{E}_n = E_n - (i/2)\Gamma_n \approx \epsilon_n - (i/2)A_n^2 \]
Interaction between resonances

\[ \mathcal{H} = \mathcal{H}^0 + V \]

Real \( V \)

\( V = 0 \)

\( V \neq 0 \)

Imaginary \( V \)

Real \( V \): energy repulsion (weak, possible crossing at \( V \neq 0 \))

width attraction

Imaginary \( V \): energy attraction

width repulsion

broad resonance (Dicke)


One-body decay

- Continuum channel \[ |c; E\rangle_N = c_j^\dagger(\epsilon_j)|\alpha; N - 1\rangle \]
  - State \( \alpha \) in \( N-1 \) nucleon daughter
  - Particle in continuum state \( j \)
  - Energy \( E = E_\alpha + \epsilon_j \)

- Transition Amplitude

\[ A_i^c(E_\alpha + \epsilon_j) = a_j(\epsilon_j) \langle \alpha; N - 1 | b_j | 1; N \rangle \]
Single-particle scattering problem

The same non-Hermitian eigenvalue problem

\[ h u_l = \frac{1}{2\mu} \left\{ -\frac{d^2}{dr^2} + \frac{l(l + 1)}{r^2} + 2\mu \left[ V(r) + \alpha \frac{Z z}{r} \right] \right\} u_l(r) = \epsilon u_l(r), \]

Internal states: \( u_l(r) \)

External states: \( \epsilon = \frac{k^2}{2\mu} \)

\[ F_l(r) = kr j_l(kr) \]

\[ G_l(r) = kr n_l(kr) \]

Single-particle decay amplitude

\[ a^j(\epsilon) = \langle 0 | c_j(\epsilon) V b_j^\dagger | 0 \rangle = \sqrt{\frac{2\mu}{\pi k}} \int_0^\infty dr F_l(r) V(r) u_l(r) \]

Single particle decay width: (requires definition and solution for resonance energy)

\[ \gamma_j(\epsilon) = 2\pi |a^j(\epsilon)|^2 \]
Two-body decays

Sequential process via one-body interaction

Direct process via two-body interaction

\[ A^c(E) = \langle c, \epsilon_1, \epsilon_2 | H_{s.p.} | 1; N \rangle \]

\[ A^c(E) = \langle c, \epsilon_1, \epsilon_2 | H_{2\text{body}} | 1; N \rangle \]
Oxygen Isotopes
Continuum Shell Model Calculation
- sd space, HBUSD interaction
- single-nucleon reactions
Pairing in two-body decay

Example: $^4$He $^5$Li $^6$Be

Pairing energy: $2 \Delta = 0.57 - (1.97) = 2.54$ MeV

Single-particle level

$$ \epsilon_f = \epsilon_0 - i \frac{\gamma}{2} $$

$$ f(\epsilon) = \frac{1}{2\pi} \frac{\gamma}{(\epsilon - \epsilon_0)^2 + \gamma^2/4} $$

Single-particle states in continuum

$$ \psi(\vec{r}; \epsilon) = \sqrt{f(\epsilon)} \psi^0(\vec{r}; \epsilon) $$

scattering function (real $\epsilon$)

Two-body problem (Cooper phenomenon)

$$ \left[ h(1) + h(2) + V(1,2) \right] \psi(1,2) = E \psi(1,2) $$

$$ h\psi^0(\vec{r}, \epsilon) = \epsilon \psi^0(\vec{r}, \epsilon) $$
Pairing interaction

\[ \langle \varepsilon_1, \varepsilon_2 | U | \varepsilon_2', \varepsilon_1' \rangle = -G \sqrt{f_1 f_2 f_1' f_2'} \]

\[ G = - \int \mathrm{d}r_1 \mathrm{d}r_2 \ U (1, 2) \left( \Psi^0 (r_1, \varepsilon_1) \Psi^0 (r_2, \varepsilon_2) \right)^2 \]

Solution

\[ \Psi^*(1, 2) = \int \mathrm{d}\varepsilon_1 \mathrm{d}\varepsilon_2 \ C (\varepsilon_1, \varepsilon_2) \Psi (\varepsilon_1; \varepsilon_1) \Psi (\varepsilon_2; \varepsilon_2) \]

\[ (\varepsilon_1 + \varepsilon_2 - E) \ C (\varepsilon_1, \varepsilon_2) = G \sqrt{f(\varepsilon_1) f(\varepsilon_2)} \ C_0 \]

\[ C_0 = \int \mathrm{d}\varepsilon_1' \mathrm{d}\varepsilon_2' \ C (\varepsilon_1', \varepsilon_2') \sqrt{f(\varepsilon_1') f(\varepsilon_2')} \]

Secular equation

\[ \frac{1}{G} = \int \mathrm{d}\varepsilon_1 \mathrm{d}\varepsilon_2 \ \frac{f(\varepsilon_1) f(\varepsilon_2)}{\varepsilon_1 + \varepsilon_2 - E - i\alpha} \]

Analytic continuation: Resonance

\[ \varepsilon = E - \frac{i}{2} \Gamma \]

\[ \varepsilon = 2\varepsilon_0 - G - \frac{i}{2} \Gamma = E_0 - \frac{i}{2} \Gamma \]

\[ \Gamma = G^2 F (\xi) \]

\[ F (E) = 2\pi G \int \mathrm{d}\varepsilon_1 \mathrm{d}\varepsilon_2 \ f(\varepsilon_1) f(\varepsilon_2) \delta (\varepsilon_1 + \varepsilon_2 - E) \]