Electromagnetic logs in $B \rightarrow X_s \ell^+ \ell^-$

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Outline

✗ Definition of the observable

✗ Calculation of the quark level decay rate: effective Hamiltonian, RGE, matrix elements

✗ Details on QED bremsstrahlung and origin of the log\(m_b/m_l\) enhanced corrections

✗ Phenomenology
Intermediate cc resonances

✗ Presence of intermediate resonances forces cuts on the dilepton invariant mass: $B \to X_s(J/\psi, \psi') \to X_s \ell^+ \ell^-$

✗ Three distinct regions ( $s = (p_1+p_2)^2$ ):
  - very low-$s$ region: $0.04 \text{ GeV}^2 < s < 1 \text{ GeV}^2$
  - low-$s$ region: $1 \text{ GeV}^2 < s < 6 \text{ GeV}^2$
  - high-$s$ region: $s > 14 \text{ GeV}^2$

✗ We assume the subtraction of background coming from subdominants $J/\psi$ decays: $J/\psi \to X \ell^+ \ell^-$
Hadronic invariant mass cut

✗ Background from the double semileptonic decay:
  \( b \rightarrow e^- \nu c (\rightarrow s \nu e^+) = b \rightarrow s e^- e^+ + \text{missing energy} \)

✗ Cuts on the invariant mass of the \( X_s \) system:
  \( m_{X_s} < 2.1 \text{ GeV} \)

✗ Parton level at lowest order: \( m_{X_s} = m_s \)
  bremsstrahlung: \( m_s < m_{X_s} < m_b \)
  non-perturbative effects: phase space (\( m_B - m_b = \Lambda \))
  Fermi motion

✗ We need: \( \epsilon_{X_s} = \int_{m_K}^{2.1} \frac{d \Gamma}{d m_{X_s}} / \int \frac{d \Gamma}{d m_{X_s}} = (93 \pm 4)\% \)
Hadronic invariant mass cut

- $s < (m_B - m_{X_s})^2$
- high-$s$ region is unaffected
- low-$s$ region:
  - not optimal cut
  - $cc$ background
  - $C_7$-$C_9$ interference
  - more events

$M_{X_s} < 2.1 \text{ GeV}$
Calculation of the decay rate

\[ \Gamma(B \to X_s l^+ l^-) = \Gamma(b \to s l^+ l^-) + \text{power corrections} \]

\[ \times \]

- The quark level decay rate is known at NNLO in QCD and at NLO in QED

\[ \times \]

- \(1/m_b^2\), \(1/m_b^3\) power corrections (\(\lambda_1\) and \(\lambda_2\))

\[ \times \]

- \(1/m_c^2\) power corrections (\(\lambda_2\))

\[ \times \]

- The parameter \(\lambda_1\) drops in further normalization and only the better known \(\lambda_2\) survives
Effective Hamiltonian

$$H_{\text{eff}} = -4 \frac{G_F}{\sqrt{2}} V_{tb} V_{ts}^* \left[ \sum_{i=1}^{10} C_i P_i + \sum_{i=3}^{6} C_i P_i Q + C_b P_b \right]$$

$$P_1 = (\bar{s}_L \gamma_\mu T^a c_L)(\bar{c}_L \gamma^\mu T^a b_L) \quad P_2 = (\bar{s}_L \gamma_\mu c_L)(\bar{c}_L \gamma^\mu b_L)$$

$$P_7 = \frac{e}{16 \pi^2} m_b (\bar{s}_L \sigma^{\mu\nu} b_R) F_{\mu\nu} \quad P_8 = \frac{g_s}{16 \pi^2} m_b (\bar{s}_L \sigma^{\mu\nu} T^a b_R) G_{\mu\nu}^a$$

$$P_9 = (\bar{s}_L \gamma_\mu b_L) \sum (l \gamma^\mu l) \quad P_{10} = (\bar{s}_L \gamma_\mu b_L) \sum (l \gamma^\mu \gamma_5 l)$$

We neglect contributions proportional to $V_{ub} V_{us}$: they are suppressed by $|C_2 V_{ub} V_{us} / (C_9 V_{tb} V_{ts})| \sim 0.5\%$
Effective Hamiltonian

\[ C_2(\mu_0) \sim O(1) \quad \langle P_2 \rangle \sim O(\alpha_{em}) \]

\[ C_{9,10}(\mu_0) \sim O(\alpha_{em}) \quad \langle P_{9,10} \rangle \sim O(1) \]

\[ C_2 \text{ mixes into } C_9 \text{ at } O(\alpha_{em}); \text{ hence produces a large log not associated to } \alpha_s \]

\[ C_9(\mu_b) \text{ starts, therefore, at order } O(\alpha_{em}/\alpha_s) \]

\[ \text{the organization of perturbation theory is screwed: } \\
LO = \alpha_{em}/\alpha_s, \text{ NLO } = \alpha_{em}, \text{ NNLO } = \alpha_{em} \alpha_s \]
RGE running and QED effects

✗ Perturbative corrections come in powers of coupling constants and of \( L = \log(M_H/M_L) \)

✗ All powers of \( c_s = \alpha_s L \) must be resummed at any given order using RG technology. All large logs are absorbed in \( c_s = O(1) \)

✗ Each QED log, \( \alpha_{em} L = c_s \alpha_{em}/\alpha_s \), gets replaced by \( f(c_s) \alpha_{em}/\alpha_s \), where the function \( f(c_s) \) is obtained by solving the RGE's

✗ This procedure is an unavoidable consequence of resumming QCD logs but not QED ones (the latter being small since \( \alpha_{em} L << 1 \))

✗ We expand in \( \alpha_s \) and \( k = \alpha_{em}/\alpha_s \sim \alpha_{em} L \)
RGE running and QED effects

✗ Solve the RGE's for $\alpha_s$ and $\alpha_{em}$:

$$
\mu \frac{d \tilde{\alpha}_s}{d \mu} = -2 \tilde{\alpha}_s^2 \sum_{n,m=0} \beta_{nm}^s \tilde{\alpha}_s^n \tilde{\alpha}_{em}^m \\
\mu \frac{d \tilde{\alpha}_{em}}{d \mu} = +2 \tilde{\alpha}_{em}^2 \sum_{n,m=0} \beta_{nm}^e \tilde{\alpha}_{em}^n \tilde{\alpha}_s^m
$$

✗ Express the RGE's for the WC's

$$
\mu \frac{d}{d \mu} \tilde{C}(\mu) = \hat{y}^T(\mu) \tilde{C}(\mu) \\
\hat{y} = \sum_{n+m \geq 1} \hat{y}^{(nm)} \tilde{\alpha}_s^n \tilde{\alpha}_{em}^m
$$

in terms of $\eta = \alpha_s(\mu_0)/\alpha_s(\mu_b)$:

$$
\frac{d}{d \eta} \tilde{C} = \frac{1}{\eta} \hat{W} \tilde{C} + O(\tilde{\alpha}_s^3, \tilde{\kappa}_s^3, \tilde{\alpha}_s^2 \tilde{\kappa}_s^2)
$$

✗ Solve perturbatively the RGE's: $\tilde{C}(\mu_b) = \hat{R} \tilde{C}(\mu_0)$
Matrix elements (virtual)

\( \langle P_i \rangle = H_i^9 \langle P_9 \rangle_{\text{tree}} + H_i^7 \frac{\langle P_7 \rangle_{\text{tree}}}{\tilde{\alpha}_s \kappa} + H_i^{10} \langle P_{10} \rangle_{\text{tree}} \)

<table>
<thead>
<tr>
<th>( i )</th>
<th>( H_i^9 )</th>
<th>( H_i^7 )</th>
<th>( H_i^{10} )</th>
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<tr>
<td>( i = 1, 2 )</td>
<td>( \tilde{\alpha}_s \kappa f_i(\hat{s}) - \tilde{\alpha}_s^2 \kappa F_i^9(\hat{s}) )</td>
<td>( -\tilde{\alpha}_s^2 \kappa F_i^7(\hat{s}) )</td>
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<tr>
<td>( i = 3 - 6 (Q) )</td>
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<tr>
<td>( i = 7 )</td>
<td>0</td>
<td>( \tilde{\alpha}_s \kappa )</td>
<td>0</td>
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<tr>
<td>( i = 8 )</td>
<td>( -\tilde{\alpha}_s^2 \kappa F_8^9(\hat{s}) )</td>
<td>( -\tilde{\alpha}_s^2 \kappa F_8^7(\hat{s}) )</td>
<td>0</td>
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<tr>
<td>( i = 9 )</td>
<td>( 1 + \tilde{\alpha}<em>s \kappa f</em>{9}^{\text{pen}}(\hat{s}) )</td>
<td>0</td>
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<tr>
<td>( i = 10 )</td>
<td>0</td>
<td>0</td>
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</tbody>
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Matrix elements (real)

\( \times \) QCD bremsstrahlung:

\[
|\langle P_9 \rangle_{\text{tree}}|^2 \Rightarrow |\langle P_9 \rangle_{\text{tree}}|^2 \left[ 1 + 8 \tilde{\alpha}_s \omega_9^{(1)}(\hat{s}) + 16 \tilde{\alpha}_s^2 \omega_9^{(2)}(\hat{s}) \right]
\]

\[
|\langle P_i \rangle_{\text{tree}}|^2 \Rightarrow |\langle P_i \rangle_{\text{tree}}|^2 \left[ 1 + 8 \tilde{\alpha}_s \omega_i(\hat{s}) \right] \quad (i = 7, 10)
\]

\[
\text{Re}(\langle P_7 \rangle_{\text{tree}} \langle P_9 \rangle_{\text{tree}}^*) \Rightarrow \text{Re}(\langle P_7 \rangle_{\text{tree}} \langle P_9 \rangle_{\text{tree}}^*) \left[ 1 + 8 \tilde{\alpha}_s \omega_{79}(\hat{s}) \right]
\]

\( \times \) QED bremsstrahlung:

\[
|\langle C_i(\mu_b) P_i \rangle_{\text{tree}}|^2 \Rightarrow |C_i(\mu_b) \langle P_i \rangle_{\text{tree}}|^2 \left[ 1 + 8 \tilde{\alpha}_e \omega_i^{\text{em}}(\hat{s}) \right] \quad (i = 7, 9, 10)
\]

\[
\text{Re}(C_7 C_9^* \langle P_7 \rangle_{\text{tree}} \langle P_9 \rangle_{\text{tree}}^*) \Rightarrow \text{Re}(C_7 C_9^* \langle P_7 \rangle_{\text{tree}} \langle P_9 \rangle_{\text{tree}}^*) \left[ 1 + 8 \tilde{\alpha}_s \omega_{79}^{\text{em}}(\hat{s}) \right]
\]

The functions \( \omega_i^{\text{em}}(\hat{s}) \) contain terms enhanced by \( \log(m_b/m_l) \)
Structure of the expansion

✗ Amplitude:

\[ A = \kappa \left[ A_{\text{LO}} + \tilde{\alpha}_s A_{\text{NLO}} + \tilde{\alpha}_s^2 A_{\text{NNLO}} + O(\tilde{\alpha}_s^3) \right] + \kappa^2 \left[ A_{\text{em}} + \tilde{\alpha}_s A_{\text{em}}^{\text{NLO}} + \tilde{\alpha}_s^2 A_{\text{em}}^{\text{NNLO}} + O(\tilde{\alpha}_s^3) \right] + O(\kappa^3) \]

note that \( A_{\text{LO}} \sim 0.03 \) is anomalously small compared to \( A_{\text{NLO}} \sim 4 \)

✗ Decay width:

\[ |A|^2 = \kappa^2 \left[ A_{\text{LO}}^2 + \tilde{\alpha}_s^2 A_{\text{LO}} A_{\text{NLO}} + \tilde{\alpha}_s^2 A_{\text{NLO}}^2 \right] + \kappa^2 \left[ \tilde{\alpha}_s^2 A_{\text{LO}} A_{\text{NNLO}} + \tilde{\alpha}_s^3 (2 A_{\text{NLO}} A_{\text{NNLO}} + ...) \right] + \kappa^3 \left[ 2 A_{\text{LO}} A_{\text{em}} + \tilde{\alpha}_s 2 ( A_{\text{NLO}} A_{\text{em}}^{\text{NLO}} + A_{\text{NLO}} A_{\text{em}}^{\text{LO}} + A_{\text{LO}} A_{\text{em}}^{\text{NLO}} + ...) \right. \\
+ \tilde{\alpha}_s^2 2 ( A_{\text{NLO}} A_{\text{em}}^{\text{NLO}} + A_{\text{NLO}} A_{\text{em}}^{\text{NNLO}} + A_{\text{NNLO}} A_{\text{em}}^{\text{LO}} + A_{\text{LO}} A_{\text{em}}^{\text{NNLO}} + ...) \right. \\
+ \tilde{\alpha}_s^3 2 ( A_{\text{NLO}} A_{\text{em}}^{\text{NNLO}} + A_{\text{NNLO}} A_{\text{em}}^{\text{NLO}} + ...) \]
Decay width

\[ \langle H_{\text{eff}} \rangle = \frac{4 G_F V_{tb} V_{ts}^*}{\sqrt{2}} \left[ A_V(\hat{s}) \langle P_9 \rangle_{\text{tree}} + A_\sigma(\hat{s}) \frac{\langle P_7 \rangle_{\text{tree}}}{\tilde{\alpha}_s \kappa} + A_A(\hat{s}) \langle P_{10} \rangle_{\text{tree}} \right] \]

\[ \frac{d \Gamma (B \to X_s e e)}{d \hat{s}} = \frac{m_{b,\text{pole}}^5 G_F^2 V_{tb}^2 V_{ts}^2}{48 \pi^3} (1 - \hat{s})^2 \]  
\[ \times \left[ (1 + 2 \hat{s})(A_V^2 + A_A^2) + 4 \left( 1 + \frac{2}{\hat{s}} \right) A_\sigma^2 + 12 A_V A_\sigma \right] \]

\[ \times \hat{s} \text{ is normalized to } m_{b,\text{pole}} \]
\[ \times \text{huge uncertainty coming the } m_b \text{ factor in the normalization} \]
\[ \times \text{NLO and NNLO matrix elements of } P_1 \text{ and } P_2 \text{ depend} \]
\[ \text{on } m_{c,\text{pole}} \]
\[ \times \text{Uncertainties from } m_{b,\text{pole}} \text{ and } m_{c,\text{pole}} \text{ can be almost completely removed} \]
Branching ratio

The $m_{b,pole}$ factors are removed by normalizing to the semileptonic $B \rightarrow X_{u}e\nu$ rate (better than $B \rightarrow X_{c}e\nu$ because it avoids the appearance of phase space factors involving $m_{c,pole}$):

$$\frac{d \ BR(B \rightarrow X_{s}ll)}{d \ \hat{s}} = BR_{b \rightarrow c e \nu}^{exp} \left| \frac{V_{cb}}{V_{ub}} \right|^{2} \frac{1}{C} \frac{d \ \Gamma(B \rightarrow X_{s}ll)/d \ \hat{s}}{\Gamma(B \rightarrow X_{u}e\nu)}$$

$$C = \left| \frac{V_{ub}}{V_{cb}} \right|^{2} \frac{\Gamma(B \rightarrow X_{c}e\nu)}{\Gamma(B \rightarrow X_{u}e\nu)}$$

We adopt $C=0.58 \pm 0.01$ [Bauer,Ligeti,Luke,Manohar,Trott]

We assume 100% correlation between the errors on $C$ and $m_{c}$
Comment on the schemes for $m_b$ and $m_c$

✗ Presence of renormalon ambiguities in the BR expressed in terms of $m_{b,\text{pole}}$. They are removed if the 1S or MSbar masses are adopted [Hoang, Ligeti, Manohar]

✗ We use the 1S mass for both the bottom and the charm: [Bauer, Ligeti, Luke, Manohar, Trott]

$$m_{b}^{1S} = (4.69 \pm 0.03) \text{GeV} \quad m_{b}^{1S} - m_{c}^{1S} = (3.41 \pm 0.01) \text{GeV}$$

✗ There are some subtleties in the Upsilon expansion of the pole mass in terms of the 1S mass: resummation of large $\log(\mu/m_b \alpha_s C_F)$ requires that we treat terms of order $O(\alpha_s^{n+1} \epsilon^n)$ as if they were of order $O(\alpha_s^n)$

✗ The expansion of $m_{b,\text{pole}}$ in terms of $m_{b,1S}$ impacts the BR at the 1.5%. The expansion of the charm mass has almost negligible effect
Calculation of $\langle P_9 \rangle$

✗ we adopt the NDR scheme (the divergent terms do not contain terms involving the Levi-Civita tensor)

✗ the calculation of the virtual corrections and the integration over the d-dimensional 3-particle phase is standard

✗ the integration of the bremsstrahlung kernel over the d-dimension 4-particle phase space is highly non trivial:
  - we want to remain differential in $s$
  - since photons couple to both electrons and quarks, the integration kernel is much more complicated than in the case of gluon bremsstrahlung

✗ After adding virtual and real corrections, there is a residual $1/\varepsilon$ pole that survives the integration over the low-\(s\) region (but not if we extend it to the whole PS)
Some details on the PS$_4$ integration

✓ After squaring and summing over the spins the bremsstrahlung kernel can be expressed in terms of:

$s_{12}, s_{1s}, s_{2s}, s_{1\gamma}, s_{2\gamma}, s_{s\gamma}, s_{\text{tri}}=s_{1\gamma}+s_{2\gamma}+s_{s\gamma}$

with up to two $s_{ij}$ in the denominator (→ up to $1/\varepsilon^2$ poles)

✓ We can integrate trivially over the angular variables.

The 4-particle massless phase space measure becomes:

$\overline{d\, PS_4} = \tilde{\mu}^6 \varepsilon \int_\Omega dPS_4$

\[= \tilde{\mu}^6 \varepsilon \frac{2^{-12+10\varepsilon} \pi^{-5+3\varepsilon} m_b^{4-6\varepsilon}}{\Gamma(3/2-\varepsilon) \Gamma(1-\varepsilon) \Gamma(1/2-\varepsilon)} \left( -\Delta_4 \right)^{-1/2-\varepsilon} \Theta \left( -\Delta_4 \right) \]

\[\times d\hat{s}_{12} d\hat{s}_{1s} d\hat{s}_{2s} d\hat{s}_{1\gamma} d\hat{s}_{2\gamma} d\hat{s}_{s\gamma} \delta \left( 1 - \sum \hat{s}_{ij} \right) \]

$\Delta_4 = (\hat{s}_{12} \hat{s}_{s\gamma})^2 + (\hat{s}_{1s} \hat{s}_{2s})^2 + (\hat{s}_{1\gamma} \hat{s}_{2\gamma})^2$

\[-2(\hat{s}_{12} \hat{s}_{1s} \hat{s}_{2s} \hat{s}_{s\gamma} + \hat{s}_{1s} \hat{s}_{1\gamma} \hat{s}_{2s} \hat{s}_{2\gamma} + \hat{s}_{12} \hat{s}_{1\gamma} \hat{s}_{2s} \hat{s}_{s\gamma} + \hat{s}_{12} \hat{s}_{s\gamma} \hat{s}_{2s} \hat{s}_{1\gamma})\]

[Gerhmann-de Ridder, Gehrmann, Heinrich]
More details on the PS$_4$ integration

✗ First one integrates the delta function and reduces the kernel to a set of master integrals, that are classified by their degree of divergence (negative powers of $s_{ij}$)

✗ The basic idea to factorize the Gram determinant. e.g.: choosing $s_{2s}$ as factorization variable we have:

$$-\Delta_4 = -(\hat{s}_{12} + \hat{s}_{1y})^2 [\hat{s}_{2s} + 2B \hat{s}_{2s} + C] = (\hat{s}_{12} + \hat{s}_{1y})^2 (\hat{s}_{2s}^+ - \hat{s}_{2s}) (\hat{s}_{2s} - \hat{s}_{2s}^-)$$

$$\hat{s}_{2s}^\pm = -B \pm \sqrt{\Xi}/2$$

the integrand is now (after eliminating the theta function and with proper integration bounds):

$$Ker(\hat{s}_{ij})(\hat{s}_{12} + \hat{s}_{1y})^{-1-2\epsilon} (\hat{s}_{2s}^+ - \hat{s}_{2s})^{-1/2-\epsilon} (\hat{s}_{2s}^- - \hat{s}_{2s}^-)^{-1/2-\epsilon}$$
Even more details of the PS$_4$ integration

- The change of variable \( \hat{s}_{2s} = (\hat{s}_{2s}^+ - \hat{s}_{2s}^-) \chi + \hat{s}_{2s}^- \) leads to:
  \[
  \text{Ker}(\hat{s}_{ij})(\hat{s}_{12} + \hat{s}_{1\gamma})^{-1-2\epsilon} \Xi^{-\epsilon} (1 - \chi)^{-1/2-\epsilon} \chi^{-1/2-\epsilon}
  \]

- If the kernel does not contain poles in \( s_{2s} \) (the variable used to factorize the Gram determinant), the integral over \( \chi \) can be done analytically in terms of Gamma functions.

- The choice of the subsequent order of integration should aim at extracting all the divergences as soon as possible: the latter must be factorized out in terms of poles and Gamma functions.
Final details of the PS$_4$ integration

For instance, after the above steps, the innocent-looking $1/(s_1\gamma s_2\gamma)$ kernel yields:

$$-\frac{\pi}{\epsilon} d \hat{s}_{12} \hat{s}_{12}^{-1-\epsilon} (1-\hat{s}_{12})^{1-4\epsilon} \left\{ \frac{\Gamma(1-2\epsilon)\Gamma(\epsilon)\Gamma(1-\epsilon)}{\Gamma(2-4\epsilon)} \right. 2 F_1(-\epsilon, 1-2\epsilon; 2-4\epsilon; 1-\hat{s}_{12})$$

$$-(1-\hat{s}_{12}) \int_0^1 du \int_0^1 dv \frac{u^{-2\epsilon}(1-u)^{-\epsilon} v^{1-3\epsilon}(1-v)^{-\epsilon}}{\hat{s}_{12} + (1-\hat{s}_{12}) v(1-u)} \left[ 1-(1-\hat{s}_{12}) u v \right]^\epsilon$$

After expanding in $\epsilon$, all the integrals can be done in terms of logs and polylogs
Summing everything up

✗ The total correction is given by the sum of renormalized virtual corrections and bremsstrahlung

✗ The differential correction contains a residual divergence produced by quasi-collinear photon emission from the lepton lines: this divergence remains because we remain differential in the dilpeton invariant mass while integrating out the photon

✗ If we regularized collinear singularities keeping the electron mass, the singularity would have been a log(m_{lepton}):

\[-\frac{\tilde{\alpha}_{em}}{\epsilon} \rightarrow \tilde{\alpha}_{em}\left(\log \frac{m_b^2}{m_l^2} + \text{non-log terms}\right)\]

✗ Note that this logs are still quite small and do not need to be resummed
From NDR to mass regularization

✗ To change regulator at this stage, we need to construct an intermediate IR-safe observable

✗ We consider a differential BR in which $s$ is identified as:

$$(p_1+p_2+p_γ)^2 \quad \text{if } p_γ \parallel (p_1 \text{ or } p_2)$$

$$(p_1+p_2)^2 \quad \text{otherwise}$$

✗ We need the differential BR with emission of a collinear photon in the NDR and mass regularization schemes:

$$\frac{d \Gamma^{(\epsilon, m)}_{\text{coll}, 3}}{d \hat{S}} \quad \frac{d \Gamma^{(\epsilon, m)}_{\text{coll}, 2}}{d \hat{S}}$$

$$(p_1+p_2+p_γ)^2 \quad (p_1+p_2)^2$$

✗ The conversion term is then the double difference:

$$T_S = \left( \frac{d \Gamma^{(\epsilon)}_{\text{coll}, 3}}{d \hat{S}} - \frac{d \Gamma^{(\epsilon)}_{\text{coll}, 2}}{d \hat{S}} \right) - \left( \frac{d \Gamma^{(m)}_{\text{coll}, 3}}{d \hat{S}} - \frac{d \Gamma^{(m)}_{\text{coll}, 2}}{d \hat{S}} \right)$$
From NDR to mass regularization

- Technically this program can be implemented using the splitting function of the electron, \( f_\gamma(x,E) \), where \( E \) and \( xE \) are the energies of the initial electron and of the emitted photon.

- \( f_\gamma(x,E) \) contains a collinear singularity that can be regularized in dimreg or by assuming a non-vanishing electron mass.

- Direct calculation yields:

\[
f^{(e)}_\gamma(x,E) = 4 \tilde{\alpha}_{em} \left[ \frac{1 + (1 - x)^2}{x} \left( -\frac{1}{2\epsilon} + \log \frac{E}{\mu} + \log(2 - 2x) \right) - \frac{(2 - x)^2}{2x} \log \frac{2 - x}{x} \right]
\]

\[
f^{(m)}_\gamma(x,E) = 4 \tilde{\alpha}_{em} \left[ \frac{1 + (1 - x)^2}{x} \left( \log \frac{E}{m} + \log(2 - 2x) \right) - 1 + x - \frac{x^2}{2} \log x - \frac{(2 - x)^2}{2} \log(2 - x) \right]
\]

soft singularity  
collinear singularity
From NDR to mass regularization

- The differential decay width for $b \to s \, e \, e \, \gamma_{\text{coll}}$ is:

$$d \Gamma^{\epsilon, m}_{\text{coll}}(\hat{s}_{12}, \hat{s}_{1s}, \hat{s}_{2s}, x) = \frac{8 G_F^2 |V_{tb} V_{ts}|^2}{m_b} f^{\epsilon, m}_y(x, E_1) |\langle P_9\rangle_{\text{tree}}|^2 \, dx \, dPS_3$$

- The decay width differential in $s=(p_1+p_2+p_{\gamma,\text{coll}})^2$ is:

$$\frac{d \Gamma^{\epsilon, m}_{\text{coll}, 3}}{d \hat{s}} = \int_0^1 dx \int d \hat{s}_{1s} d \hat{s}_{2s} M_3(\hat{s}, \hat{s}_{1s}, \hat{s}_{2s}) f^{\epsilon, m}_y(x, E_1) |\langle P_9\rangle_{\text{tree}}|^2_{\hat{s}_{12} \to \hat{s}}$$

- The decay width differential in $s=(p_1+p_2)^2$ is:

$$\frac{d \Gamma^{\epsilon, m}_{\text{coll}, 2}}{d \hat{s}} = \int_0^{1-\hat{s}} \frac{d x}{\bar{x}} \int d \hat{s}_{1s} d \hat{s}_{2s} M_3(\frac{\hat{s}}{\bar{x}}, \hat{s}_{1s}, \hat{s}_{2s}) f^{\epsilon, m}_y(x, E_1) |\langle P_9\rangle_{\text{tree}}|^2_{\hat{s}_{12} \to \frac{\hat{s}}{\bar{x}}}$$. 

\[
(p_{e1}+p_{e2}+p_\gamma)^2 = (\bar{x} \, p_1 + p_2 + x \, p_1)^2 = (p_1+p_2)^2 = s_{12}
\]

\[
(p_{e1}+p_{e2})^2 = (\bar{x} \, p_1 + p_2)^2 = \bar{x} \, 2 \, p_1 \cdot p_2 = \bar{x} \, s_{12}
\]
From NDR to mass regularization

✗ Only the total correction term contains only the E-independent difference \( f^{(e)}_y - f^{(m)}_y \), hence we can perform explicitly the PS integration over \( s_1s \) and \( s_2s \):

\[
T_s = 2 \left[ \int_0^1 dx \left[ f^{(e)}_y(x) - f^{(m)}_y(x) \right] \sigma(\hat{s}) - \int_0^{1-\hat{s}} dx \frac{f^{(e)}_y(x) - f^{(m)}_y(x)}{1-x} \sigma \left( \frac{\hat{s}}{1-x} \right) \right]
\]

\[
\sigma(\hat{s}) = \frac{G_F^2 |V_{tb} V_{ts}|^2 m_b^5}{48 \pi^3} (1-\hat{s})^2 (1+2\hat{s}) + O(\epsilon)
\]

✗ Adding virtual, real and collinear terms one obtains:

\[
\left| \langle P_9 \rangle_{\text{tree}} \right|^2 \rightarrow \left| \langle P_9 \rangle_{\text{tree}} \right|^2 \left( 1 + 8 \tilde{\alpha}_{em} \omega^{em}_9(\hat{s}) \right)
\]

\[
\omega^{em}_9(\hat{s}) = \log \frac{m_b^2}{m_e^2} \left( \log (1 - \hat{s}) - \frac{(1-\hat{s})(1+4\hat{s}-8\hat{s}^2) + (1-6\hat{s}^2 + 4\hat{s}^3) \log \hat{s}}{6(1-\hat{s})^2(1+2\hat{s})} \right) + ...
\]
The correction term

- The total decay width is an IR-safe quantity, hence the integral of the log(mb/me) term over the whole s-spectrum must vanish.
Other log-enhanced corrections

✗ At the order we working only QED corrections to $<P_9>$ should be considered (up to and including $O\left(\tilde{\alpha}_s^2 \kappa^3\right)$)

✗ We decided to include all terms of order $O\left(\tilde{\alpha}_s^3 \kappa^3 \log \frac{m_b^2}{m_e^2}\right)$

✗ We have corrections to $|<P_7>|^2$, $|<P_{10}>|^2$, Re[$<P_7><P_9>^*$], $|<P_{1,2}>|^2$ and Re[$<P_{1,2}><P_9>^*$]

✗ The dominant ones stems from $\omega_{9,10}^{em}$ and contribute in equal amount to the final log-enhanced contribution
Is this correction physical?

✗ The critical point is the lepton reconstruction at Babar and Belle

✗ Muons are completely disentangled from collinear photons

✗ Electrons are more problematic: photon emitted at angles smaller then ~ 2° (θ < 0.035, φ < 0.05) from the direction of either electrons are combined with the electron candidate to form a “recovered electron”

✗ The inclusion of photons emitted in a cone around the electron direction is equivalent to set an upper limit on the invariant mass of the photon-electron system:

\[
\log \frac{m_b^2}{m_e^2} \to \log \frac{m_b^2}{\Lambda^2} \quad \text{with} \quad \Lambda \lesssim O(m_\mu)
\]
Numerics: inputs

The most critical input parameters are:

\[ |V_{ts} V_{tb}/V_{cb}|^2 = 0.967 \pm 0.009 \]

\[ BR(B \rightarrow X_c e \bar{\nu}) = 0.1061 \pm 0.0017 \]

\[ m_b^{1S} = (4.69 \pm 0.03) \text{ GeV} \]

\[ m_b^{1S} - m_c^{1S} = (3.41 \pm 0.01) \text{ GeV} \]

\[ C = 0.58 \pm 0.01 \]

New results for the top mass

CDF: \[ M_{t, \text{pole}} = (173.5 \pm 2.7_{\text{stat}} \pm 3.0_{\text{syst}}) \text{ GeV} \]

D0: \[ M_{t, \text{pole}} = (172.1 \pm 5.2_{\text{stat}} \pm 4.9_{\text{syst}}) \text{ GeV} \]

Average: \[ M_{t, \text{pole}} = (173.1 \pm 3.5) \text{ GeV} \]
Numerics: results in the low-s region

✗ Including QED corrections to the Wilson coefficients:

[Bobeth,Gambino,Gorbahn,Haisch]

\[ BR(B \rightarrow X_s l l) = 1.54 \times 10^{-6} \]

✗ Including all QED corrections:

\[
BR(B \rightarrow X_s e e) = (1.63 \pm 0.09_{\text{scale}} \pm 0.07 m_t \pm 0.025 m_c \pm 0.014 m_b \\
\quad \pm 0.015_{\text{CKM}} \pm 0.03_{BR_{sl}} \pm 0.03_{C}) \times 10^{-6}
\]

\[= (1.63 \pm 0.12) \times 10^{-6} \]

\[
BR(B \rightarrow X_s \mu \mu) = (1.57 \pm 0.09_{\text{scale}} \pm 0.07 m_t \pm 0.025 m_c \pm 0.014 m_b \\
\quad \pm 0.015_{\text{CKM}} \pm 0.03_{BR_{sl}} \pm 0.03_{C}) \times 10^{-6}
\]

\[= (1.57 \pm 0.12) \times 10^{-6} \]

✗ Experimental results (e-µ average):

Belle: \( (1.49 \pm 0.50^{+0.41}_{-0.32}) \times 10^{-6} \)

Babar: \( (1.8 \pm 0.7 \pm 0.5) \times 10^{-6} \)

WA: \( (1.60 \pm 0.51) \times 10^{-6} \)
Impact on $C_7 > 0$ scenarios

$x$ Use $\text{BR}(B \to X_s \gamma) = \left( 3.52 \pm 0.3 \right) \times 10^{-4}$ to constrain the Wilson coefficients $C_7$ and $C_8$

$x$ Models with $C_7 > 0$ require sizable contributions to $C_9$ and $C_{10}$
TODO list

✓ Complete prediction in the high-s region

✓ log-enhanced corrections to the Forward-Backward asymmetry

✓ Better understanding of the experimental details of the measurement