Comparison of approaches for inclusive $V_{ub}$ determination

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What progress has been made (a) in the past decade?

**MODEL INDEPENDENCE**

- moved beyond lepton endpoint to theoretically cleaner cuts (hadronic invariant mass, lepton invariant mass, combined cuts, $P_+$, ...)
- SCET et. al.: unravels scales relevant for cut spectra, generalizes shape function analysis beyond leading order, sums Sudakov logs ... theoretical errors now much better understood
What progress has been made (b) in the past few years?

Systematics of theory better understood

- Further development of SCET/subleading theory
  - Perturbative and nonperturbative corrections & uncertainties are better understood.
- New (possibly large) subleading effects discovered
In principle, $V_{ub}$ is as easy as $V_{cb}$:

$$|V_{ub}| = (3.06 \pm 0.08 \pm 0.08) \times 10^{-3} \left( \frac{B(B \to X_u \ell \bar{\nu})}{0.001} \frac{1.6 \text{ ps}}{\tau_B} \right)^{1/2}$$

- 50 MeV uncertainty on $m_b(1S)$
- Perturbative uncertainty

Combine to a ~5% error

- Very clean theoretically: greatest uncertainty is b quark mass ...
- Nonperturbative effects are small

... but this requires cutting out ~100 times larger background from charm
The Classic Method: cut on the endpoint of the charged lepton spectrum

Disadvantages: • only \( \sim 10\% \) of rate
The Classic Method: cut on the endpoint of the charged lepton spectrum

Disadvantages:
- only \(~10\%\) of rate
- sensitivity to fermi motion - local OPE breaks down
Cutting on the hadronic invariant mass spectrum gives more rate, but has the same problem with fermi motion:

\[(Falk, Ligeti, Wise; Dikeman, Uraltsev)\]
But this doesn’t always happen (depends on proximity of cut to perturbative singularities) ... the local OPE holds for the leptonic $q^2$ spectrum:

(Bauer, Ligeti, ML)
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<th>% of rate</th>
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<td>$E_\ell &gt; \frac{m_B^2 - m_D^2}{2m_B}$</td>
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<td>don’t need neutrino</td>
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| “Optimized cut” | ~45% | - insensitive to $f(k^+)$  
 - lots of rate  
 - can move cuts away from kinematic limits and still get small uncertainties | |
| $P_+ > m_D^2/m_B$ | ~70% | - lots of rate  
 - theoretically simplest relation to $b \rightarrow s \gamma$ | |

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Theoretical Issues:

- fermi motion - at leading and subleading order \( (E_\ell, s_H, P_+ \text{ cuts}) \)

- Weak Annihilation (WA) \( (\text{all}) \)

- \( m_b \) - rate is proportional to \( m_b^5 \) - 100 MeV error is a \( \sim 5\% \)
  error in \( V_{ub} \). But restricting phase space increases this
  sensitivity - with \( q^2 \) cut, scale as \( \sim m_b^{10} \) \( (q^2, \text{optimized } q^2 - s_H \text{ cuts}) \)

- perturbative corrections - known (in most cases) to \( O(\alpha_s^2 \beta_0) \)
  - generally under control. When fermi motion is important,
    leading and subleading Sudakov logarithms have been
    resummed. \( (\text{all}) \)
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leading and subleading Sudakov logarithms have been
resummed. \( (all) \)

uncertainty in \( m_b \) is now at 50 MeV level

new insights into all of these in past couple of years
Shape function: issues

\[ f(\omega) \sim \langle B|\bar{b}\delta(\omega - i\hat{D}\cdot n)b|B\rangle \]

universal distribution function (applicable to all decays)

Options:

(i) model

Ex:

\[ f(k^+) = N(1 - x)^a e^{(1+a)x} \]

\( (\text{model}) \)

\( (\text{de Fazio and Neubert}) \)

\( a, N \) determined by \( \bar{\Lambda}, \lambda_1 \) (gets first two moments right .. but the uncertainty in \( f(k^+) \) is not simply given by the uncertainties in \( \bar{\Lambda}, \lambda_1 \))

It is very difficult to determine theoretical uncertainties with this approach!

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NB - the “smearing” approach

\[ d\Gamma = \int d\Gamma_{\text{parton}} \bigg|_{m_b \to m_b + \omega} f(\omega) d\omega \]
NB - the “smearing” approach is not valid beyond tree level ...

\[ d\Gamma = \int d\Gamma_{\text{parton}}\bigg|_{m_b \rightarrow m_b + \omega} f(\omega) d\omega \]

- some of the radiative corrections which are smeared should properly be included in the renormalization of the shape function
- this will cancel out in the relations between spectra, but can introduce large spurious radiative corrections in intermediate results
(ii) Better: determine from experiment: the SAME function determines the photon spectrum in $B \rightarrow X_s\gamma$ (at leading order in $1/m$)

$$\frac{1}{\Gamma} \frac{d\Gamma}{d\hat{E}_\ell}(\bar{B} \rightarrow X_u\ell\bar{\nu}_\ell) = 4 \int \theta(1 - 2\hat{E}_\ell - \omega)f(\omega)\,d\omega + \ldots$$

$$\frac{1}{\Gamma} \frac{d\Gamma}{d\hat{s}_H}(\bar{B} \rightarrow X_u\ell\bar{\nu}_\ell) = \int \frac{2\hat{s}_H^2(3\omega - 2\hat{s}_H)}{\omega^4}\theta(\omega - \hat{s}_H)f(\omega - \hat{\Delta})\,d\omega + \ldots$$

$$\frac{1}{\Gamma} \frac{d\Gamma}{d\hat{E}_\gamma}(\bar{B} \rightarrow X_s\gamma) = 2f(1 - 2\hat{E}_\gamma) + \ldots$$

and so can be measured from the photon spectrum in $\bar{B} \rightarrow X_s\gamma$:

(NB must subtract off contributions of operators other than $O_7$)

![Graph showing photon spectrum](image)
(iii) Best - avoid the shape function altogether, and just relate physical quantities! (leading order shape function cancels out between spectra)

\[ P_\gamma \equiv m_B - 2E_\gamma \]

\[
\text{ex: } \int_0^{m_B} \Delta M d_s H \frac{d\Gamma_u}{ds_H} \propto \frac{|V_{ub}|^2}{|V_{tb}V_{ts}^*|^2} \int_0^\infty dP_\gamma W_{sH}(\Delta M, P_\gamma) \frac{d\Gamma_s}{dP_\gamma}
\]

(Rothstein, Leibovich, Low)
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\[ W \text{ has an expansion in powers of } \alpha_s, \frac{\Lambda_{\text{QCD}}}{m_B}, \text{ with leading term known} \]

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\[ W_{sH}(\Delta_M, P_\gamma) = \theta(\Delta_M - P_\gamma) + \theta(P_\gamma - \Delta_M) \frac{\Delta_M^3 (2P_\gamma - \Delta_M)}{P_\gamma^3} + O(\alpha_s) + O(\Lambda_{\text{QCD}}/m_B) \]
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\[ \int_{m_B}^{m_B} dM_s H \frac{d\Gamma_u}{ds_H} \propto \frac{|V_{ub}|^2}{|V_{tb}V_{ts}^*|^2} \int_0^\infty dP_\gamma W_{s_H}(\Delta M, P_\gamma) \frac{d\Gamma_s}{dP_\gamma} \]

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W has an expansion in powers of \( \alpha_s, \Lambda_{QCD}/m_B \), with leading term known

- (theoretical) systematic errors accumulate when you include intermediate unphysical quantities like the shape function (i.e. large perturbative corrections cancel out between spectra)
- shape function can’t fit true spectra, which have resonances - only makes sense when smeared over resonance region
(Ligeti’s maxim: better data shouldn’t make life harder)

$$\int_{x_i^{\text{cut}}}^{x_i} d x_i \frac{d \Gamma(B \to X_u l \bar{\nu})}{d x_i} = \int d E_\gamma W(x_i^{\text{cut}}, E_\gamma) \frac{d \Gamma(B \to X_s \gamma)}{d E_\gamma}$$

$W$ can be sensibly defined and calculated; the individual spectra can’t.
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\[
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& P_\gamma \equiv m_B - 2E_\gamma
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similarly,

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\text{ex:} & \int_{0}^{\Delta P} dP_+ \frac{d\Gamma_u}{dP_+} \propto \frac{|V_{ub}|^2}{|V_{tb}V_{ts}^*|^2} \int_{0}^{\Delta P} dP_\gamma W_{P^+}(\Delta P_+, P_\gamma) \frac{d\Gamma_s}{dP_\gamma} \\
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(Mannel and Recksiegel; Bosch, Neubert, Lange, Paz)
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\]

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P_+ \equiv m_X - |E_X|
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- \(P_+\) cut requires \(B \rightarrow X_s\gamma\) photon spectrum over a smaller region than \(s_H\) cut
- not a big difference in practical terms (\(W, f(k^+)\) both suppress large \(P_\gamma\) region) but theoretically cleaner

\[
W_{sH}(\Delta M, P_\gamma) = \theta(\Delta M - P_\gamma) + \theta(P_\gamma - \Delta M) \frac{\Delta_3^M (2P_\gamma - \Delta M)}{P_\gamma^3}
\]
Leading logs - to sum, or not?

SCET allows very elegant RGE resummation:

\[ W_{P+}^{\text{NLL}}(\Delta, P_\gamma) = T(a) \left\{ 1 + \frac{C_F \alpha_s(m_b)}{4\pi} H(a) + \frac{C_F \alpha_s(\mu_i)}{4\pi} \left[ 4f_2(a) \ln \frac{m_b(\Delta - P_\gamma)}{\mu_i^2} - 3f_2(a) + 2f_3(a) \right] \right\} \]
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at 2 loops:

\[ W^{(\alpha_s^2)}_{P_+} = \frac{C_F \alpha_s^2(m_b)}{(4\pi)^2} \left[ (0.83\beta_0 + 3.41) \ln^2 \frac{m_b}{\Delta - P_{\gamma}} + (4.67\beta_0 - 19.1) \ln \frac{m_b}{\Delta - P_{\gamma}} - (5.19\beta_0 + c_0) \right] \]

(Hoang, Ligeti and ML)

\[ \mathcal{O}(\log^2) : \mathcal{O}(\log) : \mathcal{O}(\log^0) = 1 : 0.87 : (-0.86 - 0.02c_0) \]

not a good expansion!
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at 2 loops:

- \textit{leading log} \quad \textit{next-to-leading log} \quad \textit{NNLL} \\
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- large Sudakov double logs \( \alpha_s^n \log^m(m_b/\mu), \ m = n + 1, \ldots, 2n \) cancel from \( W \)
- \( \log m_b/\mu \sim \log 3 \) is not large enough to justify leading log expansion - more justified to stick to fixed order perturbation theory (cf summing logs of \( m_c/m_b \) in exclusive \( B \rightarrow D^* \ell \bar{\nu}_\ell \))
W: Nonperturbative corrections

(Bauer, ML and Mannel; Leibovich, Ligeti and Wise; Burrell, ML and Williamson; Stewart and Lee; Mannel and Tackmann; Bosch, Neubert, Paz; Beneke, Campanario, Mannel and Pecjak, ...)

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  • Models give expected magnitude of corrections (naively, $O(\Lambda_{QCD}/m)$ could be 5% or 50%!)
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• Corrections are largest for the $E_l$ endpoint spectrum (but improve as cuts are loosened), better for $s_H$ and $P_+$

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SLAC/INT Workshop
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• Corrections are largest for the $E_l$ endpoint spectrum (but improve as cuts are loosened), better for $s_H$ and $P_+$
• Weak annihilation effects can be large - wide variation in estimates of size
Subleading effects (with small WA):

(a) spectra

- **Leading order (model)**
- **Subleading order (2 models)**
- **charm limit**

(b) integrated spectra

- **Bosch, Neubert, Lange, Paz**
Theoretical Issues:

- Weak annihilation ... in local OPE \((q^2, \text{optimized } q^2 - s_H \text{ cuts})\)

\[
O \left(16\pi^2 \times \frac{\Lambda_{QCD}^3}{m_b^3} \times \frac{f_B}{0.2 \text{ GeV}} \right) \sim 0.03 \left(\frac{B_2 - B_1}{0.1}\right)
\]

\(\sim 3\% \text{ (?? guess!)}\) contribution to rate at \(q^2 = m_b^2\)

- an issue for all inclusive determinations
- relative size of effect gets worse the more severe the cut
- no reliable estimate of size - can test by comparing charged and neutral B’s, comparing D and D_s semileptonic widths, looking for consistency between different determinations
Theoretical Issues:

- Weak annihilation ... in nonlocal OPE  \((E_\ell, s_H, P_+ \text{ cuts})\)

- enhanced in shape function region to \(O(\Lambda_{QCD}/m_b)^2\)
- concentrated in large \(q^2\) region
- can easily be >20% shift to integrated rate for \(E_\ell > 2.3\) GeV (smaller effect for other spectra since more rate included)
Theoretical Issues:

- Weak annihilation ... in nonlocal OPE \((E_\ell, s_H, P_+ \text{ cuts})\)

\[
O \left( 4\pi \alpha_s(\mu_i) \times \frac{\Lambda_{QCD}}{m_b} \times \epsilon \right)
\]

- hard to power count ... estimates of size vary by almost 2 orders of magnitude!

Lee and Stewart: up to 180% of LEADING term for lepton endpoint! (smaller for \(s_H\) and \(P_+\)) - would completely mess up shape function expansion

Bosch, et. al.; Neubert; Beneke et. al.: colour suppression \(\Rightarrow \epsilon << 1 + \text{no factor of } 4 \Rightarrow \text{negligible effect (smaller than other } 1/m \text{ effects)
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- WA effects largest  
- reduced phase space - duality issues? |
| $s_H < m_D^2$ | ~80% | lots of rate | depends on $f(k^+)$ (and subleading corrections)  
- need shape function over large region |
| $q^2 > (m_B - m_D)^2$ | ~20% | insensitive to $f(k^+)$ | very sensitive to $m_b$  
- WA corrections may be substantial  
- effective expansion parameter is $1/m_c$ |
| “Optimized cut” | ~45% | - insensitive to $f(k^+)$  
- lots of rate  
- can move cuts away from kinematic limits and still get small uncertainties | - sensitive to $m_b$ (need +/- 60 MeV for 5% error in best case) |
| $P_+ > m_D^2/m_B$ | ~70% | - lots of rate  
- theoretically simplest relation to $b \to s\gamma$ | depends on $f(k^+)$ (and subleading corrections) |
Bottom line(s):

• there is no “best method” - each has its own sources of uncertainty
  • local OPE: $b \rightarrow c$ experience gives us confidence in framework, but we are pushing things to lower momentum scales for $V_{ub}$ - perturbative, nonperturbative effects are more significant
  • nonlocal OPE: reasonable model estimates suggest things are OK, but no experimental test of framework

• we only believe $V_{cb}$ because of all the checks. Our confidence in $V_{ub}$ will grow if different methods give compatible results (i.e. limits WA contributions)

• experiments can help beat down theoretical uncertainties
  • improved measurement of $B \rightarrow X_s \gamma$ photon spectrum - lowering cut reduces effects of subleading corrections, as well as sensitivity to details of $f(k^+)$
  • test size of WA (weak annihilation) effects - compare $D^0$ & $D_s$ S.L. widths, extract $|V_{ub}|$ from $B^{\pm}$ and $B^0$ separately

• $V_{ub}$ wall is likely to be at the $\sim5\%$ level via these methods, assuming no inconsistencies