Omnès Dispersion Relations for Heavy-to-light Semileptonic Decays

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1. Exclusive semileptonic decays

2. Dispersive techniques
   - Dispersive bounds
   - Omnes dispersion relations

3. A twisted comment
Outline

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Exclusive semileptonic decay

Vector current matrix element

\[ \langle \pi(k)|V_\mu|B(p)\rangle = \]

\[ f^+(q^2) \left( p_\mu + k_\mu - \frac{m_B^2 - m_\pi^2}{q^2} q_\mu \right) + f^0(q^2) \frac{m_B^2 - m_\pi^2}{q^2} q_\mu \]

where

\[ q^2 = (p - k)^2 = m_B^2 + m_\pi^2 - 2m_BE \]

E is pion energy in B rest frame.

\[ f^+(0) = f^0(0) \]

Differential decay rate

\[ \Gamma(B^0 \rightarrow \pi^- l^+ \nu) = \frac{G_F^2 |V_{ub}|^2}{24\pi^3} \int_0^{q_{max}^2} |k_\pi|^3 |f^+(q^2)|^2 \]
$B \rightarrow \pi$ form factors

- $N_f = 0$
  - UKQCD (1999)
  - Abada et al. (2000)
  - El-Khadra et al. (2001)
  - JLQCD (2001)

- $N_f = 2+1$
  - Fermilab (2004)
  - HPQCD (2004)

Hashimoto ICHEP04
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Dispersive bounds

Two-point function

$$\Pi_{\mu\nu}(q^2) = i \int d^4 x \ e^{iq \cdot x} \langle 0 | TV_{\mu}(X) V_{\nu}^+(0) | 0 \rangle$$

$$= \left( \frac{q_\mu q_\nu}{q^2} - g_{\mu\nu} \right) \Pi_T(q^2) + \frac{q_\mu q_\nu}{q^2} \Pi_L(q^2)$$

Dispersion relations, eg:

$$\frac{\partial \Pi_L}{\partial q^2} = \frac{1}{\pi} \int_0^\infty dt \ \frac{\text{Im} \ \Pi_L(t)}{(t - q^2)^2}$$

- LHS: evaluate in pQCD
- RHS: insert states, restrict to $B\pi$ and $B^*$ to get inequalities
Dispersive bounds: history

- Bourrely, Machet, de Rafael NPB189 (1981) 157 showed how to get bounds on $|f^{+,0}(q^2)|$ and further (and now more important) how to input known values $f^{+,0}(q_i^2)$ to obtain strengthened upper and lower bounds.

- Boyd Grinstein Lebed PRL74 (1995) 4603: used inputs from pole forms, BSW, HM $\chi$PT pole-dominance.

- Lellouch NPB479 (1996) 353: used lattice QCD inputs and $f^{+}(0) = f^{0}(0)$.

- Arnesen, Grinstein, Rothstein, Stewart 2005, Stewart CKM2005, Grinstein CKM2005: add constraint from $|V_{ub}|f^{+}(0)$ from SCET/factorisation in $B \to \pi\pi$. 
$f^+(q^2)$

(a) no extra points

$\begin{align*}
\frac{q^2}{m_B^2} & \\
\end{align*}$

(b) $f_{\text{pole}}(q_{\text{max}}^2)$

HM$\chi$PT two points

(c) $f^+(0)_{\text{BSW}}$

(d) $f_{\text{pole}}^+(q_{\text{max}}^2)$

suppressed zero in plot!
Lattice-constrained bounds 1996

\[ f^0(|q^2|) \quad f^+(q^2) \]

Lattice points plus \( f^+(0) = f^0(0) \)

Curves are bounds after including lattice errors
A precision model independent Exclusive $V_{ub}$:

unquenched Lattice (FNAL, HPQCD) + SCET

$B \rightarrow \pi \ell \bar{\nu}$ form factor

$q^2$ relevant for nonleptonic

$\frac{d\Gamma}{dq^2}$

Dispersion relations bound the shape of the form factor

$$\chi^{(n)} \geq \frac{3}{2\pi} \int_{(m_B + m_\pi)^2}^{\infty} dt \ t^{-n-3} k(t) |F(t)|^2$$

(Boyd, Grinstein, Lebed; ...)

(These bounds are also used for excl. $V_{cb}$)

Taken from Iain Stewart at CKM2005
Data (avg. Belle, Babar, Cleo):

J. Dingfelder (WGII)

\[ \text{Br}(B \to \pi \ell \bar{\nu}) = (1.39 \pm 0.12) \times 10^{-4} \]

- \( B \to \pi \ell \bar{\nu} \)  
  (Lattice + SCET+ dispersion)
- \( B \to X_u \ell \bar{\nu} \)  
  (OPE, shape function analysis, HFAG '04 avg.)

\[ |V_{ub}| = 4.08 \pm 0.22 \pm 0.40 \]

\[ |V_{ub}| = 4.70 \pm 0.44 \]

\[ |V_{ub}|(q^2 \geq 16) = (3.87 \pm 0.70 \pm 0.22^{+0.62}_{-0.48}) \times 10^{-3} \text{ (Belle, FNAL)} \]

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Omnès dispersion relations

- Alternative to dispersive bounds technique
- Try to use minimal and model-independent input
  - Mandelstam hypothesis of maximum analyticity
  - Watson’s theorem

\[
\frac{f^+(s + i\epsilon)}{f^+(s - i\epsilon)} = \frac{T(s + i\epsilon)}{T(s - i\epsilon)} = e^{2i\delta(s)}, \quad s > s_{th}
\]

- Needs knowledge of phase shift \(\delta(s)\) in elastic \(B\pi \rightarrow B\pi\) scattering in \(J^P = 1^-\) and isospin-1/2 channel

\[
T(s) = \frac{8\pi is}{\lambda^{1/2}(s)}(e^{2i\delta(s)} - 1)
\]
Omnès: one subtraction

- JMF-Nieves PLB505 (2001) 82 found $\delta(s)$ from $T(s)$ in an on-shell BS scheme with kernel determined by tree-level HM\(\chi\)PT
  - $\delta(s)$ depends on $g_{BB^*\pi}$
- Once-subtracted result

\[
f(q^2) = f(0) \exp \left[ \frac{q^2}{\pi} \int_{s_{\text{th}}}^{\infty} \frac{\delta(s)}{s(s - q^2)} \, ds \right]
\]

- Fitted to lattice data for $f^+$ and $f^0$, with free parameters $f^+(0) = f^0(0)$ and $g_{BB^*\pi}$
- Con: need $\delta(s)$ far above threshold: $\Longrightarrow$ add extra resonances to keep $f^+(0)$ small
Omènes: more subtractions

- Pallante and Pich NPB592 (2001) 294 applied to FSI in $K \rightarrow \pi\pi$ decays: lessened need to know phase-shifts by multiple subtractions at a single point, requiring amplitude and many derivatives

- JMF-Nieves 2005 (in prep): use Omènes with multiple subtractions at many different points — plug in $f^+(q^2)$ wherever you know it
\[ q_{\text{max}}^2 \]

\[ B^* \]

\[ q^2 \]

\[ s_{\text{th}} = (m_B + m_\pi)^2 \]
Omnès representation

\[ f^+(q^2) = \left( \prod_{j=0}^{n} [f^+(q_j^2)]^{\alpha_j(q^2)} \right) \exp \left\{ I_\delta(q^2; \{q_j^2\}) \prod_{k=0}^{n} (q^2 - q_k^2) \right\} \]

\[ I_\delta(q^2; \{q_j^2\}) = \frac{1}{\pi} \int_{s_{th}}^{+\infty} \frac{ds}{(s-q_0^2) \cdots (s-q_n^2)} \frac{\delta(s)}{s - q^2} \]

\[ \alpha_j(q^2) = \prod_{k=0, k\neq j}^{n} \frac{q^2 - q_k^2}{q_j^2 - q_k^2} \]

\[ \alpha_j(q_i^2) = \delta_{ij}, \quad \sum_{k=0}^{n} \alpha_k(q^2) = 1 \]
Explicit formula for $f^+(q^2)$

- With *many* subtractions, only need $\delta(s)$ for $s$ close to $s_{th}$.
- We consider $B^*$ as a bound state with $m_{B^*}$ not far from $s_{th}$.
  Hence $\delta(s) \approx \pi$ in $\mathcal{I}_\delta$.
- Omnès factor $\mathcal{I}_\delta$ can then be integrated exactly . . .

Explicit formula for $f^+(q^2)$

$$f^+(q^2) \approx \frac{1}{s_{th} - q^2} \prod_{j=0}^{n} \left[ f^+(q_j^2)(s_{th} - q_j^2) \right]^{\alpha_j(q^2)}, \quad n \gg 1$$

with (as before)

$$\alpha_j(q^2) = \prod_{k=0, k \neq j}^{n} \frac{q^2 - q_k^2}{q_j^2 - q_k^2}$$
Explicit formula for \( f^+(q^2) \)

- With *many* subtractions, only need \( \delta(s) \) for \( s \) close to \( s_{th} \).
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\]

with (as before)

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\alpha_j(q^2) = \prod_{k=0, k\neq j}^{n} \frac{q^2 - q_k^2}{q_j^2 - q_k^2}
\]
Multiply-subtracted Omnès fit

Fit to lattice data at high $q^2$ and LCSR point at $q^2 = 0$

\[ f^{+(B \to \pi)}(q^2) \]

Graph showing the fit with data points and the formula for $f^{+(B \to \pi)}(q^2)$.
6-subtracted Omnès fit: NRCQM plus LCSR

\[ E_\pi / \text{GeV} \]

\[ q^2 / \text{GeV}^2 \]

Albertus, JMF, Hernández, Nieves, Verde-Velasco in preparation
NRCQM plus LCSR

- Nonrelativistic constituent quark model used at large $q^2$ to give 5 input points in $18 \text{ GeV}^2 \leq q^2 \leq q^2_{\text{max}}$
- Input LCSR value for $f^+(0) = 0.28 \pm 0.05$ from Khodjamirian et al PRD62 (2000) 114002
- Use $\tau_{B^0} = (1.536 \pm 0.014) \times 10^{-12} \text{ s}$ and $B_{\text{exp}}(B^0 \rightarrow \pi^- l^+ \nu_l) = (1.33 \pm 0.22) \times 10^{-4}$ PDG 2004
- Theory error from $g_{BB^*\pi} f_{B^*}$ input to NRCQM and $f^+_{\text{LCSR}}(0)$, plus from changing $q-\bar{q}$ potential in NRCQM
- Find
  \[
  \frac{\Gamma(B^0 \rightarrow \pi^- l^+ \nu_l)}{|V_{ub}|^2} = (0.50 \pm 0.20) \times 10^{-8} \text{ MeV}
  \]
  and
  \[
  |V_{ub}| = 0.0034 \pm 0.0003 \text{ (exp)} \pm 0.0007 \text{ (theory)}
  \]
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Chiral extrapolation

Issue of chiral extrapolation in lattice QCD calculations of heavy-to-light exclusive semileptonic decays.

Consider $D \rightarrow \pi$ case.

BK parametrisation of form factors
Becirevic and Kaidalov PLB478 (2000) 417

\begin{align*}
    f^+(q^2) &= \frac{f(0)}{(1 - q^2/m_D^2)(1 - \alpha q^2/m_D^{2*})} \\
    f^0(q^2) &= \frac{f(0)}{1 - q^2/\beta m_D^{2*}}
\end{align*}
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\[ f^+(q^2) = \frac{f(0)}{(1 - q^2/m_{D^*}^2)(1 - \alpha q^2/m_{D^*}^2)} \]
\[ f^0(q^2) = \frac{f(0)}{1 - q^2/\beta m_{D^*}^2} \]

- satisfies $f^+(0) = f^0(0)$
- satisfies heavy-quark scaling
- incorporates $D^*$ pole
- has parameters for more structure/poles
Chiral extrapolation

Use ‘HQET’ form factors

\[ \langle P | V^\mu | D \rangle = \sqrt{2m_D} [v^\mu f_{||}(E) + p^\mu_\perp f_{\perp}(E)] \]

(E is pion energy)

- interpolate to set of fixed \( E \) using BK
- chiral extrapolation of \( f_{||,\perp} \) (with \( S\chi PT \))
- return to \( f^{+,0} \) and extrapolate to full kinematic range with BK
Twisting

Twisted BC could let you line up the black points (fix $E$), avoiding one use of a form-factor ansatz

Or at least let you calculate at more values of $E$ to reduce ansatz-dependence
Conclusions

- Dispersion techniques maximise usefulness of known values of form factors at a few points.
- A few (but not too few) values at widely-separated points are better than many values very close together.
- Suggested one way twisted boundary conditions could help in lattice calculations of heavy-to-light decay form factors.