From Pairing to BEC: calculations using the exact renormalization group (hep-ph/0406249)

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36,000 students
800 physics UGs
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Outline

1. Introduction
2. Exact Renormalization Group
3. Zero Range Pairing Model
4. Mean Field: Exact Results
5. Numerical Results
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Motivation

- Enormous progress in effective field theories for low energy nuclear/nucleon physics
- Up to few body systems for “real nuclei”.
- Relies on separation of scales.
- Too many similar scales in nuclear matter!
- No real theory of nuclear matter!
- Try to find a simple field-theoretical approach.
Exact Renormalization Group

**Basic Object**
- Work with effective action (see Weinberg, QTF II).
- Legendre transform of usual $W_j = \ln Z_j$.
- Functional of *classical* field.
- Generator of 1PI Green’s functions.

**Idea**
- Introduced by Wetterich [PLB301 (1993) 90].
- Reviews see hep-ph/0005122, cond-mat/0309101.
- Add artificial running (RG) to problem.
Scalar field

For scalar boson field $\phi$ normally use

$$Z_J = \int D\phi \: e^{i(S[\phi]+J\cdot\phi)}.$$

Add a term $\exp(-i/2\phi \cdot R \cdot \phi)$.

$R$ depends on scale $k$, suppresses modes with $q \leq k$.

Quantity to look at is the Legendre transform of $W[J] = -\ln Z_J$,

$$\Gamma[\phi_c] = W[J] - J \cdot \phi_c + \frac{1}{2} \phi_c \cdot R \cdot \phi_c$$

Thus $\phi_c \equiv \delta W[J] / \delta J$.

This implies also that $\delta \Gamma[\phi_c] / \delta \phi_c = J$.

$\Gamma$ is the effective action in the limit $R \to 0$.

Physical interpretation: no source, $\delta \Gamma[\phi_c] / \delta \phi_c = 0$. 

Birse, Krippa, McGovern, Walet

From Pairing to BEC
evolution

We can easily find evolution of $W$ with $k$, and thus $\Gamma$ with $k$ (for constant $\phi_c$, see later),

$$\partial_k \Gamma = \frac{i}{2} \text{Tr} \left[ (\partial_k R) \frac{\delta \phi_c}{\delta J} \right]$$

From definition of $\Gamma$ (and $\phi_c$) we have

$$\frac{\delta J}{\delta \phi_c} = - (\Gamma^{(2)} - R), \quad \Gamma^{(2)} = \frac{\delta^2 \Gamma}{\delta \phi_c \delta \phi_c}$$

We thus get non-perturbative ("exact") one loop structure

$$\partial_k \Gamma = - \frac{i}{2} \text{Tr} \left[ (\partial_k R) (\Gamma^{(2)} - R)^{-1} \right]$$
Fundamental equations

- $\Gamma[\phi_c] = W[J] - J \cdot \phi_c + \frac{1}{2} \phi_c \cdot R_k \cdot \phi_c$
- $R_k$ runs with “scale” $k$.
- $\partial_k \Gamma = \frac{i}{2} \text{Tr}[\partial_k R_F ((\Gamma^{(2)} - R)^{-1})_{FF}] - \frac{i}{2} \text{Tr}[\partial_k R_B ((\Gamma^{(2)} - R)^{-1})_{BB}]$
- 2 by 2 block matrix in case of pairing (ph mixing)
- Solve ERG by parametrisation (can’t solve for functional).

Properties of $R$

- full effective action for $k \rightarrow 0$
- classical action of FT as $k \rightarrow \infty$
- Choose $k$ to add mass-gap for low-energy modes ($q < k$)
- $\partial_k R$ also UV regulator (bonus).
Attractive force for fermions: pairing

- Weak attractions: BCS
- Strong attraction: BEC

- One type of fermion $\psi$ (~neutron matter)
- Chemical potential $\mu$

\[ \mathcal{A} = \int d^4x [\psi^\dagger (i\partial_t + \mu + \frac{\nabla^2}{2M}) \psi + \frac{1}{2} g (\psi^T \sigma_2 \psi)(\psi^\dagger \sigma_2 \psi^T)] \]

- Boson for correlated fermion pairs (and gap) $\phi$ through Hubbard-Stratonovich transformation.
Effective action

\[
\Gamma[\psi, \psi^\dagger, \phi, \phi^\dagger, \mu, k] = \int d^4x \left[ \phi^\dagger \left( Z_\phi (i \partial_t) + \frac{Z_m}{2m} \nabla^2 \right) \phi - U(\phi, \phi^\dagger) 
+ \psi^\dagger \left( Z_\psi (i \partial_t + \mu) + \frac{Z_M}{2M} \nabla^2 \right) \psi 
- Z_g g \left( \frac{i}{2} (\psi^T \sigma_2 \psi) \phi^\dagger - \frac{i}{2} (\psi^\dagger \sigma_2 \psi^T) \phi \right) \right]
\]

- Bosons have become dynamical
- \(U\) contains 2\(\mu\) term
- Many running couplings!
Expand $U$ about equilibrium to 2nd order in constant background

$\rho_0 = \phi_c^\dagger \phi_c$: (Note $\rho_0 \propto \Delta^2$!)

$$U = u_0 + u_1 (\phi^\dagger \phi - \rho_0) + \frac{1}{2} u_2 (\phi^\dagger \phi - \rho_0)^2.$$  

Two phases: symmetric where minimum at $\rho_0 = 0$; condensed phase where $u_1 = 0$ at $\rho_0 \neq 0$. Work at fixed density rather than fixed $\mu$ (want to study BEC, where $\mu < 0$).
Solve system of coupled ODE’s
Basic Diagrams

- $g$
- $u_1$
- $u_2$
- $u_3$
- ?
Here every internal line is a “full” propagator, a box \( \partial_k R \) insertion is required once for each internal line.

- a: \( Z_\psi, Z_m \)
- b: \( u_1, Z_\phi, Z_M \)
- c: \( u_2 + \ldots \)
- d: \( Z_g \)
Structure of equations

Carry out all energy integrals (0-th component) in closed form.

Regulator only contributes to three-momentum integral.

\( R_B = \frac{k^2}{2m} f(q/k) \) (\( f(0) = 1 \), \( f(\infty) = 0 \)).

Use smoothed step function for \( f \):

\[
f(x) = \frac{\text{erf}((x+1)/\sigma) + \text{erf}((x-1)/\sigma)}{2\text{erf}(1/\sigma)}
\]
Fermionic regulator

More tricky; positive for particle state and negative for hole states (since we do not work in a “sensible” particle-hole formalism, but relative to bare vacuum).

We use

$$R_F(q, k, p_F, \mu) = \text{sgn}(q - \mu) \frac{k^2}{2m} f\left(\frac{(q - p_F)}{k}\right)$$

$$p_{\mu} = \sqrt{2M\mu}, \quad p_F = (3\pi^2 n)^{1/3}.$$  
In absence of gap, \(p_F = p_{\mu}\). With gap, Fermi surface will shift (to keep density constant), or disappear completely for BEC.
Initial Conditions

- Follow evolution from large $k = K$ to $k \approx 0$.
- Initial conditions derived from matching to evolution in vacuum only.
- Analytical solution $E_{FR}(q, p_F, k) = (q^2/(2M) - \mu) + R_F$

\[
\begin{align*}
    u_1(K) &= -\frac{M}{4\pi a} + \int \frac{d^3q}{(2\pi)^3} \left[ \frac{1}{E_{FR}(q,0,0)} - \frac{1}{E_{FR}(q,0,K)} \right] \\
    T(p = 0) &= \frac{4\pi a}{M} = -\frac{1}{u_1(0)}
\end{align*}
\]

- Use zero-density result $T(p = 0) = \frac{4\pi a}{M} = -\frac{1}{u_1(0)}$.
- Difference of linearly divergent terms, so extreme fine tuning problem.
Gapped phase

- As soon as $u_1$ hits zero, we start running the gap to keep $u_1$ at zero.
- We also start running $\mu$ to keep $n$ fixed.
- Leads to additional terms in evolution
- Equations are no longer closed! E.g., $\partial_k u_2$ contains $u_3 \partial_k \Delta^2$.
- In mean field we can write down closed expressions (next transparency).
- Use those to close equations (e.g., use $u_3$ from there).
Equations can be solved exactly in approximation where we neglect $\Gamma_{BB}^{(2)} - R$ term in running.

Is just mean field theory.

Marani et al cond-mat/9703160, Papenbrock and Berstch nucl-th/9811077. (And substantially earlier!)

Agrees with numerics (next section)
Effective potential

Full mean field effective potential (no expansion, $k = 0$)

$$U^{MF}(\Delta, \mu) = \frac{p_{\Delta}^5}{2M\pi} \left[ \frac{1}{8ap_{\Delta}} - \frac{1}{15}(1+x^2)^{3/4}P_{3/2}^1\left(-\frac{x}{\sqrt{1+x^2}}\right) \right]$$

$P_l^m(y)$ associated Legendre function; $p_{\Delta} = \sqrt{2M\Delta}, \ x = \mu/\Delta$.

Minimise w.r.t. $\Delta$ at fixed $N$, solve for $\mu$ and $\Delta$.

Log singularity at small $\Delta$ gives small $p_Fa$ result

$$\Delta \approx \frac{8}{e^2\varepsilon_F} \exp\left(-\frac{\pi}{2p_F|a|}\right).$$
Approach to numerics

- Solve ODEs (ignore running of all $Z$'s but boson $Z_{\phi}$).
- Crucial (and difficult point) to study evolution at constant density
- Still slightly preliminary: effect of bosons not fully understood!
- Start in symmetric phase; rather trivial (unphysical) transition to broken phase.
- Studied various $R_{B,F}$'s!
Numerical solution of the evolution equations for infinite $a_0$, starting from $K = 16 \text{ fm}^{-1}$.

Blue: full solution, magenta: “mean field”. Transition to condensed phase at $k_{\text{crit}} = 1.2 \text{ fm}^{-1}$.

- Contribution of boson loops small–tricky point!
- $u_2$ and $Z_{\phi}$ go to zero as $(-\ln(k))^{1/2}$ with boson loops!
Crossover from BCS to BEC

- $p_F$ defined by density!
- Crossover from BCS to BEC.
- Little difference between mean field (red) and bosonic (green) results. :-(
- Problems with convergence in small gap regime.
- Different behaviour at small gap?
Wishlist

- Better understanding of role of Goldstone bosons.
- Complete analysis of $\Gamma$! (Wave function renormalisation constants, and coupling constant).
- Add momentum dependent forces (effective range).
- Treatment of $ph$ channels. Issues with bosonisation?
- Three body forces...