Nuclear structure issues
and double beta decay

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The observation of $\beta\beta 0\nu$ decay would immediately tell us that neutrinos are massive Majorana particles. But without accurate calculations of the nuclear matrix elements that determine the decay rate it would be difficult to reach quantitative conclusions about neutrino masses and hierarchies.
If (or when) the $0\nu\beta\beta$ decay is observed two problems must be resolved:

a) What is the mechanism of the decay, i.e., what kind of virtual particle is exchanged between the affected nucleons (or quarks)?

b) How to relate the observed decay rate to the fundamental parameters, i.e., what is the value of the corresponding nuclear matrix element?
All these diagrams can contribute to the $0\nu \beta\beta$ decay

We will consider this graph

The other graphs might contribute with similar rates if the exchanged heavy particles have $\sim$TeV mass. See Cirigliano et al. hep-ph/0406199 for a possible `diagnostic tool'.
\[ \frac{1}{T_{1/2}} = G(E_0, Z) M_{\text{nucl}}^2 \langle m_{\beta\beta}\rangle^2 \], where \( G(E_0, Z) \) is known phase space factor, \( M_{\text{nucl}} \) is the nuclear matrix element, and \( \langle m_{\beta\beta}\rangle = \sum |U_{ei}|^2 m_i e^{i\alpha(i)} \). The shaded region reflects the spread due to the unknown Majorana phases, the red lines take into account the errors in oscillation parameters.
The decay proceeds between the $0^+$ ground states of the initial and final even-even nuclei, and is governed by the matrix element

$$|M_{0\nu}| \equiv M_{0\nu}^{GT} - \frac{g_N^2}{g_A^2} M_{0\nu}^{SF} = \langle f | \sum_{l,k} H(r_{lk}, A) \tau_i^+ \tau_k^+ \left( \mathbf{\sigma}_i \cdot \mathbf{\sigma}_k - \frac{g_N^2}{g_A^2} \right) |i \rangle \ . \ (1)$$

Here the 'neutrino potential' is

$$H_k(r, A_k) = \frac{2R_N}{\pi r} \int_0^\infty dq \frac{q \sin(qr)}{\omega(\omega + A_k)} \ , \ (2)$$

$$A_{1(2)} = E_m - (M_i + M_f)/2 \pm (\epsilon_1 - \epsilon_2)/2 \ ,$$

To a good approximation

$$H_k(r, A_k) \sim \frac{e^{-1.15 A_{mk} \eta_k}}{\eta_k} \ . \ (3)$$

Short range repulsion are typically taken into account by using

$$f(r) H(r) f(r) \ \text{instead of} \ \ H(r) \ , \ f(r) = 1 - e^{-\gamma r^2}(1 - \gamma_2 r^2) \ . \ (4)$$
## Candidate Nuclei for Double Beta Decay

<table>
<thead>
<tr>
<th>Candidate Nuclei</th>
<th>Q (MeV)</th>
<th>Abund. (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{48}\text{Ca} \rightarrow ^{48}\text{Ti}$</td>
<td>4.271</td>
<td>0.187</td>
</tr>
<tr>
<td>$^{76}\text{Ge} \rightarrow ^{76}\text{Se}$</td>
<td>2.040</td>
<td>7.8</td>
</tr>
<tr>
<td>$^{82}\text{Se} \rightarrow ^{82}\text{Kr}$</td>
<td>2.995</td>
<td>9.2</td>
</tr>
<tr>
<td>$^{96}\text{Zr} \rightarrow ^{96}\text{Mo}$</td>
<td>3.350</td>
<td>2.8</td>
</tr>
<tr>
<td>$^{100}\text{Mo} \rightarrow ^{100}\text{Ru}$</td>
<td>3.034</td>
<td>9.6</td>
</tr>
<tr>
<td>$^{110}\text{Pd} \rightarrow ^{110}\text{Cd}$</td>
<td>2.013</td>
<td>11.8</td>
</tr>
<tr>
<td>$^{116}\text{Cd} \rightarrow ^{116}\text{Sn}$</td>
<td>2.802</td>
<td>7.5</td>
</tr>
<tr>
<td>$^{124}\text{Sn} \rightarrow ^{124}\text{Te}$</td>
<td>2.228</td>
<td>5.64</td>
</tr>
<tr>
<td>$^{130}\text{Te} \rightarrow ^{130}\text{Xe}$</td>
<td>2.533</td>
<td>34.5</td>
</tr>
<tr>
<td>$^{136}\text{Xe} \rightarrow ^{136}\text{Ba}$</td>
<td>2.479</td>
<td>8.9</td>
</tr>
<tr>
<td>$^{150}\text{Nd} \rightarrow ^{150}\text{Sm}$</td>
<td>3.367</td>
<td>5.6</td>
</tr>
</tbody>
</table>

**Issues include:** Q value, abundance, ease of purification (chemical and isotopic), radioactivity (incl. cosmogenesis), & experimental ease of use.
Table 1: Summary of experimentally measured $2\nu\beta\beta$ half-lives and matrix elements ($^{136}\text{Xe}$ is an important exception where a limit is quoted).

<table>
<thead>
<tr>
<th>Isotope</th>
<th>$T_{1/2}^{2\nu}$ (y)</th>
<th>References</th>
<th>$M_{GT}^{2\nu}$ (MeV$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{48}\text{Ca}$</td>
<td>$(4.2 \pm 1.2) \times 10^{19}$</td>
<td>BAL96,BRU00</td>
<td>0.05</td>
</tr>
<tr>
<td>$^{76}\text{Ge}$</td>
<td>$(1.3 \pm 0.1) \times 10^{21}$</td>
<td>KLA01a,AVI91,AAL96</td>
<td>0.15</td>
</tr>
<tr>
<td>$^{82}\text{Se}$</td>
<td>$(9.2 \pm 1.0) \times 10^{19}$</td>
<td>ELL92,ARN98</td>
<td>0.10</td>
</tr>
<tr>
<td>$^{96}\text{Zr}$</td>
<td>$(1.4^{+3.5}_{-0.5}) \times 10^{19}$</td>
<td>ARN99,KAW93,Wieser01</td>
<td>0.12</td>
</tr>
<tr>
<td>$^{100}\text{Mo}$</td>
<td>$(8.0 \pm 0.6) \times 10^{18}$</td>
<td>DAS95,EJI91a,EJI91c,DES97,ALS97,ASH01</td>
<td>0.22</td>
</tr>
<tr>
<td>$^{116}\text{Cd}$</td>
<td>$(3.2 \pm 0.3) \times 10^{19}$</td>
<td>ARN96,DAN00,EJI95</td>
<td>0.12</td>
</tr>
<tr>
<td>$^{128}\text{Te}$</td>
<td>$(7.2 \pm 0.3) \times 10^{24}$</td>
<td>BER93,CRU93</td>
<td>0.025</td>
</tr>
<tr>
<td>$^{130}\text{Te}$</td>
<td>$(2.7 \pm 0.1) \times 10^{21}$</td>
<td>BER93</td>
<td>0.017</td>
</tr>
<tr>
<td>$^{136}\text{Xe}$</td>
<td>$&gt; 8.1 \times 10^{20}$ (90% CL)</td>
<td>GAV00</td>
<td>&lt;0.03</td>
</tr>
<tr>
<td>$^{150}\text{Nd}$</td>
<td>$7.0^{+11.8}_{-0.3} \times 10^{18}$</td>
<td>DES97,ART95</td>
<td>0.07</td>
</tr>
<tr>
<td>$^{238}\text{U}$</td>
<td>$(2.0 \pm 0.6) \times 10^{21}$</td>
<td>TUR91</td>
<td>0.05</td>
</tr>
</tbody>
</table>

(1) deduced from the geochemically determined half-life ratio $^{128}\text{Te}/^{130}\text{Te}$
(2) geochemical result includes all decay modes; other geochemical determinations only marginally agree
(3) radiochemical result, again for all decay modes

$$M_{GT}^{2\nu} = \sum m \frac{\langle f | \hat{\sigma} \tau^+ | m \rangle \cdot \langle m | \hat{\sigma} \tau^+ | i \rangle}{E_m - (M_i + M_f)/2}$$
Spread of calculated m.e. for $0\nu$ decay
Histogram from Bahcall et al. hep-ph/0403167
Two basic ways to evaluate $M^{0\nu}$:

a) Shell model (SM) - limited number of s.p. states but all (or most) configurations of valence nucleons included. Spectroscopy of low-lying states well described. Problem: what is the effect of far away s.p. states?

b) Quasiparticle random phase approximation (QRPA) - easy to include many s.p. states, but only simple configurations are included. Only some spectroscopy data can be checked, e.g. the $2\nu$ decay rate.

Main question: which method is more correct, or less wrong?
Shell Model Problem

- Define a valence space
- Derive an effective interaction
  \[ \mathcal{H}\psi = E\psi \rightarrow \mathcal{H}_{\text{eff}}\psi_{\text{eff}} = E\psi_{\text{eff}} \]
- Build and diagonalize the Hamiltonian matrix.

In principle, all the spectroscopic properties are described simultaneously (Rotational band AND \(\beta\) decay half-life).

F. Nowacki
$T_{1/2}$ fixed, thus $\langle m_\nu \rangle \sim 1/M_{\text{nucl}}$

F. Nowacki, preliminary, private comm
### $\beta$ decay systematics

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>$^{128}Sn$</th>
<th>$^{130}Sn$</th>
<th>$^{132}Sb$</th>
<th>$^{132}Te$</th>
<th>$^{133}Te$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transition</td>
<td>$0^+ \to 1^+$</td>
<td>$0^+ \to 1^+$</td>
<td>$4^+ \to 3, 4, 5^+$</td>
<td>$0^+ \to 1^+$</td>
<td>$3^+_2 \to \frac{1}{2}, \frac{3}{2}, \frac{5}{2}^+$</td>
</tr>
<tr>
<td>$T_{1/2}^{\text{exp.}}$</td>
<td>59.07m</td>
<td>3.72m</td>
<td>2.79m</td>
<td>3.2d</td>
<td>12.5m</td>
</tr>
<tr>
<td>$T_{1/2}^{\text{calc.}}$ (0.74)</td>
<td>32.21m</td>
<td>2.47m</td>
<td>1.56m</td>
<td>1.73d</td>
<td>6.42m</td>
</tr>
<tr>
<td>Renorm.</td>
<td>0.54</td>
<td>0.6</td>
<td>0.55</td>
<td>0.54</td>
<td>0.53</td>
</tr>
</tbody>
</table>

| | $^{134}Te$ | $^{135}Xe$ | $^{136}Cs$ |
| | $0^+ \to 1^+$ | $\frac{3}{2}^+ \to \frac{1}{2}, \frac{3}{2}, \frac{5}{2}^+$ | $5^+ \to 4, 5, 6^+$ |
| | 41.8m | 9.14h | 13.16d |
| | 29.19m | 7.07h | 8.1d |
| | 0.62 | 0.63 | 0.57 |

Our valence space leads to a renormalization of the $\sigma\tau$ operator of a factor 0.57.
2ν half-lifes

<table>
<thead>
<tr>
<th>Parent nuclei</th>
<th>$^{48}\text{Ca}$</th>
<th>$^{76}\text{Ge}$</th>
<th>$^{82}\text{Se}$</th>
<th>$^{130}\text{Te}$</th>
<th>$^{136}\text{Xe}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{1/2}^{2\nu}(g.s.)$ th.</td>
<td>3.7E19</td>
<td>1.15E21</td>
<td>3.4E19</td>
<td>4E20</td>
<td>6E20</td>
</tr>
<tr>
<td>$T_{1/2}^{2\nu}(g.s.)$ exp</td>
<td>4.2E19</td>
<td>1.4E21</td>
<td>8.3E19</td>
<td>2.7E21</td>
<td>&gt; 8.1E20</td>
</tr>
</tbody>
</table>

$2\nu$ strength function in $^{130}\text{Te}$ and $^{136}\text{Xe}$

F.Nowacki, preliminary, private comm
QRPA procedures:

a) Solve the BCS equations and get $u_i$ and $v_i$. Adjust $g_{\text{pair}}$ to get experimental gaps.

b) For each multipole (in the intermediate odd-odd nucleus) solve the QRPA eqs. Adjust relevant parameters, e.g. to get the giant GT resonance at the correct energy ($g_{\text{ph}}$), or to get the $2\nu$ rate ($g_{\text{pp}}$).

c) For “realistic” interaction the QRPA eqs. for the $1^+$ multipole are near QRPA instability. Potential problem.
dependence of the $2\nu$ matrix element on the strength of the
isoscalar ($\lambda^{10}$) and isovector ($\lambda^{10}$) pairing for $^{48}\text{Ca}$, shell model, $fp$ shell

\begin{align*}
\lambda^{01} &= 1.0, \quad \lambda^{10} = 0.8 \\
|M^{2\nu}| &= 0.093 \\
S(\beta^+) &= 1.44 \\
|M_{\text{closed}}| &= 0.556 \\
\end{align*}

\begin{align*}
\lambda^{01} &= 0.9, \quad \lambda^{10} = 0.9 \\
|M^{2\nu}| &= 0.061 \\
S(\beta^+) &= 1.37 \\
|M_{\text{closed}}| &= 0.211 \\
\end{align*}

\begin{align*}
\lambda^{01} &= 1.0, \quad \lambda^{10} = 1.0 \\
|M^{2\nu}| &= 0.081 \\
S(\beta^+) &= 1.26 \\
|M_{\text{closed}}| &= 0.288 \\
\end{align*}

\begin{align*}
\lambda^{01} &= 0.9, \quad \lambda^{10} = 1.0 \\
|M^{2\nu}| &= 0.054 \\
S(\beta^+) &= 1.28 \\
|M_{\text{closed}}| &= 0.078 \\
\end{align*}

\begin{align*}
\lambda^{01} &= 1.1, \quad \lambda^{10} = 1.1 \\
|M^{2\nu}| &= 0.046 \\
S(\beta^+) &= 1.20 \\
|M_{\text{closed}}| &= 0.063 \\
\end{align*}

\[\langle ab|V|c,d\rangle_{JT=0,1} \rightarrow \lambda^{01} \langle ab|V|c,d\rangle_{JT=0,1}\]

\[\langle ab|V|c,d\rangle_{JT=1,0} \rightarrow \lambda^{10} \langle ab|V|c,d\rangle_{JT=1,0}\]
Points to consider:

a) Does it matter how many s.p. one includes?

b) Does it matter which parametrization of the $G$-matrix one uses?

c) Does it matter whether the $g_A$ is quenched?

d) Does it matter whether one uses one of the generalization of QRPA?

e) What is the best way to fix adjustable parameters?

For attempt to answer some of these questions see Rodin et al. Phys. Rev. C68, 044302 (2003)
Matrix elements $M^{0\nu}$ are calculated for different sets of s.p. states (from 2 $h\omega$ to 4 $h\omega$), and different parametrizations of the G-matrix.

For each case pairing is adjusted to the same gap. In addition, for each case $g_{pp}$ is adjusted so that $M^{2\nu}$ has the same experimental value.

The calculation is repeated with QRPA and “renormalized QRPA”. Also, the calculation is repeated for $g_A=1.26$ and $g_A=1$.

Since (at least in this work) the resulting $M^{0\nu}$ are quite similar, the results are averaged, giving also a spread.
When $M^{2\nu}$ is fixed, instead of $g_{pp}$, $M^{0\nu}$ becomes independent on the number of levels.
From Rodin et al. and Rodin, private comm.
Rodin, private comm.
Multipole decomposition for $^{76}$Ge, 12 and 21 s.p. levels

Rodin, private comm
Multipole decomposition for $^{100}$Mo, 13 and 21 s.p. levels, note that 1+ has an opposite sign here

Rodin, private comm
Comparison of $M^{0\nu}$ of Rodin et al. (RQRPA) and Nowacki et al. (SM, private comm., preliminary)

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>RQRPA</th>
<th>SM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{76}\text{Ge}$</td>
<td>2.3-2.4</td>
<td>1.6</td>
</tr>
<tr>
<td>$^{82}\text{Se}$</td>
<td>1.9-2.1</td>
<td>1.7</td>
</tr>
<tr>
<td>$^{96}\text{Zr}$</td>
<td>0.3-0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>$^{100}\text{Mo}$</td>
<td>1.1-1.2</td>
<td>0.3</td>
</tr>
<tr>
<td>$^{116}\text{Cd}$</td>
<td>1.2-1.4</td>
<td>1.9</td>
</tr>
<tr>
<td>$^{130}\text{Te}$</td>
<td>1.3</td>
<td>2.0</td>
</tr>
<tr>
<td>$^{136}\text{Xe}$</td>
<td>0.6-1.0</td>
<td>1.6</td>
</tr>
</tbody>
</table>

Except for $^{100}\text{Mo}$ the agreement between these very different calculations is reasonably good.
Spread of calculated m.e. for $0\nu$ decay
Histogram from Bahcall et al. hep-ph/0403167
Attempts to include the effects of far away single-particle states

This can be done by properly defining the effective operators (e.g. in the shell model), or by including such states explicitly (e.g. in QRPA).

We made at attempt to shed light on the issue by performing calculations in an exactly solvable semi-realistic model.
SU(5) Model: two levels \((fpg_{9/2} \text{ and } sdg_{7/2})\) at energy \(\epsilon\):

\[ H = \epsilon \hat{N}_2 - G \sum_{a,b=1}^{2} (S_{pp}^{\dagger} S_{pp}^{a} + S_{nn}^{\dagger} S_{nn}^{b} + g_{pp} S_{pp}^{\dagger} S_{pp}^{b} + g_{ph} \vec{T} \cdot \vec{T}) , \]  

(1)

\(S\) are isovector pair operators,

\[ S_{pp}^{\dagger} = \frac{1}{2} \sum_{\alpha \in a} \hat{j}_{\alpha} [\pi_{\alpha}^{\dagger} \pi_{\alpha}]^{0} , \quad S_{nn}^{\dagger} = \frac{1}{2} \sum_{\alpha \in a} \hat{j}_{\alpha} [\nu_{\alpha}^{\dagger} \nu_{\alpha}]^{0} , \quad S_{pn}^{\dagger} = \frac{1}{\sqrt{2}} \sum_{\alpha \in a} \hat{j}_{\alpha} [\pi_{\alpha}^{\dagger} \nu_{\alpha}]^{0} . \]  

(2)

When \(g_{pp} \neq 1\) isospin is violated (analog of the SU(4) violation)

The double beta decay operators (Fermi) are:

\[ M_{\kappa} = \sum_{i,j} O_{\kappa}(i, j) \tau_{i}^{\dagger} \tau_{j}^{\dagger} , \quad O_{2\nu}(i, j) = 1 , \quad O_{0\nu}(i, j) = \frac{1}{|\vec{r}_{i} - \vec{r}_{j}|} , \]  

(3)

Effective two body operators are obtained from

\[ \langle \bar{\Psi}_{k} | M_{\kappa}^{eff} | \bar{\Psi}_{l} \rangle = \langle \Psi_{k} | M_{\kappa} | \Psi_{l} \rangle \]  

and the ‘effective charges’ are determined in the two (or two + four) particle systems.
What effective operator does?
Full black – exact in full space, dashed green – bare operator in small space,
dotted blue – fitted in two nucleon system, red – two + four nucleon system
SU(5) Model: two levels ($fpg_{9/2}$ and $sdg_{7/2}$) at energy $\epsilon$; comparison with QRPA:

In our model the interaction acts only in $0^+$ intermediate states. For all other multipoles $0\nu$ expressions reduce to their BCS form. The final result can be written as

$$\langle 0_f | M_\kappa | 0_i \rangle_{QRPA} = (0^+ \text{ part})$$

$$+ \sum_{\alpha, \beta, J} u_{\alpha n}^J u_{\beta p}^J u_{\beta p}^J u_{\alpha n}^J ([\beta \beta]^J | O_\kappa | [\alpha \alpha]^J) \times \left[ \hat{j}_{\alpha}^J \hat{j}_{\beta}^J \delta_{J,0} - \frac{\hat{j}_{\alpha}^2 \delta_{\alpha,\beta}}{\hat{j}_{\alpha}^2} \right],$$

where the "$0^+$ part" refers to the matrix element when only the $0^+$ multipole in the particle-hole decomposition of the operator is included. For $2\nu$ decay the rest of the expression vanishes, but for $0\nu$ decay there is a simple contribution from the other multipoles that is independent of the proton-neutron interaction.
Full lines - exact, dashed lines QRPA, lines labeled by the distance between shells $\epsilon$, at $g_{pp} = 1$ isospin symmetry is restored and $M_{2\nu}$ vanishes, but $M_{0\nu}$ is nonvanishing
When $g_{pp}$ is eliminated, one can plot $M_{ov}$ vs $M_{2v}$.

$G$ renormalized means that the $G$ is adjusted to give the same gap in small and big spaces. Note that $\varepsilon = 40$ is close to the realistic value.
Conclusions

• I presented arguments against taking the spread of all calculations as a measure of the uncertainty in the nuclear matrix element.

• I have also shown that, by insisting that the $2\nu$ decay is described properly, the dependence within QRPA and its generalizations on many external choices is greatly reduced.

• I have also shown that with such procedure the QRPA and SM agree much better.

• Finally, in a simple model that allows exact solution, QRPA works very well and simple effective operators improve the result (but are not perfect).