Mean-field study of nuclear structure
with semi-microscopic interaction

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Ref.: H.N., P.R.C 68, 014316 (’03)
H.N. & M. Sato, N.P.A 699, 511 (’02); 714, 696 (’03)
I. Introduction

Properties of unstable nuclei • important topic of today’s nuclear physics
(and also of astrophysics!)

A basic question: How to understand exotic properties of unstable nuclei?
— in a unified manner together with stable nuclei

Mean-field approximation • good 1st approx. to nuclear structure

• saturation properties \{ basis to understand
• shell structure \[ fundamental properties of nuclei

\[ \downarrow \]
• s.p. orbits • necessity of MF \[ ⇒ important also in studying unstable nuclei

Mean-field in unstable nuclei?

• s.p. potential — expressed by an appropriate function?

  \textit{e.g.} Woods-Saxon potential (central part) \[ V(r) = \frac{V_0}{1 + \exp\left(\frac{r-R}{a}\right)} \]

  \[ R \leftrightarrow \text{radius}, \quad a \sim 0.65 \text{ fm} \leftrightarrow E \sim -8 \text{ MeV} \quad \leftarrow \text{‘self-consistency’} \ V(r) \propto \rho(r) \]

  \[ R, a \text{ for unstable nuclei? — not known} \]

  \[ ⇒ \text{self-consistent theory \textit{(e.g. HF)} \}

• HF & HFB interaction (non-relativistic, hereafter)
  — conventionally used int. & parameters good enough?
Conventional MF int. (Skyrme, Gogny) — parameters fitted mainly to stable nuclei

⋯ applicability to unstable nuclei?

- **Skyrme int.**
  
  lots of parameter-set — which one is recommended?
  
  zero-range form ⋯ justified by DME for stable nuclei
  
  — well tested for unstable nuclei?

- **Gogny int.**
  
  D1S: standard parameter-set → unphysical true minimum
  
  (problematic particularly in \( n \)-rich nuclei)

⇒ reinvestigate effective int.

**Fully microscopic int.** (e.g. G-matrix) — not yet suitable for MF calculations

- quantitative understanding of saturation — insufficient

- origin of LS splitting? ⋯ significant contribution of higher-order terms?

  Ref.: K. Suzuki et al., P.R.C 36, 804 (’87)
  
  S. C. Pieper & V. R. Pandharipande, P.R.L 70, 2541 (’93)

  — desirable to incorporate at the MF order

⇒ “semi-microscopic” approach

  importing both microscopic & phenomenological information
II. M3Y-type effective interaction & nuclear matter properties

Basic properties of low-energy effective int.?

← microscopic & phenomenological information

Nuclear structure models

- Shell model
- Cluster model
- Mean-field models
  - RPA

Nuclear reaction models

- Folding model
- DWBA

Microscopic theories

Phenomenology

Basic characters?

(tuning of parameters, at last)
Our strategy ... M3Y interaction ⇒ modify phenomenologically

- connection to microscopic theory: M3Y int. ← fit to $G$-matrix
  problems of original M3Y int. (← $G$-matrix)
  1) saturation  2) LS splitting ⇒ modification required

- connection to other models:
  DDM3Y in folding model
  D. T. Khoa et al., P.R.C 56, 954 ('97)

  shell model — USD vs. M3Y
  B. A. Brown et al., Ann. Phys. 182, 191 ('88)
Interaction form:

\[ v_{12} = v^C_{12} + v^LS_{12} + v^TN_{12} + v^DD_{12}; \]

\[ v^C_{12} = \sum_n (t^n_{SE} P_{SE} + t^n_{TE} P_{TE} + t^n_{SO} P_{SO} + t^n_{TO} P_{TO}) f_n^C(r_{12}), \]

\[ v^LS_{12} = \sum_n (t^n_{LSE} P_{TE} + t^n_{LSO} P_{TO}) f_n^{LS}(r_{12}) L_{12} \cdot (s_1 + s_2), \quad L_{12} = r_{12} \times p_{12} \]

\[ v^TN_{12} = \sum_n (t^n_{TNE} P_{TE} + t^n_{TNO} P_{TO}) f_n^{TN}(r_{12}) r_{12}^2 S_{12}, \quad S_{12} : \text{tensor operator} \]

\[ v^DD_{12} = t^{DD}(1 + x^{DD} P_{\sigma})[\rho(r_1)]^\alpha \delta(r_{12}). \]

\[ f_n(r) = e^{-\mu_n r}/\mu_n r \text{ for M3Y-type int.} \quad \mu_n : \text{range parameter} \]

Ref.: G. Bertsch et al., N.P.A 284, 399 (’77); N. Anantaraman et al., N.P.A 398, 269 (’83)

cf. \[ f_n^{C,LS}(r) = \delta(r), \nabla^2 \delta(r) \rightarrow \text{Skyrme int.} ; \quad f_n^C(r) = e^{-(\mu_n r)^2}, f_n^LS = \nabla^2 \delta(r) \rightarrow \text{Gogny int.} \]

- \[ v^DD_{12} \text{ introduced with contact form} \]
  (original M3Y \cdots \quad v^DD_{12} = 0 \rightarrow \text{unable to produce saturation})

- \[ \text{mainly focus on central part;} \]
  LS/tensor part enhanced/quenched only by an overall factor (\rightarrow \text{future problem})

- \[ \text{longest-range term of } v^C_{12} \quad \text{kept to be OPEP form} \]
Parameter-sets: Ref. H.N., P.R.C 68, 014316 (’03)

‘M3Y-P0’: original version based on Paris $NN$ int. — no saturation
    N. Anantaraman et al., N.P.A 398, 269 (’83)

(‘M3Y-P1’: naive modification → saturation
    a part of shortest-range central force → $\rho$-dep. contact force)

‘M3Y-P2’: including some adjustment to finite (doubly magic) nuclei
    saturation — TE channel primarily responsible
    moderately reproduce biniding energies of doubly magic nuclei
        (⋯ shown later)

    LS $\rightarrow \times 1.8$, tensor $\rightarrow \times 0.12$ (: $^{208}$Pb s.p. levels)

    still keeping OPEP part
Nuclear matter properties at & around saturation point ↔ $v_{12}^C + v_{12}^{DD}$ ↔ gross properties

Nuclear matter energy \( E \equiv E/A \); \( E = \mathcal{E}(\rho, \eta_t, \eta_s, \eta_{st}) \)

\[
\rho = \sum_{\sigma} \rho_{\tau\sigma} = \rho_{p\uparrow} + \rho_{p\downarrow} + \rho_{n\uparrow} + \rho_{n\downarrow}; \quad \eta_s = \left( \sum_{\sigma} \sigma \rho_{\tau\sigma} \right) / \rho = (\rho_{p\uparrow} - \rho_{p\downarrow} + \rho_{n\uparrow} - \rho_{n\downarrow}) / \rho;
\]

\[
\eta_t = \left( \sum_{\sigma} \tau \rho_{\tau\sigma} \right) / \rho = (\rho_{p\uparrow} + \rho_{p\downarrow} - \rho_{n\uparrow} - \rho_{n\downarrow}) / \rho; \quad \eta_{st} = \left( \sum_{\sigma} \sigma \tau \rho_{\tau\sigma} \right) / \rho = (\rho_{p\uparrow} - \rho_{p\downarrow} - \rho_{n\uparrow} + \rho_{n\downarrow}) / \rho
\]

1. Saturation point \( \rho_0 \ & \ & \mathcal{E}_0 \ (= -b_{\text{vol}}) \ldots \min. \ of \ \mathcal{E} \)

\[
\left. \frac{\partial \mathcal{E}}{\partial \rho} \right|_{\text{sat.}} = 0 \rightarrow \rho_0, \ \mathcal{E}_0; \quad \begin{cases} \rho_0 \approx 0.16 \text{fm}^{-3} \leftarrow \text{density dist.} \ & \ & \text{radii} \\
\mathcal{E}_0 \approx -16 \text{MeV} \leftarrow \text{binding energies} \end{cases}
\]

origin of saturation? (← micro. theory)

H. A. Bethe, Ann. Rev. Nucl. Sci. 21, 93 (’71)

- SE channel: attractive for low \( q \), repulsive for high \( q \) (due to ‘hard core’)
  \( \rightarrow \) repulsion at \( k_F \gtrsim 2 \text{fm} \) (\( q \): mom. transfer)

- TE channel: attractive for low \( q \), second-order effects of tensor force
  \( \rightarrow \) repulsion at \( k_F \gtrsim 1.5 \text{fm} \ldots \) primarily responsible for saturation
Symmetric nuclear matter:

Neutron matter:

Contribution of SE & TE channels:
A problem of Gogny D1S int. — collapse in $n$-matter \( (\mathcal{E} \to -\infty \text{ as } \rho \to \infty) \)

$\to$ unphysical true minimum in finite nuclei

\[ n \& p \text{ concentrate separately} \]

This config. seems to have been avoided practically in stable nuclei, but could be serious in numerical cal. for $n$-rich nuclei.

why? \[ x^{DD} = 1 \to \text{no } \rho\text{-dep. in SE channel} \]
\[ \Rightarrow \text{quick fix} \to x^{DD} < 0.94 \quad (\leftarrow \text{no unphys. min. in } n\text{-matter}) \]
keeping \( \mathcal{E} \) of sym. nucl. matter, \( a_t \) (sym. energy), \( \Delta(\epsilon_F) \to \text{`D1S}_{\alpha'} \)
2. Landau-Migdal parameters around saturation point (↔ curvature, etc.)

\[ v_{12}(k_1 \approx k_2 \approx k_{F0}) \approx N_0 \sum_\ell [f_\ell + f'_\ell (\tau_1 \cdot \tau_2) + g_\ell (\sigma_1 \cdot \sigma_2) + g'_\ell (\sigma_1 \cdot \sigma_2)(\tau_1 \cdot \tau_2)] P_\ell (\cos \theta_{12}) \]

\( N_0 : \) level density at the Fermi surface, \( \cos \theta_{12} \equiv \hat{k}_1 \cdot \hat{k}_2 \)

- \( f_1 \leftrightarrow \) effective mass (\( k \)-mass) \( M^* \) \( \Rightarrow M^* \approx 0.7M \leftrightarrow \) s.p. level spacing for \( |\varepsilon - \varepsilon_F| \gtrsim 20 \text{MeV} \), \( E_x \) of GQR

- \( f_0 \leftrightarrow \) incompressibility \( \mathcal{K} = 9 \left. \frac{\partial^2}{\partial \rho^2} \mathcal{E} \right|_{\text{sat.}} \left( = \frac{3k_{F0}^2}{M^*}(1 + f_0) \right) \) \( \mathcal{K} \approx 210 - 230 \text{MeV} \) (non-rel. models) \( \leftrightarrow E_x \) of GMR

- \( f'_0 \leftrightarrow \) volume sym. energy \( a_t = \frac{1}{2} \left. \frac{\partial^2}{\partial \eta_t^2} \mathcal{E} \right|_{\text{sat.}} \left( = \frac{k_{F0}^2}{6M^*}(1 + f'_0) \right) \) \( a_t(= b_{\text{sym}}) \approx 30 \text{MeV} \leftrightarrow \) binding energy of \( Z < N \) nuclei
cf. surface sym. energy

- \( g'_0 \approx 0.9 - 1.2 \left( \frac{k_{F0}^2}{6M^*}(1 + g'_0) = \frac{1}{2} \left. \frac{\partial^2}{\partial \eta_s^2} \mathcal{E} \right|_{\text{sat.}} \right) \leftrightarrow E_x \) of GTR
cf. in pion unit, \( g'_{NN} \approx 0.6 \) Ref.: T. Suzuki & H. Sakai, P.L.B 455, 25 ('99)

- \( g_0 \lesssim 0.5 ? \left( \frac{k_{F0}^2}{6M^*}(1 + g_0) = \frac{1}{2} \left. \frac{\partial^2}{\partial \eta_s^2} \mathcal{E} \right|_{\text{sat.}} \right) \leftrightarrow E_x \) of spin excitation

- \( f'_1 \leftrightarrow \kappa_{\text{GDR}}, g'_{\ell} \text{ (in } T(M1)) \) \( \kappa_{\text{GDR}} \approx 0.4 - 0.7 ? \)
Comparison of nucl. matter properties:

<table>
<thead>
<tr>
<th></th>
<th>SLy5</th>
<th>D1S</th>
<th>M3Y-P1</th>
<th>M3Y-P2</th>
<th>Exp.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_{F0}$ (fm$^{-1}$)</td>
<td>1.330</td>
<td>1.334</td>
<td>1.358</td>
<td>1.340</td>
<td>1.3 – 1.4</td>
</tr>
<tr>
<td>$E_0$ (MeV)</td>
<td>−15.98</td>
<td>−16.01</td>
<td>−15.99</td>
<td>−16.14</td>
<td>≈ −16</td>
</tr>
<tr>
<td>$K$ (MeV)</td>
<td>229.9</td>
<td>202.9</td>
<td>225.7</td>
<td>220.4</td>
<td>210 – 230</td>
</tr>
<tr>
<td>$M^*/M$</td>
<td>0.697</td>
<td>0.697</td>
<td>0.641</td>
<td>0.652</td>
<td>≈ 0.7</td>
</tr>
<tr>
<td>$a_t$ (MeV)</td>
<td>32.03</td>
<td>31.12</td>
<td>30.35</td>
<td>30.61</td>
<td>≈ 30</td>
</tr>
<tr>
<td>$g_0$</td>
<td>1.123</td>
<td>0.466</td>
<td>0.046</td>
<td>0.113</td>
<td>≲ 0.5?</td>
</tr>
<tr>
<td>$g'_0$</td>
<td>−0.141</td>
<td>0.631</td>
<td>0.891</td>
<td>1.006</td>
<td>0.9 – 1.2</td>
</tr>
</tbody>
</table>

SLy5: Skyrme int. with LS current contribution

$g_0 \ll g'_0 \sim 1 \cdots$ not necessarily obtained with conventional int.

M3Y-type int. give reasonable values

← significant contribution of OPEP to $g'_0$
III. Mean-field calculations with M3Y-type interaction in finite nuclei

**Numerical method** — **“Gaussian expansion method”** (GEM)

developed by Kamimura *et al.* for few-body calculations

**Basis:** \( \varphi_{\alpha \ell jm}(r) = R_{\alpha \ell j}(r)[Y^{(\ell)}(\hat{r})\chi_{\sigma}]_{m}^{(j)} \); \( R_{\alpha \ell j}(r) = N_{\alpha \ell j} r^{\ell+2p_{\alpha}} \exp\left[-(r/\nu_{\alpha})^{2}\right] , \quad \alpha = \alpha(\nu_{\alpha}, p_{\alpha}) \)

\[
\begin{cases}
\text{h.o.(-equivalent) bases} & \text{← fixed } \nu_{\alpha} : \text{real, } p_{\alpha} = 0, 1, \ldots \\
\text{Gaussian expansion method by Kamimura *et al.*} \\
\Rightarrow & \text{← } \nu_{\alpha} : \text{real/complex with geometric progression, } p_{\alpha} = 0 \\
& \text{Ref.: E. Hiyama *et al.*, Prog. Part. Nucl. Phys. 51, 223 (’03)} \\
& \text{hybrid bases, etc.}
\end{cases}
\]

2-body int. matrix elements ← Fourier transform.

⇒ solve HF/HFB eq. as generalized eigenvalue problem ⇒ iteration

**Advantages:**

- **ability to describe** \( E \)-dep. exponential asymptotics
  ← superposition of multi-range Gaussians

- **tractability of various 2-body interactions** ⋯ suitable to studying effective int.
  central, LS, tensor parts; function form of \( r \) — delta, Gauss, Yukawa, *etc.*

⇒ self-consistent MF (HF/HFB) calculations with spherical symmetry
cf. wave-function asymptotics for a bound neutron:

\[
\text{for large } r \quad \cdots \quad R_j(r) \approx \frac{e^{-\kappa r}}{r}; \quad \kappa = \sqrt{2M|E|} \quad (E < 0) \quad \text{small } |E| \to \text{long tail}
\]

\[E\text{-dep. exponential asymptotics} \quad \text{— required to describe halo-like structure}\]

- h.o. basis-set \cdots \text{impractical}
  \[\therefore R(r) \sim (\text{polynomial of } r) \times (\text{single-range Gaussian})\]

- transformed harmonic oscillator

Ref.: M. V. Stoitsov \textit{et al.}, P.R.C 58, 2092 (’98)

\[R_{THO}(r) \propto R_{HO}(f(r)) \quad f(r): \text{appropriate function of } r\]

\[\to \text{iterative change}\]

\[\cdots \text{practical only for zero-range int.}\]

- radial mesh \cdots \text{inefficient particularly for finite-range int.}

- \textbf{GEM} \cdots \text{practically good enough, available for finite-range int.}
Demonstration of GEM — application to HF calculation in O-isotopes $^{16}\text{O}$, $^{24}\text{O}$ & $^{28}\text{O}$

Ref.: H.N. & M. Sato, N.P.A 699, 511 (’02); 714, 696 (’03)

- Basis-set

\[
\begin{align*}
\text{h.o.(-equivalent) basis-set} & \quad \cdots \quad b = 1.70 \text{fm} \quad (\leftarrow \hbar \omega = 41.2 \cdot 24^{-1/3} \text{MeV}) \\
\text{GEM basis-set} & \quad \cdots \quad \nu_{\text{max}} = 5.7, 10, 31 \text{fm} \quad \text{for } K = 7, 10, 15 \\
\text{hybrid basis-set} & \quad \cdots \quad \text{GEM set + 1-node h.o. basis}
\end{align*}
\]

$K$: # of bases

$\Rightarrow$ comparison

- Interaction: Skyrme SLy4 & Gogny D1S

(— no essential difference in the following results)

- $^{16}\text{O}$ — stable; $^{24}\text{O}$ — beside drip-line; $^{28}\text{O}$ — unbound, but calculable in HF

  $\rightarrow$ wave-function asymptotics is important in $^{24,28}\text{O}$

  (→ advantage of GEM & hybrid bases)

cf. HFB — oscillatory asymptotics for some s.p. states

$\rightarrow$ complex GEM will be advantageous (work in progress)
Convergence for $E$ & radii (SLy4)

Density distribution (D1S)
Application of M3Y-type int. to doubly magic nuclei (by spherical HF cal.)

Binding energies & radii of doubly magic nuclei:

<table>
<thead>
<tr>
<th></th>
<th>SLy5</th>
<th>D1S</th>
<th>M3Y-P2</th>
<th>Exp.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
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<tr>
<td>$^{16}$O:</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E$ (MeV)</td>
<td>$-128.6$</td>
<td>$-129.5$</td>
<td>$-127.1$</td>
<td>$-127.6$</td>
</tr>
<tr>
<td>$\sqrt{\langle r^2 \rangle}$ (fm)</td>
<td>$2.59$</td>
<td>$2.59$</td>
<td>$2.60$</td>
<td>$2.61$</td>
</tr>
<tr>
<td></td>
<td></td>
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<tr>
<td>$^{40}$Ca:</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>$E$ (MeV)</td>
<td>$-344.3$</td>
<td>$-344.5$</td>
<td>$-338.7$</td>
<td>$-342.1$</td>
</tr>
<tr>
<td>$\sqrt{\langle r^2 \rangle}$ (fm)</td>
<td>$3.29$</td>
<td>$3.36$</td>
<td>$3.37$</td>
<td>$3.47$</td>
</tr>
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<td></td>
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<tr>
<td>$^{48}$Ca:</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E$ (MeV)</td>
<td>$-416.0$</td>
<td>$-416.8$</td>
<td>$-411.8$</td>
<td>$-416.0$</td>
</tr>
<tr>
<td>$\sqrt{\langle r^2 \rangle}$ (fm)</td>
<td>$3.44$</td>
<td>$3.50$</td>
<td>$3.52$</td>
<td>$3.57$</td>
</tr>
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<td></td>
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<tr>
<td>$^{208}$Pb:</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E$ (MeV)</td>
<td>$-1630.3$</td>
<td>$-1637.8$</td>
<td>$-1635.8$</td>
<td>$-1636.4$</td>
</tr>
<tr>
<td>$\sqrt{\langle r^2 \rangle}$ (fm)</td>
<td>$5.37$</td>
<td>$5.51$</td>
<td>$5.53$</td>
<td>$5.49$</td>
</tr>
</tbody>
</table>

S.p. levels around $^{208}$Pb:
Exp. data on unstable nuclei → what information of effective int.?

- Appreciable deviation from $Z/N = 1$ — global & local
e.g. $\rho(r)$ & $Y_p(r)$ by HF cal. with M3Y-P2 int. $(Y_p = \rho_p/\rho = (1 + \eta_t)/2)$

$Y_p \cdots r$-dep. in the vicinity of drip line (cf. skin)

→ wide variety of information of eff. int.?

• Shell structure far from $\beta$-stability \( \cdots \) “shell evolution”\?

\[
\begin{cases}
\sigma\tau\text{-channel of effective int.?} \\
\text{volume effect (} \rightarrow \text{r-dep. unimportant)} \\
\text{effects of tensor force?}
\end{cases}
\]

T. Otsuka et al., P.R.L. 87, 082502 (’01)
Exp. evidence on new magic numbers in unstable nuclei

kink in $S_n$: \( \text{e.g. } N = 16 \)
A. Ozawa \textit{et al.}, P.R.L. 84, 5493 (’00)

1st excitation energy:
R. Kanungo \textit{et al.}, P.L.B 528, 58 (’02)

\[ \Rightarrow N = 16, \ 32 \text{ magicity near } n\text{-drip line} \]

(other numbers also suggested)
“Shell evolution” for $N = 16$ isotones $\Leftarrow \varepsilon_n(0d_{3/2}) - \varepsilon_n(1s_{1/2})$ difference

Note: OPEP central part
dots pure $\sigma\tau$-channel

$\Rightarrow N = 16$ magicity in $p$-deficient nuclei — correlating to $\sigma\tau$-channel?

Otsuka et al.’s claim confirmed via MF calculations
role of OPEP revealed

Note: shell model int. dots no rearrangement for orbits in inert core
$\Rightarrow$ tends to give stronger $Z$-dep.
“Shell evolution” for $N=32$ isotones $\leftrightarrow \varepsilon_n(0f_{5/2}), \varepsilon_n(1p_{1/2})-\varepsilon_n(1p_{3/2})$ difference

○ slope of $\varepsilon_n(0f_{5/2})-\varepsilon_n(1p_{3/2})$ $\leftrightarrow$ $\sigma\tau$-channel

○ M3Y-P2 $\rightarrow$ inversion of $0f_{5/2}$ & $1p_{1/2}$ reproduced! (without core-pol.)

○ Z-dep. of $\Delta\varepsilon_n(0f_{5/2}) \rightarrow N = 32$ magicity at $Z \sim 20$?

$N = 34$ magicity? ⋯ not supported
IV. Summary

M3Y-type int. (e.g. M3Y-P2)

- $G$-matrix $\rightarrow$ fit (M3Y) $\rightarrow$ phenomenological modification
  - saturation properties irrelevant to spin d.o.f. — as good fit as in conventional int.
  - properties of stable doubly-magic nuclei — as good fit as conventional int.
  - $g' (\leftrightarrow \sigma\tau$ excitation) — reasonable value obtained naturally,
  - in contrast to conventional int.
  - shell structure far from $\beta$-stability — notable difference from conventional int.
  - seems to account for new magic number $N = 16$ ($& N = 32$)
  - role of OPEP (central part) clarified

Problems:
- pairing properties ($\leftrightarrow \text{SE channel at } \rho < \rho_0$)?
- non-central (LS & tensor) forces — not yet investigated carefully
- applications to other models or beyond MF ($\rightarrow$ universality?)
  — e.g. optical model, RPA, shell model
Comparison of bases ($s_{1/2}$)

S.p. wave functions in WS pot. by GEM

$^{16}$O

$E$-dep. exponential asymptotics practically fulfilled!