Bloch-Horowitz Schemes

A Few-Body Application

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T. Luu
Outline

- Bloch-Horowitz primer

- Solving BH “exactly” for s-shell nuclei
  - Non-perturbative method
  - Perturbative method

- Applying LO BH term to p-shell nuclei
  - Above and beyond the G-matrix

- Applying BH as a cluster expansion
  - Separation of A-body forces

- Conclusions
Bloch-Horowitz Equation

\[ H = \sum_{i<j}^{A} (T_{ij} + V_{ij}) \]
\[ = \sum_{i}^{A} T_{i} + \sum_{i<j}^{A} V_{ij} - \frac{P_{CM}^2}{2M_{A}}. \]

Non-Rel. Intrinsic Hamiltonian

\[ H_{\text{eff}}(E) = P\{H + H \frac{1}{E - QH} QH\}P, \]
First Step: Curing the Asymptotic behavior

\[ H_{eff} = P \left\{ \frac{E}{E - TQ} [T_{eff} + V_{eff}] \frac{E}{E - QT} \right\} P, \]

\[ T_{eff} = T + T \frac{1}{E} QT, \]

\[ V_{eff} = V + V \frac{1}{E - QH} QV. \]

Define new states

\[ |\Omega\rangle = \frac{E}{E - QT} P |\Omega\rangle \]

\[ < \Omega_f | H_{eff} | \Omega_i > = < \Omega_f | T_{eff} + V_{eff} | \Omega_i >. \]

New states have nice asymptotic behavior

\[ |\Omega\rangle \sim \exp(-\gamma r)/\gamma r \]
Applying BH to s-shell nuclei

Deuteron: Can solve for $V_{\text{eff}}$ in terms of included-space HO overlaps

$$V_{\text{eff}} = t_{12} - t_{12} G_0 \left[ \Gamma_0 + \Gamma_\infty \right]^{-1} G_0 t_{12},$$

$$G_0 = \frac{1}{E - T},$$

$$\Gamma_0^{-1} = \left\{ P \frac{1}{E - T} P \right\}^{-1}$$

$$t_{12} = V_{12} + V_{12} G_0 t_{12}$$

$$\Gamma_\infty = PG_0 t_{12} G_0 P,$$

$$= \quad + \quad + \quad + \quad + \ldots$$
Exact effective interaction has the following properties:

1. Gives the correct binding energy
2. Binding energy is independent of $\Lambda$
3. Binding energy is independent of oscillator parameter $b$
Three-body system

\[ V_{eff}^3 = V + V \frac{1}{E - QT} Q V_{eff}^3. \]

\[ | \tilde{\Psi}_\sigma >_{12} = V_{12} | \tilde{\Omega}_\sigma > + V_{12} \frac{1}{E - QT} Q | \tilde{\Psi}_\sigma >, \]

\[ | \tilde{\Psi}_\sigma >_{23} = V_{23} | \tilde{\Omega}_\sigma > + V_{23} \frac{1}{E - QT} Q | \tilde{\Psi}_\sigma >, \]

\[ | \tilde{\Psi}_\sigma >_{31} = V_{31} | \tilde{\Omega}_\sigma > + V_{31} \frac{1}{E - QT} Q | \tilde{\Psi}_\sigma >, \]

\[ | \tilde{\Psi}_\sigma >_{12} = \left( V_{12,eff}^{(2+1)} + V_{12,eff}^{(3+0)} \right) | \tilde{\Omega}_\sigma >. \]
What is actually solved:

\[ V_{12,\text{eff}}^{(2+1)} = V_{12,\text{eff}}^{(3+0)} = V_{12,\text{eff}}^{(2+1)} + V_{12,\text{eff}}^{(2+1)} \frac{1}{E - QT - Q\Pi V_{12,\text{eff}}^{(2+1)}} Q\Pi V_{12,\text{eff}}^{(2+1)} \]

\[ V_{\text{eff}} = V_{12,\text{eff}}^{(2+1)} + V_{12,\text{eff}}^{(2+1)} \frac{1}{E - QT} Q\Pi V_{\text{eff}}^{(2+1)} \]
Three-body results
Four-body system

Similar procedure:

\[
V_{\text{eff}} = V_{12,\text{eff}}^{(2+2)} + V_{\alpha,\text{eff}} + V_{\beta,\text{eff}}
\]

Invoke Faddeev-Yakubovsky decompositions

3-body eff.

\[
V_{\alpha,\text{eff}} = T_{12,\text{eff}}^{(3+1)} \frac{1}{E - QT} Q \Pi V_{12,\text{eff}}^{(2+2)} + T_{12,\text{eff}}^{(3+1)} \frac{1}{E - QT} Q \Pi (-\Pi_{34} V_{\alpha,\text{eff}} + V_{\beta,\text{eff}})
\]

\[
V_{\beta,\text{eff}} = \bar{T}_{12,\text{eff}}^{(2+2)} \frac{1}{E - QT} Q \bar{\Pi} V_{12,\text{eff}}^{(2+2)} + \bar{T}_{12,\text{eff}}^{(2+2)} \frac{1}{E - QT} Q \bar{\Pi} (1 - \Pi_{34}) V_{\alpha,\text{eff}}
\]

4-body eff.

One day, with help from Andreas. . .
Is the few-body problem perturbative?

For certain ranges of $b$, solving the integral equations (i.e. summing diagrams to all orders) is overkill.
How does $V_{12,\text{eff}}$ compare to $G$?

$$V_{12,\text{eff}} = \frac{1}{E - QT} Q + \frac{1}{E - QT} QV_{12} \frac{1}{E - QT} Q + \frac{1}{E - QT} QV_{12} \frac{1}{E - QT} QV_{12} \frac{1}{E - QT} Q \cdots$$

Both represent a ladder summation, however...

- There is no ‘starting energy’—$E$ is the self-consistent energy
- $Q$ is the many-body Pauli operator $\Rightarrow V_{12,\text{eff}}$ is multi-valued, i.e. depends on $\Lambda$ of spectator nucleons
- $T$ represents the many-body kinetic operator ($T=\sum T_{ij}$) $\Rightarrow$ cluster recoil is incorporated

$\Rightarrow V_{12,\text{eff}}$ is an $A$-body operator
Dealing with $V_{12,\text{eff}}$

$$V_{12,\text{eff}} = t_{12} - t_{12} G_0 \left[ \Gamma_0 + \Gamma_\infty \right]^{-1} G_0 t_{12},$$

$$G_0 = \frac{1}{E-T},$$

$$\Gamma_0 = \left\{ P \frac{1}{E-T} P \right\},$$

$$t_{12} = V_{12} + V_{12} G_0 t_{12},$$

$$\Gamma_\infty = PG_0 t_{12} G_0 P,$$

$$t_{12}(p`,p; E-\Sigma q_i^2/2\mu)$$

Matrix elements of $t_{12}$ involve A-dimensional nested integrals?

Fortunately, the answer is NO!
Dealing with $t_{12}$

\[
\int dq_1 \cdots dq_{A-2}q_1^2 \cdots q_{A-2}^2 R_{n_1'l_1}(q_1)R_{n_1l_1}(q_1) \cdots R_{n'_{A-2}l_{A-2}}(q_{A-2})R_{n_{A-2}l_{A-2}}(q_{A-2})t_{12}'(p', p; E - \sum_{i=1}^{A-2} \frac{q_i^2}{2\mu}) = \\
(-1)^{n_1+n_1'+\cdots+n_{A-2}+n'_{A-2}} \sqrt{\cdots} \times \sum_{m_1,m'_1=0}^{n_1,n_1'} (\cdots) \cdots \sum_{m_{A-2},m'_{A-2}=0}^{n_{A-2},n'_{A-2}} (\cdots) \times \prod_{i=2}^{A-2} \cdots \\
\times \left[ \int_0^\infty d\rho \rho^{A-3+2} \sum_{i=1}^{A-2} (l_i+m_i+m'_i+1) e^{-\rho^2 t_{12}'(p', p; E - \frac{\rho^2}{2\mu b^2})} \right] \\
= \overline{t_{12}'}(p', p; E) \quad \text{“Mean-field” interaction}
\]
What does \( \overline{t_{12}^{p'}}(p', p; E) \) look like?

\[ G_0 t_{12}(p', p; E) G_0 \]

\( E = -20 \text{MeV} \) \( b = 1.2 \text{ fm} \) \( ^1S_0 \)
Finally, some results. . .

Av18  S-shell

\[ \Lambda \rightarrow \Lambda + 2 \]

\[ \Lambda = 0 \quad \Lambda = 2 \]
Some P-Shell nuclei

5-body

\( (3/2^-,1/2) \)

\( \Lambda \rightarrow \Lambda + 2 \)

\( \Lambda = 0 \)

\( \Lambda = 2 \)

6-body

\( (1^+,0) \)

\( \Lambda \rightarrow \Lambda + 2 \)

\( \Lambda = 0 \)

\( \Lambda = 2 \)

7-body

\( (3/2^-,1/2) \)

Av8′
What about excited states?

5-body system 2hw

\[ b = 1.46 \quad (1/2^-, 1/2) \]

\[ b = 1.41 \quad (3/2^-, 1/2) \]
\[ b = 1.46 \] \( (1/2^-, 1/2) \) 

\[ b = 1.41 \] \( (3/2^-, 1/2) \) 

\[ b = 1.46 \] \( (1/2^-, 1/2) \) 

\[ b = 1.41 \] \( (3/2^-, 1/2) \)
6-body system 2hw

(1^+, 0) -> (2^+, 1) with b = 1.58
(2^+, 0) -> (0^+, 1) with b = 1.58
(0^+, 1) -> (3^+, 0) with b = 1.58

Exn. ⁶Li

b = 1.55

Avg'
7-body system $0_{hw}$

$(7/2^-, 1/2)$

$(1/2^-, 1/2)$

$(3/2^-, 1/2)$
BH as a cluster expansion

Ansatz: Hierarchy in number of particles interacting in Q space

\[ H_{\text{eff}}(E) = H + H \frac{1}{E - QH} QH \]

\[ = H_{\text{eff}}^2(E) + H_{\text{eff}}^3(E) + \cdots \]

\[ \sim H_{\text{eff}}^2(E) + H_{\text{eff}}^3(E) + H_{\text{eff}}^4(E) \]
Why should cluster expansion work?

• Shell model interactions, though only 2-body in nature and phenomenological, work really well
• Short-ranged clustering is qualitatively unlikely: Pauli exclusion principle excludes s-wave interactions with A>4
• NCSM have shown systematic improvements by including 3eff on top of 2eff
• GFMC calculations show alpha clustering in $^8$Be
• Analogy with EFT?
What do these interactions look like?

$$H_{\text{eff}}(E) = H + H \frac{1}{E - QH} QH$$

$$= H_{\text{eff}}^2(E) + H_{\text{eff}}^3(E) + \cdots$$

$$\sim H_{\text{eff}}^2(E) + H_{\text{eff}}^3(E) + H_{\text{eff}}^4(E)$$

$$H_{12} + H_{12} \frac{1}{E - QH_{12}} QH_{12}$$

$$H_{12} = T_{12} + V_{12}$$

Higher A-body cluster interactions are a little bit trickier, but follow similar pattern...
Why is this the way to go?

• Only need to solve up to the $A=4$ body system
• However, need to solve for multiple $\Lambda$ to account for spectator dependence (i.e. multi-valued)
• Need to solve for wide range of energies $E$
Conclusion

- Can solve effective interaction ‘exactly’ for s-shell nuclei non-perturbatively by using Faddeev techniques
- Can solve perturbatively for 2- and 3-body system
- LO BH calculation suggest 4-body system is perturbative as well
- LO BH calculation on p-shell nuclei not so impressive, even though interaction is multi-valued and includes cluster recoil
- For 0hw and 2hw calculations, LO BH gives spectra in the right ballpark, but convergence is still an issue
- BH as a cluster expansion is feasible, and may be the way to go—just need to do it and see