Equation of state under the conditions of NSE

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EOS for NSE deals with

- Large variety of nuclides: at low entropies ($S \lesssim 2$) all the nuclides up to the neutron-dreep boundary could be important (appropriate mass formulae are required)
- Excitation of nuclides (partition functions!)
- Degeneration of free neutrons and protons
- Full Fermi-Dirac statistics for $e^-e^+$ component
- Non-ideal effects: Coulomb interactions in multi-charge plasma
  Nucleon-nucleon strong interactions at $\rho \gtrsim 10^{13}$ g/ccm
Our Project

To analyze the impact of above issues on thermodynamics of collapsing matter

To construct 3-parametric tables for free energy and to supply them with a smooth interpolation procedure (spline interpolation algorithm)

To begin a new series of calculation of the collapse focusing on the first neutrino-transparent stage and kinetics of neutronization

\[
\begin{align*}
\text{Free energy: } F(T, \rho, \theta) \\
\theta \equiv N_n^0 / N_p \\
P &= \rho^2 \left( \frac{\partial F}{\partial \rho} \right)_{T, \theta} \\
S &= -\left( \frac{\partial F}{\partial T} \right)_{T, \theta} \\
E &= F + TS
\end{align*}
\]
Equations for NSE

\[ Y_i = \frac{n_i m_u}{\rho} \]

\[ \psi = \frac{\mu - mc^2}{kT} \]

\[ Y_{A,Z} = \omega_{A,Z} \mu_{A,Z}^{3/2} \frac{m_u}{\rho \lambda_T^3} \exp \left[ (A-Z)\psi_n + Z\psi_p + \frac{Q_{A,Z}}{kT} \right] \]

\[
\begin{cases}
Y_n + \sum_{A,Z} (A-Z)Y_{A,Z} = \frac{\theta}{1+\theta} \\
Y_p + \sum_{A,Z} Z Y_{A,Z} = \frac{1}{1+\theta}
\end{cases}
\]
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<th>$I_{gs}$</th>
<th>$N_{ex}$</th>
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* Stable nuclides are in bold. Formally, the isotopes $^{48}$Ca, $^{50}$V, $^{76}$Ge, and $^{82}$Se are unstable. Their half-decay lives are $5.1 \times 10^{16}$ y, $1.5 \times 10^{17}$ y, $1.09 \times 10^{21}$ y, and $1.21 \times 10^{20}$ y, respectively.
Partition Functions

For $T_9 < 40$:
1. Experimentally known states (up to $\sim 100$ per nucleus)
2. Fermi-gas level-density model as provided by Rauscher et al. (1997)

For $T_9 > 40$, we use a simple analytical formula:

$$\langle E_{ex} \rangle = Q_{A,Z} \vartheta(T), \quad 0 \leq \vartheta(T) \leq 1, \quad \vartheta(T) = a - \frac{b}{T} - \frac{c}{T^2}$$

$$\langle E_{ex} \rangle = T \frac{d \ln \omega}{d \ln T}$$

$$\ln \omega(T) = d - \frac{Q_{A,Z}}{T} \left( a - \frac{b}{2T} - \frac{c}{3T^2} \right)$$

For each $(A, Z)$ $b, c, d$ come from continuity of $\omega, \omega'$ and $\omega''$ at $T_9 = 40$
\[ \gamma = \left( \frac{\partial \ln P}{\partial \ln \rho} \right)_{s, \theta} \]

\[ \theta = \frac{30}{26} \]

\( \gamma_{\text{min}} = 0.94 \)

\( \gamma_{\text{min}} = 1.04 \)

----- no excitation
\[ S_b = \left( \frac{m_u}{k} \right) S = \text{entropy per nucleon} \]
\[ \frac{dE}{TdS} + PdV = TdS + \Psi dY_l \quad \left( V \equiv \frac{1}{\rho} \right) \]

\[ Y_l = Y_e = \frac{1}{1+\theta} \quad \text{lepton charge per nucleon} \]

\[ \Psi = -\frac{1}{m_u} \left( \mu_n - \mu_p - \mu_e \right) = -\frac{kT}{m_u} \left( \psi_n - \psi_p - \psi_e + \frac{Q_n}{kT} \right) \]

\[ Q_n = c^2 \left( m_n - m_p - m_e \right) = 0.78235 \text{ MeV} \]

\[ \psi_i = \frac{\mu_i - mc^2}{kT} \quad \text{For adiabatic process } (dE + PdV = 0) : \]

\[ \frac{dS}{dY_l} = -\frac{\Psi}{T} \]

For \( \Psi > 0 \), \( dS > 0 \) when \( dY_l < 0 \) (\( d\theta > 0 \))
$\xi_{n,p} > 1$

$$\xi_{n,p} \equiv \frac{Y_{n,p}}{\sum Y_{A,Z}} = 1$$
\[ S_b = \frac{(m_u/k)S}{S_e + S_\gamma + S_n + S_p + \sum S_{A,Z}} = 1 \]
$\lg \rho = \begin{cases} 7.7 & \text{for } 9.5 \leq T_g \leq 10.7 \\ 11.4 & \text{for } 11.8 \leq T_g \leq 12.15 \\ 12.4 & \text{for } 12.4 \leq T_g \leq 12.6 \\ 12.8 & \text{for } 12.8 \leq T_g \leq 12.95 \end{cases}$

$(A, Z)$

$S_b = 2.0 \quad \theta = 30/26$

$\delta^a$

$e$, $\gamma$, $n$, $p$

$T_g$
\[ \lg \rho = \]

\[ S_b = 4.0 \quad \theta = 30/26 \]

\[ (A, Z) \]

\[ n \]

\[ p \]

\[ \gamma \]

\[ e \]
Coulomb Interaction

Free Energy Formalism:

\[ F(\rho, T, \{Y_i\}) = F^{id}(\rho, T, \{Y_i\}) + \Delta F(\rho, T, \{Y_i\}) \]

New set of equilibrium concentrations \(\{Y_i\}\) due to the free energy minimization procedure.

One component plasma (OCP)

\[ \Delta F_{ii} = \frac{kT}{m_u} Y_{A,Z} f(\Gamma), \quad \Gamma = \left( \frac{Z e}{kT} \right) \left( \frac{4\pi}{3} n_{A,Z} \right)^{1/3} \]

Multicomponent plasma (MCP)

\[ \Gamma_e = \frac{e^2}{kT} \left( \frac{4\pi}{3} n_e \right)^{1/3} \]

Average Ion Model

\[ \Delta F_{ii} \propto (\sum_i Y_i) f(\langle \Gamma \rangle), \quad \langle \Gamma \rangle = \frac{\sum_i Y_i \Gamma_i}{\sum_i Y_i}, \quad \Gamma_i = Z_i^{5/3} \Gamma_e \]

Linear Mixing Rule

\[ \Delta F_{ii} \propto \sum_i Y_i f(\Gamma_i) \]

Complicated Mixing

\[ \Delta F_{ii} \propto \sum_{i,j} Y_i Y_j \varepsilon_{ij} f(\Gamma_{ij}) \]
Influence of Coulomb Interaction (CI) on $\gamma$
Tracks of collapsing stars

- EOS without PF
- EOS with PF
- EOS with PF and CI

$M_{\text{core}} = 1.4M_\odot$

$M_{\text{core}} = 2M_\odot$
Conclusions

1. Partition functions are not very crucial for EOS and can result in moderate changes in entropy and $\gamma$ only for $\rho > 10^{10}$ g/ccm

2. For low entropies ($S_b \leq 2$), the neutron-rich nuclei can have a strong impact on EOS by contributing to entropy and by lowering $Y_n$ and $Y_p$

3. Free nucleon degeneracy is a mild effect since at low $T$ nearly all $n$ and $p$ are packed in heavy nuclei

4. For $3 < T_g < 100$, the Coulomb Interaction distinctly influences on EOS (specifically on $\gamma$) only at $\rho > 10^{11}$ g/ccm. Further study of this issue is under way.

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