Magnetic Fields in Neutron Stars*

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*(Based on works with Kirk Buckley and Max Metlitski: PRL, 2004 and PRC, 2004)*
I. Historical Introduction. Motivation.

1. Conventional picture The extremely dense interior in neutron stars is mainly composed of neutrons, with a small amount of protons and electrons in beta equilibrium. The neutrons form $^{3}P_{2}$ Cooper pairs and Bose condense to a superfluid state, while the protons form $^{1}S_{0}$ Cooper pairs and Bose condense to form a superconductor.

2. Type-II superconductivity It is generally assumed that the proton superconductor is a type-II superconductor, which means that it supports a stable lattice of magnetic flux tubes in the presence of a magnetic field (Abrikosov lattice). This belief is based on simple estimations of the coherence length $\xi$ and the London penetration depth $\lambda$ which ambiguously imply a type-II superconductivity.

3. Landau-Ginzburg parameter

$$\lambda = \sqrt{\frac{m_{c}c^{2}}{4\pi q^{2}n_{p}}}$$
$$\xi = \sqrt{\frac{\hbar^{2}}{2m_{c}n_{p}a}}$$
$$m_{c} = 2m, \quad q = 2e$$
Typically, the Landau-Ginzburg parameter \( \kappa = \lambda/\xi \) is introduced. In a conventional superconductor, if \( \kappa < 1/\sqrt{2} \) then the superconductor is type-I and vortices attract. If \( \kappa > 1/\sqrt{2} \) then vortices repel each other and the superconductor is type-II.
4. “Naive” Moral:

⇒ TYPE I SUPERCONDUCTOR
implies ⇒ No Field in the Bulk

⇒ TYPE II SUPERCONDUCTOR
implies ⇒ Abrikosov Lattice Structure
Superfluidity or superconductivity – which is the dominant form of energy loss at low temperatures. He coined the name "superfluid" for this phenomenon. The key was provided by John Bardeen, Leon Cooper and Robert Schrieffer, whose 1957 "BCS theory" showed that pairs of electrons with identical spins and momentum would become bound together and would act as single bosons, having macroscopic wavefunctions.

Besides being of tremendous theoretical interest in modern society, superconducting magnets are vehicles for developing key technological discoveries of superfluids and superconductors: quantum mechanics on the macroscopic scale. For instance, superfluid tritium-3 He – was manifested in 1938 by Pyotr Kapitsa and his collaborator, for instance, in magnetic resonance imaging technique and superconducting magnets are increasingly used for bending the paths of charged particles in particle accelerators.

In 1972, Bardeen, Cooper and Schrieffer received the 1972 Nobel Prize in Physics. For this work they took advantage of the Pauli principle from Fermi-Dirac statistics to condense into the lowest-lying energy single–particle state at low temperatures. Electrons, however, obey Fermi statistics and are described by the preferred term "i-Dirac statistics" for this phenomenon.

Heike Kamerlingh Onnes received the 1913 Nobel Prize in Physics. In their theory the Cooper pairs are also used for bending the paths of charged particles in particle accelerators. Besides being of tremendous interest in modern society, the understanding of superfluids and superconductors is important for the development of novel applications in modern technology.
5. Life is much more complicated:

FIG. 1. The intermediate state of a thin slab of indium, in which the superconducting regions (black) are decorated with niobium (black). The applied field $H_a$ is close to the critical field $H_c$ ($h = H_a/H_c = 0.931$). Adapted from Haenssler and Rinderer [3].

(Superconducting region is black, normal phase is bright)
Fig. 2.17a-f. Intermediate state of the same Pb disk as in Fig. 2.9 in perpendicular magnetic field for decreasing values of $\tilde{H} = H/H_c$: (a) $\tilde{H} = 0.84$, (b) $\tilde{H} = 0.74$, (c) $\tilde{H} = 0.58$, (d) $\tilde{H} = 0.42$, (e) $\tilde{H} = 0.32$, (f) $\tilde{H} = 0.21$. Normal phase is bright, $T = 4.2$ K, (NI)$_T$ transition. (Courtesy of A. Kiendl)
Fig. 2.8a-f. Intermediate-state structure of a Pb film in perpendicular magnetic field for increasing values of $H$. (a) 95 G, (b) 132 G, (c) 178 G, (d) 218 G, (e) 348 G, (f) 409 G (normal domains are bright, $9.3 \mu m$).
Fig. 2.11. Intermediate state of a monocrystalline tin film of 29 μm thickness at h = 2 and T = 1.2 K, (NI) transition, normal regions are dark [2.21]

(Superconducting region is bright, normal phase is dark)
II. Recent Development


The periodic timing behavior of PSR B182811 and correlated changes in beam profile have been interpreted as due to precession with a period of 1 yr and an amplitude of $\sim 3^\circ$.


Abstract: “I show that the standard picture of the neutron star core containing coexisting neutron and proton superfluids, with the proton component forming a type II superconductor threaded by flux tubes, is inconsistent with observations of long-period ($\sim 1$ yr) precession in isolated pulsars. I conclude that either the two superfluids coexist nowhere in the stellar core, or the core is a type I superconductor rather than type II. Either possibility would have interesting implications for neutron star cooling and theories of spin jumps (glitches).”
3. The summary of the Link’s paper:

a). The estimates show that a neutron star core containing coexisting neutron vortices and proton flux tubes cannot precess with a period of \( \sim 1 \text{ yr} \).

b). The fraction of the neutron components moment of inertia that is pinned against flux tubes must be \( \ll 10^{-8} \). Hence, observations require that neutron vortices and proton flux tubes coexist nowhere in the star.

c). Either the stars magnetic field does not penetrate any part of the core as the Abrikosov vortices corresponding to a type II superconductor,

d). or: at least one of the hadronic fluids is not superfluid. (This latter possibility appears unlikely in the face of pairing calculations which predict coexisting neutron and proton superfluids in the outer core.)

e). If the core is a type I superconductor, at least in those regions containing vortices, the magnetic flux could exist in macroscopic normal regions that surround superconducting regions that carry no flux. In this case, the magnetic field would not represent the impediment to the motion of vortices that flux tubes do, and the star could precess with a long period.
III. Type I instead of Type II

1. Main goal
Motivated by the previous papers, we suggest a possible scenario which would lead to the type I behavior in spite of the fact that $\lambda, \xi$ remain the same as in the “naive” estimates suggesting type II behavior.

2. Main idea. Analogies with other fields
a). There are many situations where the standard picture will be qualitatively modified. For example, if there is a second component (such as a neutron component in our specific case), it may be energetically favorable for the cores of vortices to be filled with a nonzero condensate of this second component, as it was originally suggested in the cosmological context by Witten (cosmic strings).

b). There are numerous examples of physical systems where such phenomena occurs: superconducting cosmic strings in cosmology, magnetic flux tubes in the high $T_c$ superconductors (antiferromagnetic condensate fills the vortex core), Bose-Einstein condensates (two component BEC systems), superfluid $^3He(A$ and $B$), and high baryon density quark matter.

c). Our main assumption is: such a nontrivial vortex
structure occurs due to the strong interaction between two superfluids (n and p) in two component system. d). If it happens, the vortex-vortex interaction pattern changes (repulsion ⇒ attraction). This automatically changes the type II ⇒ type I behavior.

3. Possible Resolution of the Paradox

a). The vortex-vortex (attracting) interaction due to the nontrivial core structure would resolve the apparent discrepancy (B.Link) between the observation of long period precession (I.Stairs) and the typical parameters of the neutron stars which naively suggest type-II superconductivity in neutron stars.

b). Specifically, on a macroscopic distance scale, the magnetic flux must be embedded in the superconductor. This would mean that the superconductor is in an intermediate state as opposed to the vortex state of the type-II superconductor.

c). The superconducting domains will then exhibit the Meissner effect, while the normal domains will carry the required magnetic flux.

d). Source for a nontrivial vortex core structure may have a different nature (Aurel Bulgac).
IV. Vortex Structure

1. Effective Landau-Ginsburg free energy

\[ F = \int d^2r \left( \frac{\hbar^2}{2m_c} |(\nabla - \frac{iq}{\hbar c} A)\psi_1|^2 + |\nabla \psi_2|^2 \right) 
\]

\[ + \left( \frac{B^2}{8\pi} - \mu_1 |\psi_1|^2 - \mu_2 |\psi_2|^2 \right) \]

\[ + \left( \frac{a_{11}}{2} |\psi_1|^4 + \frac{a_{22}}{2} |\psi_2|^4 + a_{12} |\psi_1|^2 |\psi_2|^2 \right) \]

where \( m_c = 2m \) and \( q = 2e \), \( \mu_i \) is the Bose chemical potential of the \( i^{th} \) component and \( a_{ij} = 4\pi\hbar^2 l_{ij}/m_c \).

The Bose chemical potentials \( \mu_1, \mu_2 \) determined by the Cooper pair densities \( n_1 \equiv |\psi_1|^2 \) and \( n_2 \equiv |\psi_2|^2 \) and coefficients \( a_{ij} \).

\[ n_1 = \frac{a_{22}\mu_1 - a_{12}\mu_2}{a_{11}a_{22} - a_{12}^2}, \quad n_2 = \frac{a_{11}\mu_2 - a_{12}\mu_1}{a_{11}a_{22} - a_{12}^2}. \]
2. Few Important Remarks:

a) Original symmetry is: $U(1) \times U(1)$ corresponding to the conservation of two species $\psi_1$ and $\psi_2$;

b) In the limit $\mu_i = \mu$, $a_{ij} = a$ the symmetry is $U(2)$. The vacuum manifold is given by the 3-sphere $|\psi_1|^2 + |\psi_2|^2 = \mu/a$;

c) A very small $U(2)$ violating change in the chemical potentials $\mu_1$ and $\mu_2$ that violates the $U(2)$ symmetry produces a very large asymmetry of proton and neutron Cooper pair densities $n_1$, $n_2$ by selecting a particular vacuum state on the original degenerate manifold.

d) In particular, if $a_{ij} = a$, but the chemical potentials are slightly different, $\mu_1 = \mu_2 - \delta\mu$, $\delta\mu > 0$ then neutrons condense, $n_2 = \mu_2/a$, $n_1 = 0$, which corresponds a very large asymmetry of proton and neutron Cooper pair densities $n_2/n_1 = \infty$. 
3. Main Assumption:
We assume that free energy $\mathcal{F}$ is mainly $U(2)$ symmetric, and the large asymmetry of proton and neutron Cooper pair densities (known to realize in nature) can be achieved in the effective lagrangian approach by small explicit $U(2)$ violation: $\mu_1 = \mu - \delta \mu$, $\mu_2 = \mu + \delta \mu$, where $\delta \mu/\mu \ll 1$, and $a_{11} = a_{22} = a$, $a_{12} = a - \delta a$, where $\delta a/a \ll 1$.

$$n_1 \approx \frac{\mu}{2a} - \frac{\delta \mu}{\delta a}, \quad n_2 \approx \frac{\mu}{2a} + \frac{\delta \mu}{\delta a}, \quad 0 < n_2/n_1 < \infty$$

a) Technical motivation for the assumption: We knew (based on the previous experience) that in this case a nontrivial vortex structure occurs. It leads to the unusual vortex-vortex interaction features.
b) More physical (rather than technical) explanation: The original isotopical symmetry could not disappear without a trace, it must be hidden somewhere.... In particular, in a similar problem in QCD with $N_c = 2$ (fundamental quarks) or $N_c = 3$ (adjoint quarks), the fermi surfaces for different flavors could be very different. However, the original symmetry can be used to calculate the different Cooper pair condensates.
4. Equations of Motion

- The Landau-Ginzburg equations of motion following from the free energy are:

\[
\frac{\hbar^2}{2m_c}(\nabla - \frac{iq}{\hbar c}A)^2 \psi_1 = -\mu_1 \psi_1 + a|\psi_1|^3 + (a - \delta a)|\psi_2|^2 \psi_1,
\]

\[
\frac{\hbar^2}{2m_c}\nabla^2 \psi_2 = -\mu_2 \psi_2 + a|\psi_2|^3 + (a - \delta a)|\psi_1|^2 \psi_2,
\]

\[
\nabla \times (\nabla \times A) = \frac{-iq\hbar}{4\pi} \left( \psi_1^* (\nabla - \frac{iq}{\hbar c}A) \psi_1 - h.c. \right)
\]

- In previous calculations it has been assumed that the neutron order parameter \( \psi_2 \) will remain at its vacuum expectation value \( |\psi_2| = \text{const.} \) in the vicinity of the proton vortex (this is not the case in many similar systems.)

- So, anticipating a non-trivial behavior of the neutron field \( \psi_2 \), we adopt the following ansatz for the fields describing a proton vortex with a unit winding number:

\[
\psi_1 = \sqrt{n_1} f(r) e^{i\theta}, \quad \psi_2 = \sqrt{n_2} g(r), \quad A = \frac{\hbar c a(r)}{q} \frac{\hat{\theta}}{r}
\]
• We wish to find the asymptotic behavior of fields $\psi_1$, $\psi_2$ and $A$ far from the proton vortex core, as this will determine whether distant vortices repel or attract each other. The asymptotic behavior can be found analytically

$$f(r) = 1 + F(r), \quad g(r) = 1 + G(r), \quad a(r) = 1 - rS(r),$$

so that far away from the vortex core, $F, G, rS \ll 1$ and $F, G, S \to 0$ as $r \to \infty$. This allows us to linearize far from the vortex core the equations of motion corresponding to the free energy to obtain:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}\right) \begin{pmatrix} F \\ G \end{pmatrix} = M \begin{pmatrix} F \\ G \end{pmatrix}$$

where matrix $M$ mixing the fields $F$ and $G$ is,

$$M = \frac{4mc}{\hbar^2} \begin{pmatrix} a & a - \delta a \\ a - \delta a & a \end{pmatrix} \cdot \begin{pmatrix} n_1 & 0 \\ 0 & n_2 \end{pmatrix}$$

$$S'' + \frac{1}{r} S' - \frac{1}{r^2} S = \frac{1}{\lambda^2} S' \quad \Rightarrow \quad S = \frac{C_A}{\lambda} K_1(r/\lambda)$$
5. Standard picture:

In previous works the influence of the neutron condensate on the proton vortex was neglected, which formally amounts to setting the off-diagonal term $M_{12}$ to 0. In that case,

$$F = C_F K_0(\sqrt{2}r/\xi), \quad G = 0.$$ 

It is estimated that $\lambda \sim 80$ fm and $\xi \sim 30$ fm, which leads to $\kappa = \lambda/\xi \sim 3$ i.e. $\kappa > 1/\sqrt{2}$. Therefore, distant vortices repel each other leading to

**type-II behavior.**

*This is the standard picture of the proton superconductor in neutron stars that is widely accepted in the astrophysics community.*
6. Beyond the standard picture:

• The standard procedure described above is inherently flawed since the system exhibits an approximate $U(2)$ symmetry, This makes the mixing matrix $\mathcal{M}$ nearly degenerate.

The general solution is:

$$
\begin{pmatrix} F \\ G \end{pmatrix} = \sum_{i=1,2} C_i K_0(\sqrt{\nu_i} r) \mathbf{v}_i
$$

where $\nu_i$ and $\mathbf{v}_i$ are the eigenvalues and eigenvectors of matrix $\mathcal{M}$, and $C_i$ are constants.

• In the limit $\gamma = n_1/n_2 \ll 1$ and $\epsilon = 2\delta a/a \ll 1$ one can estimate the eigenvalues and eigenvectors of the matrix $\mathcal{M}$ as:

$$
\nu_1 \simeq \frac{2\epsilon}{\xi^2}, \quad \mathbf{v}_1 \simeq \begin{pmatrix} -1 \\ \gamma \end{pmatrix},
$$

$$
\nu_2 \simeq \frac{2}{\gamma \xi^2}, \quad \mathbf{v}_2 \simeq \begin{pmatrix} 1 \\ 1 \end{pmatrix}
$$
• The physical meaning of solution: there are two modes in our two component system. The first mode describes fluctuations of relative density (concentration) of two components and the second mode describes fluctuations of overall density of two components.

• Notice that $\nu_1 \ll \nu_2$, and hence the overall density mode has a much smaller correlation length than the concentration mode. Therefore, far from the vortex core, the contribution of the overall density mode can be neglected, and one can write:

$$\left( \begin{array}{c} F \\ G \end{array} \right) (r \to \infty) \approx C_1 K_0(\sqrt{2\epsilon} r/\xi) \cdot \left( \begin{array}{c} -1 \\ y \end{array} \right)$$

(1)

• The most important result: a typical scale is of order $\xi/\sqrt{\epsilon}$ - the correlation length of the concentration mode. Since $\epsilon \ll 1$, this distance scale can be much larger than the proton correlation length $\xi$, which is typically assumed to be the radius of the proton vortex core.
Here we show the numerical solution of the profiles of the proton vortex \( f(\tilde{r}) \), neutron condensate \( G(\tilde{r})/\gamma \), and \( a(\tilde{r}) \) (electromagnetic field) as a function of the dimensionless radial coordinate \( \tilde{r} = r/\xi \), where \( \xi \) is the coherence length. We have used \( \kappa = 3, n_1/n_2 = 0.05 \), and \( \epsilon = 0.02 \) in this numerical solution.
V. Vortex - Vortex Interaction

1. Few general remarks

- If the interaction between two vortices is repulsive, it is energetically favorable for the superconductor to organize an Abrikosov vortex lattice with each vortex carrying a single magnetic flux quantum. This is classic type-II behavior. If the interaction between two vortices is attractive, it is energetically favorable for \( n \) vortices to coalesce and form a vortex of winding number \( n \). This is type-I behavior.

- The Landau-Ginzburg parameter \( \kappa = \lambda / \xi \) is introduced. In a conventional superconductor, if \( \kappa < 1/\sqrt{2} \) then the superconductor is type-I and vortices attract. If \( \kappa > 1/\sqrt{2} \) then vortices repel each other and the superconductor is type-II.

- The typical value for a neutron star is \( \kappa \sim 3 \), so one could naively expect that the proton superfluid is a type-II superconductor.
• The case we consider here has one new element, $\epsilon$ which was not present in the standard type-I/II classification. However, we expect that analogous classification (when one compare $\lambda$ and actual vortex core) should remain in effect. In such an analysis $\xi$ should be replaced by the actual size of the proton vortices

$$\xi \Rightarrow \delta \sim \frac{\xi}{\sqrt{\epsilon}}.$$ 

• Therefore, we will define a new Landau-Ginzburg parameter for our case,

$$\kappa_{np} \equiv \frac{\lambda}{\delta} = \sqrt{\epsilon \frac{\lambda}{\xi}}.$$ 

• We expect type-I behavior with vortices attracting each other if $\kappa_{np} \ll 1$ and type-II behavior if $\kappa_{np} \gg 1$. For relatively small $\epsilon$ such an argument would immediately suggest that for the typical parameters of the neutron stars type-I superconductivity is realized (rather than the naively assumed type-II superconductivity).
2. Explicit calculation of the vortex-vortex interaction

• Define: \((F, G, A) = (F_1 + F_2, G_1 + G_2, A_1 + A_2)\) be the exact fields produced by two vortices at locations \(r_1\) and \(r_2\). When \(r\) is far from the cores of both vortices, \(F_i, G_i, A_i\) are small for both \(i = 1, 2\) and are known to be:

\[
F_i \simeq -G_i/\gamma \simeq -C_1 K_0(\sqrt{2\epsilon}|r - r_i|/\xi),
\]

\[
A_i \simeq \frac{\hbar c}{q\lambda} C_A K_1(|r - r_i|/\lambda) \hat{\theta}
\]

• To calculate the interaction energy of the two vortices, we divide the space into two cells \(T_1\) and \(T_2\), which contain the centers of vortices 1 and 2 respectively. The vortex-vortex interaction energy is then:

\[
\mathcal{F}_{int} = \mathcal{F}[F_1 + F_2, G_1 + G_2, A_1 + A_2] - 2\mathcal{F}[F_1, G_1, A_1]
\]
• We assume the vortex separation to be large. Since this boundary is far away from either vortex center, we can use the asymptotic expressions for the fields \((F_i, G_i, A_i)\) to explicitly calculate the integrals.

\[
\mathcal{F}_{int} \approx 2 \oint_T dS \cdot \left( \frac{\hbar^2 n_1}{m_c} F_2 \nabla F_1 + \frac{\hbar^2 n_2}{m_c} G_2 \nabla G_1 \right) + \frac{1}{4\pi} A_2 \times (\nabla \times A_1) \tag{2}
\]

Here the integral is over the boundary of cell \(T\).

• Substituting asymptotic solutions into the above, we find the vortex-vortex interaction energy per unit length to be:

\[
U(d) = \frac{2\pi \hbar^2 n_1}{m_c} \left( C_A^2 K_0(d/\lambda) - C_1^2 K_0(\sqrt{2}\epsilon d/\xi) \right)
\]
3. Interpretation of the result

- As we anticipated the relevant parameter is a new Landau-Ginzburg parameter for two component system,

\[ \kappa_{np} \equiv \frac{\lambda}{\delta} = \sqrt{\epsilon \frac{\lambda}{\xi}}. \]

- If \( \kappa_{np} > \frac{1}{\sqrt{2}} \) then vortices repel each other and the superconductor is type-II. This corresponds to the first (conventional) term in the interaction energy

\[ U(d) \simeq \frac{2\pi \hbar^2 n_1}{m_c} (K_0(d/\lambda)) \sim \exp\left(-\frac{d}{\lambda}\right) \]

- If \( \kappa_{np} < \frac{1}{\sqrt{2}} \) then the superconductor is type-I and vortices attract. This corresponds to the second term in the interaction energy

\[ U(d) \simeq -\frac{2\pi \hbar^2 n_1}{m_c} \left(K_0(\sqrt{2\epsilon d}/\xi)\right) \sim -\exp\left(-\frac{\sqrt{2\epsilon d}}{\xi}\right) \]
VI. Critical Magnetic Fields

1. Motivation. General Comments

- We realized that two component system may have type I rather than type II behavior. Therefore, the standard estimation of critical field which destroys superconductivity must be also reconsidered;

- Comparison $H_c$ with $H_{c2}$ will support our claim that our system indeed exhibits the type I superconductivity rather than type II.

- Usually one calculates the critical magnetic fields $H_c$ and $H_{c2}$. These are the physically meaningful fields above which the superconductivity is destroyed in type-I and type-II superconductors respectively. If $H_c > H_{c2}$ then the superconductor is type-I, otherwise, the superconductor is type-II.

2. Calculation of $H_c$

- $H_c$ is defined as the point at which the Gibbs free energy of the normal phase is equal to the Gibbs free energy of the superconducting phase.
• The Gibbs free energy in the presence of an external magnetic field $H$ is:

$$g(H, T) = f(B, T) - \frac{BH}{4\pi}$$

where $H$ is the external magnetic field, $B$ is the magnetic induction, $T$ is the temperature.

• For the superconducting state: $\langle |\psi_1|^2 \rangle = n_1$, $\langle |\psi_2|^2 \rangle = n_2$, and $B = 0$ (Meissner effect), the Gibbs free energy is

$$g_s(H, T) = -\frac{\mu^2}{2a} - \frac{(\delta \mu)^2}{\delta a} - \delta a \left( \frac{\mu}{2a} \right)^2.$$ 

• For the normal state, we have $\langle |\psi_1|^2 \rangle = 0$, $\langle |\psi_2|^2 \rangle = (\mu + \delta \mu)/a$, and $B = H$. The Gibbs free energy is:

$$g_n(H, T) = -\frac{H^2}{8\pi} - \frac{\mu^2}{2a} - \frac{\mu \delta \mu}{a}$$

$H_c$ is defined as the point at which $g_s(H_c) = g_n(H_c)$, i.e

$$H_c = \sqrt{8\pi \delta a} \left( \frac{\mu}{2a} - \frac{\delta \mu}{\delta a} \right) \rightarrow n_1 \sqrt{8\pi \delta a},$$
2. Calculation of $H_{c2}$

- In order to calculate $H_{c2}$, we follow the standard procedure and linearize the equations of motion for $\psi_1$ about the normal state with $\langle |\psi_1|^2 \rangle = 0$ and $\langle |\psi_2|^2 \rangle = (\mu + \delta\mu)/a$. The linearized equation of motion reads,

$$\frac{\hbar^2}{2m_c} \left( -i \nabla - \frac{q}{\hbar c} A \right)^2 \psi_1 = \omega \psi_1,$$

where $\omega \equiv (\mu + \delta\mu) \frac{\delta a}{a} - 2\delta\mu$. This is simply a Schrödinger equation for a particle in a magnetic field, with an energy of $\omega$. This is a standard quantum mechanics problem. The first Landau level is the ground state energy of $\epsilon_0(H) = \hbar |q| H / 2m_c c$. Therefore, if $\omega < \epsilon_0$, then only the trivial solution with $\psi_1 = 0$ is possible. The critical field $H_{c2}$ is defined as the point at which $\omega = \epsilon_0(H_{c2})$.

$$H_{c2} = \frac{2m_c c}{\hbar |q|} \left[ (\mu + \delta\mu) \frac{\delta a}{a} - 2\delta\mu \right] \simeq \frac{4m_c c}{\hbar |q|} \delta a \ n_1$$
3. Numerical estimates

- If $H_c < H_{c2}$ this means that it is energetically favorable for microscopic regions of the superconducting state to be nucleated as $H$ is decreased. This is type-II behavior, and this nucleation manifests itself in the form of an vortex lattice.

- If $H_c > H_{c2}$, then it is energetically favorable for macroscopic regions of the superconducting state to be present as $H$ is decreased. This is a type-I superconductor and the superconducting state persists everywhere in the material when $H < H_c$.

\[
\frac{H_{c2}}{H_c} \approx \sqrt{2} \frac{m_c c}{\sqrt{\pi \hbar}} \frac{\sqrt{\delta a}}{q} = \sqrt{2} \kappa_{np},
\]

in agreement with the results obtained from the vortex-vortex interaction calculation.
• Numerical estimate:

\[ H_c = \frac{\varphi_0}{2\pi\lambda\xi} \sqrt{\frac{\delta a}{a}}, \quad \varphi_0 = \frac{2\pi\hbar c}{q} = 2 \times 10^7 \text{G} \cdot \text{cm}^2, \]

where \( \varphi_0 \) is the quantum of the fundamental flux. If we substitute \( \lambda = 80 \text{ fm} \) and \( \xi = 30 \text{ fm} \) (typical values) in the expression for the critical magnetic field \( H_c \) is estimated to be the \( H_c \approx 10^{14} \text{ G} \).
VII. Intermediate State

1. Overview

- If the core is indeed a type-I superconductor, the magnetic field must be expelled from the superconducting region (Meissner effect).

- On the other hand: it takes a very long time to expel a typical magnetic flux from the neutron star core ($\sim 10^9$ years, G.Baym, C.Pethick, D.Pines, 1969). Therefore, if the magnetic field existed before the neutron star became a type-I superconductor, it is likely that magnetic field will remain there.

2. Resolution of the puzzle:

- The magnetic field could exist in macroscopically large regions where there are alternating domains of superconducting (type-I) matter and normal matter (Intermediate state). In this case, neutron stars could have long period precession.

- The well-know example of this phenomenon is due to Landau, 1937 who argued that in this case a domain
structure can be formed, similar to ferromagnetic systems, see commandment 5 below,

\[ a = \sqrt{\frac{\Delta d}{f(h)}}, \quad h \equiv \frac{H}{H_c}, \quad f(h \to 0) \simeq \frac{h^2}{\pi} \ln \frac{0.56}{h} \]
• Specifically, on a macroscopic distance scale, the magnetic flux must be embedded in the superconductor. This would mean that the superconductor is in an intermediate state.

• Another argument suggesting the same outcome follows from the fact that topology \( \int d^3x \vec{A} \cdot \vec{B} \) (magnetic helicity, linking number) is frozen in the environment with high conductivity; therefore, the magnetic field must remain in the bulk of the neutron star.


• The intermediate state is characterized by alternating domains of superconducting and normal matter. The superconducting domains will then exhibit the Meissner effect, while the normal domains will carry the required magnetic flux.

• The pattern of these domains is usually strongly related to the geometry of the problem.
• The simplest geometry, originally considered by Landau is a laminar structure of alternating superconducting and normal layers.

4. Lessons from condensed matter physics

• It is known that the domain morphology is not a thermodynamic state function; it depends on the path in field-temperature space through which the sample has been brought to a given point.

• For instance, cooling in zero field below $T_c$ and then applying the field, tends to produce patterns in which normal domains are embedded in a matrix of superconductor.

• When the same point in $T - H$ space is reached by cooling below $T_c$ in a fixed field the normal domains connect to the sample edges.

• These observations suggest that the patterns are not true ground states of the system– the sample is not in the thermodynamic equilibrium.
VIII. Conclusion

The consequences of this picture still remain to be explored. In particular, the standard theory of glitches (based on the ideas of type II superconductivity) has to be reconsidered.