Motivation

Large $N_{\text{flavor}}$ parametrisation of the chiral quark model

Results in the leading order of the large $N_{\text{flavor}}$ approximation (no fermions) pole trajectory and its relation to the spectral function in the $\sigma$ channel

The $\mu - T$ phase diagram

Collaborators: A. Jakovác, A. Patkós, P. Szépfalusy
Motivation

What can we expect from an effective model?

- to predict the existence of the TCP/CEP
- to give the universality class
- to help finding its location
- to give the shape of the transition line
- to tell us what is the soft mode at the CEP candidates: – scalar density fluctuation
  – sigma mode
- to give a description of the nonequilibrium dynamics near TCP/CEP
The linear sigma model with chiral fermions

extension of $SU(2) \times SU(2)$ model $2 \to N_f$

\[ L[\sigma, \pi^a, \psi] = L_M[\sigma, \pi^a] + L_F[\sigma, \pi^a, \psi] + \delta L_{ct}[\sigma, \pi^a, \psi], \]

broken symmetry phase: $\sigma(x) \rightarrow \sqrt{N} \Phi + \sigma(x), \quad m_q = g\Phi, \quad \Phi(0) = \frac{f_\pi}{2}$

\[
L_M = - \left[ \frac{\lambda}{24} \Phi^4 + \frac{1}{2} m^2 \Phi^2 \right] N - \left[ \frac{\lambda}{6} \Phi^3 + m^2 \Phi + h \right] \sigma \sqrt{N} + \\
+ \frac{1}{2} \left[ (\partial^\mu \sigma)^2 + (\partial^\mu \pi^a)^2 \right] - \frac{1}{2} \left[ m^2 + \frac{\lambda}{2} \Phi^2 \right] \sigma^2 - \frac{1}{2} \left[ m^2 + \frac{\lambda}{6} \Phi^2 \right] \pi^a \pi^a \\
- \frac{\lambda}{6\sqrt{N}} \Phi \left[ \sigma^3 + \sigma \pi^a \pi^a \right] - \frac{\lambda}{24N} \left[ \sigma^2 + \pi^a \pi^a \right]^2
\]

\[ L_F = \bar{\psi}(x) \left[ i\partial^\mu \gamma_\mu - m_q - \frac{g}{\sqrt{N}} \left( \sigma(x) + i \sqrt{2N_f} \gamma_5 T^a \pi^a(x) \right) \right] \psi(x) \]

chemical potential: $i\partial_t \Psi \to (i\partial_t + \mu) \Psi$

it is required that $m_q = g\Phi$ stays finite as $N_f = N^{1/2} \to \infty$

fermion contribution is $\mathcal{O}\left( \frac{1}{\sqrt{N}} \right)$ which precedes NLO meson effects
Leading order large N approach to the linear $\sigma$ model

used to study the properties of the $\sigma$ pole at finite temperature
A. Patkós, Zs. Sz., P. Szépfalusy, PLB 537 77; PRD 66 116004

• $\sigma$ is the quantum fluctuation of the amplitude of the chiral OP $\langle \bar{q}q \rangle$

• any change in the ground state is reflected upon the properties of $\sigma$
  $T \neq 0, n_B \neq 0$  $m_\sigma$ decreases during chiral symmetry restoration
the available phase space for the $\sigma \to 2\pi$ decay is squeezing
  threshold enhancement in the spectral function is produced
  $\to$ chance to see $\sigma$ as a sharp resonance in the matter

  T. Hatsuda, T. Kunihiro, PRL 55, 158; S. Chiku, T. Hatsuda, PRD 57, R6

Advantages of the large N expansion

+ makes strongly self-coupled theory tractable
+ the approach is insensitive to the choice of the renormalization point
+ leads to a 2$^{nd}$ order chiral transition for $h = 0$ and provides correct critical
description near $T_c \simeq 160$ MeV
+ Gives the same scenario for the $\sigma$ pole trajectory as the composite operator
formalism applied to QCD Barducci et al PRD 59, 1140224
Quantities of interest to leading order in $N$

Equation of state: $m^2 + \frac{\lambda}{6} \Phi^2(T) + m \frac{\pi}{\Phi} - \frac{h}{\Phi} = 0$

nonperturbatively renormalized

$\frac{m^2}{\lambda} + \frac{\Lambda^2}{96\pi^2} = \frac{m_R^2}{\lambda_R}$

$\frac{1}{\lambda} + \frac{1}{96\pi^2} \ln \frac{\epsilon\Lambda^2}{M_0^2} = \frac{1}{\lambda_R}$

implies $G^{-1}_\pi(p) = p^2 - \frac{h}{\Phi(T)}$

Goldstone theorem when $h \to 0$.

$\sigma$ pole: solution of $G^{-1}_\sigma(p) = 0$ on the complex plane

$$G^{-1}_\sigma(p) = p^2 - m^2 - \left[ \pi + \Phi \sqrt{\pi} + \Phi \pi \Phi + \ldots + \Phi \pi \Phi \ldots \right]$$

$$= p^2 - \frac{h}{\Phi(T)} - \frac{\lambda_R \Phi^2(T)/3}{1 - \lambda_R b_R(p)/3}$$

$b(p_0) = \pi \pi = b_0(p_0) + b_T(p_0)$ originally defined on the upper half of the $p_0$-plane.

Spectral function: $\rho_\sigma(p_0, p = 0, T) = -\frac{1}{\pi} \lim_{\epsilon \to +0} \text{Im} G_\sigma(p_0 + i\epsilon, 0, T)$
Analytical continuation to the 2\textsuperscript{nd} Riemann sheet

\[ b^>_0, R(p_0) = \frac{1}{16\pi^2} \left[ \ln \frac{m^2_{\pi}(T)}{M_0^2} - \sqrt{1 - \frac{4m^2_{\pi}(T)}{p_0^2}} \left( \text{arctanh} \sqrt{1 - \frac{4m^2_{\pi}(T)}{p_0^2}} + i\pi \right) \right] \]

\[ b^>_T(p_0) = \frac{1}{4\pi^2} \mathcal{P} \int_{m_{\pi}(T)}^{\infty} dx \frac{\sqrt{x^2 - m^2_{\pi}(T)/T^2}}{p_0^2/4T^2 - x^2} \frac{1}{\exp(x) - 1} \]

\[-\frac{i}{8\pi} \Theta(p_0 - 2m_{\pi}(T)) \sqrt{1 - \frac{4m^2_{\pi}(T)}{p_0^2}} \frac{1}{\exp(p_0/2T) - 1} \]

\( b_T(p_0) \) has discontinuity along the real axis for \( p_0 > 2m_{\pi}(T) \)

continuity across the real axis is \textit{imposed} for \( p_0 > 2m_{\pi}(T) \)

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for Im\( p_0 < 0 \):

\[ b^<_0(p_0) = b^>_0(p_0) \]

\[ b^<_T(p_0) = b^>_T(p_0) - \frac{i}{4\pi} \frac{\sqrt{1 - 4m^2_{\pi}(T)/p_0^2}}{\exp(p_0/2T) - 1} \]
Fixing $\lambda_R$ with the $\sigma$ pole at $T = 0$

With the pole parametrization $p_0 = 2m_{G0} + M_0 \exp(-i\varphi_0)$, $0 < \varphi_0 < \pi$

one solves $G^{-1}_{\sigma}(p_0, p = 0) = 0$ and determines $M_\sigma = \text{Re} p_0$, $\Gamma/2 = \text{Im} p_0$

good value of $\lambda_R$: for which we get closest to the phenomenological values

constraint on $\lambda_R$: tachyonic pole on the imaginary axis of the physical sheet

$$m_\pi(0) = 0 \quad \lambda_R = 310\quad M_\sigma/\Gamma \sim 1\quad M_\sigma = 3.5 f_\pi$$

$$m_\pi(0) \neq 0 \quad \lambda_R = 400\quad M_\sigma/\Gamma = 1.4\quad M_\sigma = 3.95 f_\pi$$

A light and not broad enough $\sigma$ is accessible in LO large N approximation.
Temperature driven pole trajectory

The $\sigma$-pole is obtained on the 2\textsuperscript{nd} Riemann sheet as the solution of:

$$\left(p_0^2 - m_\pi^2(T)\right) \left(1 - \frac{\lambda_R}{6} (b_0^< (p_0) + b_T^< (p_0))\right) - \frac{\lambda_R}{3} \Phi^2(T) = 0.$$ 

**Basic feature:** the imaginary part of the pole will eventually decrease with increasing $T$ and the pole approaches the threshold.

However, in the early stage the imaginary part is actually increasing.

The value of $m_\pi(0)$ tunes the trajectory of the pole.

With decreasing $m_\pi(0)$ the pole approaches the imaginary axis.

Below $m_\pi(0) = 61\text{MeV}$ it moves a while on the imaginary axis before switching over to the real axis.

In the chiral limit the pole goes to the origin along the imaginary axis. Mechanism for universal scaling.
\( m_\pi(T = 0) = 140 \text{ MeV} \)

**Pole trajectory**

\[ \begin{align*}
2m_\pi(T) & \text{ increases with } T \\
T^{**} \approx 0.69m_\pi(0) & : \text{ real part of the 4th quadrant pole goes below the threshold. It collides with its 1st quadrant mirror and splits up in two real poles.}
T^* \approx 1.07m_\pi(0) & : \text{ One of the poles goes over the 1st Riemann sheet and describes a stable particle}
\end{align*} \]

**Spectral function**

Threshold enhancement occurs for \( T \in (T^{**}, T^*) \)

In this interval the spectral function takes its maximum close to the threshold.

\( \rho_\sigma \) reflects the characteristics of the \( T = 0 \sigma \) pole only up to \( T < T^{**} \)
Threshold enhancement

Scalar-isoscalar spectral function:

\[
\rho_{\sigma}(p_0, \mathbf{p} = 0, T) = \frac{\lambda_R^2 \Phi^2(T) \text{Im} b_R^>(p_0) / 18\pi}{\left[ (p_0^2 - m^2_{\pi}(T)) \left( 1 - \frac{\lambda_R}{6} \text{Re} b_R^>(p_0) \right) - \frac{\lambda_R}{3} \Phi^2(T) \right]^2 + \left[ (p_0^2 - m^2_{\pi}(T)) \frac{\lambda_R}{6} \text{Im} b_R^>(p_0) \right]^2}
\]

The second term in the denominator vanishes more slowly as \( p_0 \to 2m_{\pi}(T) \)

near threshold behavior of the two terms in the denominator:

\[
(p_0^2 - m^2_{\pi}(T)) \left( 1 - \frac{\lambda_R}{6} \text{Re} b_R^>(p_0) \right) - \frac{\lambda_R}{3} \Phi^2(T) \approx p_0^2 - 4m^2_{\pi}(T)
\]

\[
(p_0^2 - m^2_{\pi}(T)) \frac{\lambda_R}{6} \text{Im} b_R^>(p_0) \approx \left[ p_0^2 - 4m^2_{\pi}(T) \right]^{1/2}
\]

Near the threshold:

\[
\rho_{\sigma}(p_0, 0, T^*) \sim \frac{1}{\sqrt{1 - \frac{4m^2_{\pi}(T^*)}{p_0^2}}}
\]
Comparison

Optimized Perturbation Theory
Y. Hidaka, O. Morimatsu, T. Nishikawa

Large $N$
Patkós A., Szép Zs., Szépfalusy P.
Method for solving the chiral quark model

fermions are taken into account perturbatively (contribution with lowest power of $g$ only)

pion pole: $m^2 + \frac{\lambda}{6} \Phi^2 + \pi + p^2 = M^2$

EoS: $\left[ m^2 + \frac{\lambda}{6} \Phi^2 + \frac{\lambda}{6N} \langle \pi^a \pi^a \rangle \right] \Phi + \frac{g}{N} \langle \bar{\psi} \psi \rangle = h$

$\sigma$-pole: $G_\sigma^{-1} = p^2 - \frac{h}{\Phi} - \frac{\lambda R \Phi^2 / 3}{1 - \frac{\lambda R}{3}} - \left[ - \frac{g}{N \Phi} \langle \bar{\psi} \psi \rangle \right] = 0$

pion propagator in the bubbles parameterized with $M^2$ (self-consistent)

analytical continuation: below the $\sigma \to 2\psi$ threshold

$T = \mu = 0 \quad g = m_q / \Phi = 2m_N / 3 f_\pi = 6.72$
The phase diagram in the chiral limit, $\hbar = 0$

Gap equation:

$$M^2 [1 - g^2 N_c B(M, m_q)] = 0 \rightarrow M = 0$$

Goldstone theorem

EoS:

$$\Phi \left[ m_R^2 + \frac{\lambda_R}{6} \Phi^2 + \frac{\lambda_R}{6} I_{tad}(M = 0) - 4 N_c \frac{g^2}{\sqrt{N}} I_{tad}^\Psi (m_q) \right] = 0$$

analytical determination of the 2nd order line $\Phi(T_c) = 0$:

$$m_R^2 + \frac{\lambda_R}{72} T_c^2 - \frac{g^2 T_c^2}{2 \pi^2} N_c \left( \text{Li}_2(-e^{\mu/T_c}) + \text{Li}_2(-e^{-\mu/T_c}) \right) = 0$$

2nd order line ends when

$$\frac{\lambda_R}{6} + \frac{g^4 N_c}{4 \pi^2} \left. \left[ \frac{\partial}{\partial n} \left( \text{Li}_n(-e^{-\mu/T_c}) + \text{Li}_n(-e^{-\mu/T_c}) \right) \right] \right|_{n=0} - \ln \frac{c T_c}{M_0 B} = 0$$

2nd order line quadratic in $\mu$

$$\text{Li}_\nu(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^\nu}$$

$$\int_0^\infty dk \frac{k^{s-1}}{e^{k-k-\mu} \pm 1} = \pm \Gamma(s) \text{Li}_s(\pm e^\mu)$$
\( T_c(\mu = 0) = 139 \) MeV

\( (T, \mu)_{TCP} = (61, 831) \) MeV

large \( g \) \( \rightarrow \) strong effect of fermions

for \( h \neq 0 \) TCP \( \rightarrow \) CEP

softening of the system

Y. Hatta, T. Ikeda PRD 67 014028

even lower value of \( T_{CEP} \)

- Z. Fodor, S. D. Katz, hep-lat/0402006

CEP: \( T_E = 162 \pm 2 \) MeV, \( \mu_E = 360 \pm 40 \) MeV \( n_f = 2 + 1 \)

\( T_c(\mu = 0) = 164 \pm 2 \) MeV

- \( T_c(\mu = 0) = (173 \pm 8)\) MeV, 2-flavors \quad Karsch et al, Nucl. Phys. B 605 (2001) 579

- curvature of the 2\textsuperscript{nd} order line:

\[
\left. \frac{T_c d^2 T_c}{2 d\mu^2} \right|_{\mu=0} = -0.101
\]

\(-0.07(3)\) lattice result

\[ \eta = \ln\left( \frac{M_{0B}}{M_{0F}} \right) \]

parameters can be arranged such as \( T_c(\mu = 0) \in (150, 170) \)

\( T_{TCP} \) doggedly stays below 70 MeV

not very sensitive to the parameters
Consistency in the $\sigma$ channel restricts $M_{0B}$!

$\lambda_R$ and $M_{0B}$ chosen such as to have realistic $\sigma$-pole position $M_\sigma \simeq 450$ MeV

Motion of poles at $\lambda_R = 0$ with decreasing values of $M_{0B}$ low scale imaginary poles appear $\rightarrow$ instability lower limit on $M_{0B}$
Why is $T_c$ so low?

- almost all effective approaches give a low value for TCP (CEP) MeV

<table>
<thead>
<tr>
<th>Authors</th>
<th>Ladder-QCD</th>
<th>CEP $=(T, \mu)^{CEP}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scavenius et al PRC 64</td>
<td>L$\sigma$M [NJL]</td>
<td>$(99, 621) [(46, 996)]$</td>
</tr>
<tr>
<td>B.-J. Schaefer et al, nucl-th/0403039</td>
<td>RG improved L$\sigma$M</td>
<td>$(52, 753)$</td>
</tr>
<tr>
<td>Barducci et al, PLB 564</td>
<td>Ladder-QCD</td>
<td>$(97, 240)$</td>
</tr>
<tr>
<td>Y. Hatta, T. Ikeda, PRD 67</td>
<td>Imprv. ladder-QCD</td>
<td>$(107, 627)$</td>
</tr>
<tr>
<td>Halasz et al, PRD 58</td>
<td>RMM</td>
<td>$(120, 700)$</td>
</tr>
<tr>
<td>J. Berges, K. Rajagopal, NPB 538</td>
<td>Inst.-induced Int.</td>
<td>$(101, 633)$</td>
</tr>
<tr>
<td>Z. Fodor, S.D. Katz</td>
<td>$n_f = 2 + 1$</td>
<td>$(162, 360)$</td>
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</tbody>
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- problems with the approach
  not yet a complete theory at $1/\sqrt{N}$ level (only one loop in fermions)

- problems with the model: not appropriate for the description of CEP in QCD
  inclusion of gluon degrees of freedom might prove important above 150 MeV


  phase transition driven by higher excited hadronic phase not by the light degrees of freedom

Behavior of the $\sigma$ pole near $T_c$

- chiral limit

$\sigma$ is the soft mode as expected

$$\text{Re} p_0 \sim \Phi(T) \quad \text{Im} p_0 \sim \Phi^2(T)$$

$$\Phi(T) \sim (T - T_c)^\beta \quad \beta = \frac{1}{2} \text{ for the } 2^{\text{nd}} \text{ order line}$$

$$\beta = \frac{1}{4} \text{ at the TCP}$$

- physical pion mass (preliminary)

$\sigma$ stays massive at the CEP


but how to explain the fattening of the potential at CEP ?

change: double $\rightarrow$ single welled potential
Possible improvement of the approach

A complete $O(1/\sqrt{N})$ treatment requires the resummation of rainbow and ladder type corrections achieved with a Dyson-Schwinger treatment of the quark propagator:

\[
\begin{align*}
\text{...} &= \text{...} + \text{...} \\
\end{align*}
\]

Expected effect: generation of an effective mass $M_q = G(T, \mu)\Phi(T, \mu)$

rough estimate: gap equation for fermions $T \rightarrow T_c (\Phi \rightarrow 0)$

\[
M_q = g\Phi(T, \mu) - \frac{g^2 M_q}{16\pi^2} B_\chi(k_0 = M_q, k = 0) \rightarrow G(T, \mu) = \frac{g}{1 + \frac{g^2}{16\pi^2} B_\chi(M_q = 0, 0)}
\]

\[
B_\chi(0, 0) = \begin{cases} 
\ln \frac{M^2(0, 0)}{M_0^2}, & \mu = T = 0 \\
\ln \frac{T^2}{M_0^2} \frac{c^2}{4\pi^2} - 2 \frac{\partial}{\partial n} \left[ \ln(e^{\mu/T}) + \ln(e^{-\mu/T}) \right] \bigg|_{n=0}, & \mu \neq 0, T \neq 0 \\
\ln \frac{T^2}{M_0^2} \frac{c^2}{16\pi^2}, & \mu = 0, T \neq 0
\end{cases}
\]

result: 10-15% increase of the $T_{TCP}$
using $M \equiv 0$ is not correct the EoS is used when obtaining it is valid only in the minimum of the potential

$$I^\pi_{tad}(M(\Phi)) \simeq I^\pi_{tad}(0) + \frac{d}{d\Phi^2} I^\pi_{tad}(M(\Phi)) \bigg|_{\Phi=0} \Phi^2 = I^\pi_{tad}(0) - \frac{T_c dM}{4\pi d\Phi^2} \bigg|_{\Phi=0} \Phi^2$$

hight temperature expansion: $I^\pi_{tad}(M) \simeq \frac{M^2}{8\pi^2} \ln \frac{cT}{M_0B} + \frac{T^2}{12} - \frac{MT}{4\pi}$

modification in the location of TCP

$$\frac{\lambda}{6} \left( 1 - \frac{T_{TCP} dM}{4\pi d\Phi^2} \bigg|_{\Phi=0} \right) +$$

$$\frac{g^4 N_c}{4\pi^2} \left[ \frac{\partial}{\partial n} \left( Li_n(-e^{-\mu_{TCP}/T_{TCP}}) + Li_n(-e^{-\mu_{TCP}/T_{TCP}}) \right) \bigg|_{n=0} - \ln \frac{cT_{TCP}}{M_0B} \right] = 0$$
Conclusions and outlook

Consistent one-loop inclusion of fermions was performed for $\mu \neq 0$

- analytic characterization of the phase structure is given in the chiral limit
- the obtained phase diagram is qualitatively correct although the transition temperature is too low
- study of the excitation spectra proves to be important to restrict the range of the renormalization scale
- improve the approach by dynamically generating the fermion mass in an attempt to push up the value of $T_c$

if works - study of the CEP position for physical pion mass
  - application of the method to the three-flavor $SU(3)_L \times SU(3)_R$ case with realistic constituent quark masses