QCD Thermodynamics from Imaginary \( \mu \)

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Motivation

- Imaginary chemical potential
- "Analyticity" of the pseudo-critical line
- \( T_c(\mu) \): \( N_f = 2, 3, 4 \) results and comparison with other methods
- Critical endpoint and its quark mass dependence for \( N_f = 3 \)
- Outlook on \( N_f = 2 + 1 \)
- Conclusions
Motivation

Fermion determinant complex for SU(3), $\mu \neq 0$

$\Rightarrow$ no standard Monte Carlo importance sampling

**Recent numerical methods:**

- Fodor, Katz
- Allton et al.
- de Forcrand, O.P.

No solution to sign problem

but tricks to ease or side-step it $\Rightarrow$ approximations, problems?

- reweighting: potential overlap problem, error analysis
- reweighting and/or Taylor expansion: dto., convergence
- imaginary $\mu$: limited range of applicability (convergence)

$\Rightarrow$ *Pursue all, cross check to control errors*
Lattice QCD at finite temperature
Karsch hep-lat/0109017

3-flavour phase diagram

The critical temperature

pure gauge: \( T_c = (271 \pm 2) \text{ MeV} \)

chiral limit of

\( N_f = 2 : \) \( T_c = \begin{cases} (171 \pm 4) \text{ MeV}, & \text{clover-impr. Wilson} \\ (173 \pm 8) \text{ MeV}, & \text{improved KS} \end{cases} \)

\( N_f = 3 : \) \( T_c = (154 \pm 8) \text{ MeV}, \) improved KS

N.B: pure gauge: cont. limit reached, 
fermions: \( a \approx 0.3 \text{ fm, still coarse} \)
Imaginary $\mu$ / fixed baryon density

$$Z_B(T, V) = \frac{1}{2\pi} \int_0^{2\pi} d\nu \, e^{-i\nu B/T} Z(\mu = i\nu, T, V)$$

measure of $Z$ positive, standard MC possible

I. compute $Z_B$ by numerical Fourier trafo

Hubbard model at large $T$ and small $B$

does not work in thermodynamic limit, QCD?

II. analytic continuation of observables

$\langle \bar{\psi}\psi \rangle$ at real and imag. $\mu$ in strong coupling
3d eff. action for Matsubara zero-modes ($T \gtrsim 2T_c$):

$$S \rightarrow S + iz \int d^3x \, \text{Tr} \, A_0 \; ; \; \; \; \; z = \frac{\mu \, N_f}{T \, 3\pi^2} \quad (\ll \frac{\mu}{T})$$

complex, but sign prob. mild, physical volumes possible

$M(\mu)$ analytic in $\mu$ (no massless modes, no transition)

$$\frac{M}{T} = c_0 + c_1 \left( \frac{\mu}{\pi T} \right)^2 + c_2 \left( \frac{\mu}{\pi T} \right)^4 + \ldots$$

fit $c_0, c_1, c_2$ to data from real and imag. $\mu$

works for $\mu \lesssim 1.5T \Rightarrow \text{Critical line in 4d?}$
QCD at complex $\mu$: general properties

$$Z(V, \mu, T) = \text{Tr} \left( e^{-\left(\hat{H} - \mu \hat{Q}\right)/T} \right); \quad \mu = \mu_R + i\mu_I$$

$\Rightarrow$ Taylor expansion is in $\bar{\mu} = \mu/T$

$\mu$ term breaks $\mathcal{T}, \mathcal{C}; \quad \mathcal{T}$ compensated by $\mu \rightarrow -\mu$

$\Rightarrow Z(\bar{\mu}) = Z(-\bar{\mu})$

**Periodicity:** $Z(\overline{3})$ transformation equivalent to shift in $\mu_I$

$$\Rightarrow Z(\bar{\mu}_R, \bar{\mu}_I) = Z(\bar{\mu}_R, \bar{\mu}_I + 2\pi/N)$$

$Z(\overline{3})$ sectors identified by Polyakov loop $\langle P(x) \rangle = |\langle P(x) \rangle| e^{i \langle \varphi \rangle}$

$$\langle \varphi \rangle = -2\pi/3$$

$$\langle \varphi \rangle = 0$$

$\bar{\mu}_R$

$\bar{\mu}_I$

$\pi$

$\pi/3$

$Z(3)$-transitions:

$$\bar{\mu}_I^c = \frac{2\pi}{3} \left( n + \frac{1}{2} \right)$$

pert./strong coupling:

1st order for deconf. phase

crossover for conf. phase

within arc:

$$\langle O \rangle = \sum_{n}^{N} c_n \bar{\mu}_I^{2n} \Rightarrow \mu_I \rightarrow i\mu_I$$
Analyticity of the (pseudo-) critical line with P. de Forcrand

phase transition from maximum of susceptibilities:

$$\chi(\beta, \bar{\mu}, V) = VN_t \langle (\mathcal{O} - \langle \mathcal{O} \rangle)^2 \rangle, \quad \mathcal{O} \in \{\text{plaq}, \bar{\psi}\psi, |P(x)|\}$$

finite volume: suscept. finite and analytic in all cases

- Location of transition:

Critical line $\beta_c(\bar{\mu})$ defined by peak $\chi_{max} \equiv \chi(\bar{\mu}_c, \beta_c)$

$$\left. \frac{\partial \chi}{\partial \beta} \right|_{\bar{\mu}_c, \beta_c} = 0, \quad \left. \frac{\partial^2 \chi}{\partial \beta^2} \right|_{\bar{\mu}_c, \beta_c} < 0.$$

Implicit function theorem:

$\chi(\beta, \bar{\mu})$ analytic $\Rightarrow \beta_c(\bar{\mu})$ analytic!

symmetries: $\chi(\bar{\mu}) = \chi(-\bar{\mu})$ $\Rightarrow \beta_c(\bar{\mu}) = \beta_c(-\bar{\mu})$

$$\beta_c(\bar{\mu}) = \sum_n c_n (a\bar{\mu})^{2n}$$
Approaching the thermodynamic limit

Def. of $\beta_c(\bar{\mu})$ not unique on finite V (e.g. $\partial_\bar{\mu} \chi = 0$)

different definitions:

Crit. line unique in thermodynamic limit! (not for crossover)

For large V it is approached arbitrarily well by $\partial_\beta \chi = 0$

- Order of transition:

  finite volume scaling:  \((\beta_c(V) - \beta_c(\infty)) \sim V^{-\sigma}\)

  \[
  \begin{align*}
  \sigma &= 1 & \text{1st order} \\
  \sigma &< 1 & \text{2nd order} \\
  \sigma &= 0 & \text{crossover}
  \end{align*}
  \]
$N_f = 2$ results

$8^3 \times 4$, KS fermions, $m_\pi \approx 300$ MeV

$(T_c(\mu = 0) \sim 170$ MeV, $a \sim 0.3$ fm)

$Z(3)$ transition: phase of $P(x)$ for $(a \mu_1)^c = \pi/12$

1st order for deconf. phase
continuous for conf. phase

⇒ schematic phase diagram: (cf. $N_f = 4$, D’Elia, Lombardo)
The deconfinement line $T_c(\mu)$: susceptibilities

plaquette

Polyakov loop

Lattice phase diagram for $\mu = i\mu_I$

control over Taylor expansion, no volume restrictions!
Comparison:

\[ a \sim 0.3 \text{ fm in all calculations} \]

<table>
<thead>
<tr>
<th>Method</th>
<th>( N_f, m_q )</th>
<th>largest lattice</th>
</tr>
</thead>
<tbody>
<tr>
<td>reweighting (FK)</td>
<td>2+1, ( am_u = 0.025 ), ( am_s = 8am_u )</td>
<td>( 8^3 \times 4 )</td>
</tr>
<tr>
<td>rew. + Taylor (A et al.)</td>
<td>2, ( am = 0.1 )</td>
<td>( 16^3 \times 4 )</td>
</tr>
<tr>
<td>imag. ( \mu ) (FP)</td>
<td>2, ( am = 0.025 )</td>
<td>( 8^3 \times 4 )</td>
</tr>
<tr>
<td>imag. ( \mu ) (EL)</td>
<td>4, ( am = 0.05 )</td>
<td>( 16^3 \times 4 )</td>
</tr>
</tbody>
</table>
**Comparison:** \( N_f = 4 \), imag. \( \mu \) vs. reweighting  

\[
\beta = \beta_c - 0.9\mu^2 + 0.5\mu^4 
\]

D’Elia, Lombardo

**Comparison:** analyt. continuation ok in SU(2)  

Giudice Papa

**Comparison:** real \( \mu \) vs. imag. \( \mu \) to leading order  

Allton et al.
$N_f = 3$ results, $T_c(\mu, m)$:

vary quark masses $m$, much more stats.

check for NLO terms in Taylor series:

$$\beta_c(a\mu, am) = \sum_{k,l=0} c_{kl} (a\mu)^{2k} (am)^l$$

⇒ evidence for $\mu^4$! no evidence for $\mathcal{O}(m^2, \mu^2 m)$

$$\frac{T_c(\mu, m)}{T_c(\mu = 0, m_c(0))} = 1 + 1.94(2) \left( \frac{m - m_c(0)}{\pi T_c} \right) + 0.602(9) \left( \frac{\mu}{\pi T_c(0, m)} \right)^2 + 0.23(9) \left( \frac{\mu}{\pi T_c(0, m)} \right)^4$$
$N_f$ dependence

N.B.: only $N_f = 3$ has $\mu^4$ correction
The critical endpoint  Phase diag. 3d: \((T, \mu, m)\)

- confined/deconfined \(\Rightarrow\) pseudo-crit. surface \(T_c(\mu, m)\)

On this surface,
- 1st. order crossover \(\Rightarrow\) line of crit. points \(T^*(\mu) = T_c(\mu, m_c(\mu))\)

Project on \((T, \mu)\):

```
\begin{center}
\begin{tikzpicture}
% TikZ code for the diagram
\end{tikzpicture}
\end{center}
```

Project on \((m, \mu)\):

```
\begin{center}
\begin{tikzpicture}
% TikZ code for the diagram
\end{tikzpicture}
\end{center}
```

\(m = 0 \Rightarrow\) true chiral phase transition

Expect: \[
\frac{m_c(\mu)}{m_c(\mu=0)} = 1 + c_1 \left( \frac{\mu}{\pi T} \right)^2 \Rightarrow c_1 \leq 9
\]
Criticality: cumulant ratios, **3d Ising universality**:

\[
B_4(m_c, \mu_c) = \frac{\langle (\delta \bar{\psi} \psi)^4 \rangle}{\langle (\delta \bar{\psi} \psi)^2 \rangle^2} \to 1.604, \quad V \to \infty
\]

(\text{\textless}, \text{\textgreater} \Rightarrow \text{first-order, crossover})

fit data to Taylor series about crit. point

\[
B_4(am, a\mu) = 1.604 + B (am - am_c(0) + A(a\mu)^2) + \ldots
\]

Chain rule: \[
\frac{dam_c}{d(a\mu)^2} = -\frac{\partial B_4}{\partial(a\mu)^2} \left( \frac{\partial B_4}{\partial am} \right)^{-1} = -A
\]

need **VERY LONG** MC runs for sufficient tunneling statistics

\begin{itemize}
  \item first-order
  \item crossover
\end{itemize}
\[ B_4(am, a\mu) = 1.604 + B \left( am - am_c(0) + A(a\mu)^2 \right) \]

\[ m_c(0) \text{ in agreement with previous result for } \mu = 0 \]

\textbf{Ising FSS} \ (\beta = \beta_c(m, \mu)):\quad \xi \propto |m - m_c(\mu)|^{-\nu}

\[ \Rightarrow B_4(L/\xi) = B_4 \left( (L/\xi)^{1/\nu} \right) \]

\[ \nu = 0.62(3) \]
\[ \nu(\text{Ising}) = 0.63 \]
Critical quark mass, $\mu_c$:

\[
\Rightarrow \frac{m_c(\mu)}{m_c(\mu = 0)} = 1 + 0.84(36) \left( \frac{\mu}{\pi T} \right)^2 + \ldots
\]

tentative comparison

Taylor exp. rew., improved action: $c_1 \sim 290...2500$ \text{Allton et al.}

$\Rightarrow$ our critical point at larger $\mu_c$

tiny change in $m \Rightarrow$ huge change in $\mu_c$
**Outlook for** $N_f = 2 + 1$

Phase diag. 4d: $(T, \mu, m_{u,d}, m_s)$

Two observables: $B^u_d(m_{u,d}, m_s, \mu), B^s_d(m_{u,d}, m_s, \mu)$

expand about $N_f = 3, \mu = 0$ critical quark mass $m_c$

$$B^i_4(m_{u,d}, m_s, \mu) = \sum_{n,l,k} b^i_{n,l,k}(m_{u,d} - m_c)^n(m_s - m_c)^l(\mu)^{2k}$$

$$B^u_d(m, m, \mu) = B^s_d(m, m, \mu) = B^3_4(m, \mu)$$

⇒ constraints: $\mu^{2k}$-coeffs. equal in all three!

⇒ leading order reduces to $\mu = 0$ investigation

⇒ strategy:

I. find $(m_{u,d}, m_s)_c$ for $\mu = 0$

II. repeat analysis for $(T, \mu, m_s)$ or $(T, \mu, m_{u,d})$
Preliminary results:

$(m_s, m_{u,d})$ phase-diagram, $\mu = 0$:

$\Rightarrow$ strong non-linearities in $m_{u,d}$

$\Rightarrow$ qualitatively consistent with our $N_f = 3$ results
Conclusions

Simulations of small baryon densities possible at finite T

⇒ analytic continuation of non-singular observables, control over systematics

• location of transition line consistent with other approaches

• (pseudo-) critical line $T_c(m, \mu)$ extremely flat, small quark mass dependence

• critical point for physical QCD not yet under control
  ⇒ strong quark mass dependence

Hard work (continuum limit!), but high T phase diagram in reach!