Inhomogeneous color superconductivity

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Summary

• Color Superconductivity and the Loff phase
• Effective gap equation for the Loff phase

In collaborations with R. Casalbuoni, M.Ciminale, R. Gatto, M. Mannarelli, M. Ruggieri
Seattle, May-June 2004
Color Superconductivity

- QCD attractive interaction in color antisymmetric antitriplet channel
- Cooper’s theorem: high density, small T: superconductivity
- $S=0$ state: Pauli principle implies antisymmetry in flavor
- At very high densities: Color Flavor Locking (CFL)
- With two flavors (large strange mass, intermediate densities: $\mu \approx 400$ MeV: 2Color superconductivity (2SC)
- Robust phenomenon: $\Delta \approx 100$ MeV, $T_c \approx 40$ MeV, possible in core of compact stars (Alford, Rajagopal, Wilczek)

\[
< 0 \mid \psi^\alpha_i \, C \, \psi^\beta_j \mid 0 > = \Delta \sum_{k=1}^{3} \epsilon_{\alpha\beta k} \epsilon_{ijk} \quad \text{CFL}
\]

\[
< 0 \mid \psi^\alpha_i \, C \, \psi^\beta_j \mid 0 > = \Delta \epsilon_{\alpha\beta 3} \epsilon_{ij3} \quad \text{2SC}
\]

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If $\delta \mu \neq 0$ it can be energetically favored a state where the Cooper pair has total momentum $2q \neq 0$.

LOFF = Larkin-Ovchinnikov & Fulde-Ferrel (1964);

$\Delta (r) \propto < 0 | \psi (r) \psi (r) | 0 > = e^{i(p_1 + p_2) \cdot r} \Delta$

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Fulde Ferrel state

- One plane wave \( \Delta(r) = \Delta \exp(2i\mathbf{q} \cdot \mathbf{r}) \)
- Results at \( T=0: \)

**BCS (homogeneous)**
\[ \delta \mu_1 < \delta \mu < \delta \mu_2 \quad \text{normal (non s.c.)} \]

\[
\delta \mu_1 = \frac{\Delta_0}{\sqrt{2}} \quad \delta \mu_2 = 0.75 \Delta_0
\]

For QCD: Alford, Bowers, Rajagopal

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**LOFF state in QCD**

\[ \delta \mu \text{ can be generated by mass effects:} \]

Kundu-Rajagopal

Or it arises from weak processes

\[ d \rightarrow u \ e \ \nu \]

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Glitches

Sudden increases of pulsar rotational frequency

Vela pls

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Structure of a Pulsar

External crust
Internal crust
Superfluid neutrons
Core

Continuous emission of e.m. radiation and slowing down

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In the neutron star normal neutrons, mainly in the metallic crust, coexist with superfluid neutrons. The latter cannot rotate due to lack of friction. There are however vortex lines pinned at the crust:

\[ n(r) = \text{number of vortex lines} \]

The only possibility for the superfluid to change its angular velocity is by means of a radial motion of v.l.s

\[ \frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{v}_r) = 0 \]

\[ \frac{\partial \Omega}{\partial t} = -2\pi \kappa n \frac{\mathbf{v}_r}{r} \]

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Starting from equal velocities
\[ \Omega = \Omega_c \]

Due to the spinning down of the star \( \Omega_c \) decreases; as long as v.l.s are pinned at the crust radial motion is forbidden and \( \Omega \) cannot decrease (the superfluid neutrons can’t spin down). When the difference \( \Omega - \Omega_c \) overcomes the pinning force, v.l.s move outwards, \( \Omega \) decreases, \( \Omega_c \) increases and there is the glitch.

In Loff phase the normal component, analogous to the crust, is provided by the crystal planes where \( \Delta = 0 \). They can supply part of the pinning force on the v.l.s.
Gap equation

For small $\Delta$, at the 2nd order phase transition a Ginzburg-Landau expansion is possible. For QCD: Bowers & Rajagopal

\[ \Omega = \alpha \Delta^2 + \beta \Delta^4 + \gamma \Delta^6 + \delta \Delta^8 \ldots \]

Crystalline structure given by

\[ \Delta(\mathbf{r}) = \sum_m \Delta_m \exp(2i \mathbf{q}_m \cdot \mathbf{r}) \]

Favored structure: cubic (fcc)

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Quasiparticles dispersion laws

For fermions d.l. is obtained solving

\[ S^{-1}\chi = 0 \]

\[
(S^{-1})^{ij}_{\alpha\beta} = \begin{pmatrix}
\delta_{\alpha\beta} \left[ \delta_{ij}(E + iv \cdot \nabla) + \delta \mu(\sigma_3)_{ij} \right] & -\epsilon_{\alpha\beta\gamma} \epsilon_{ij} \Delta(r) \\
-\epsilon_{\alpha\beta\gamma} \epsilon_{ij} \Delta^{*}(r) & \delta_{\alpha\beta} \left[ \delta_{ij}(E - iv \cdot \nabla) + \delta \mu(\sigma_3)_{ij} \right]
\end{pmatrix}
\]

where \( E \) is the quasi-particle energy and \( v \) is the Fermi velocity, that in QCD with massless quarks satisfies \( v = |v| = 1 \). Let us define

\[
\chi_{\xi}^{\alpha} = \left( \begin{array}{c}
\bar{C}_{\xi}^{\alpha} \\
-i(\sigma_2)_{\alpha\beta} \bar{F}_{\xi}^{\beta}
\end{array} \right).
\]
Performing the unitary transformation

\[ \tilde{G}^\alpha_{i} = \left( e^{i\delta_{\mu} \sigma_{3} \cdot \mathbf{r}/\hbar^{2}} \right)_{ij} G^\alpha_{j}, \quad \tilde{F}^\alpha_{i} = \left( e^{-i\delta_{\mu} \sigma_{3} \cdot \mathbf{r}/\hbar^{2}} \sigma_{2} \right)_{ij} F^\alpha_{j} \]

\[ (E + i\mathbf{v} \cdot \nabla)G - i\Delta(\mathbf{r})F = 0 \]
\[ (E - i\mathbf{v} \cdot \nabla)F + i\Delta(\mathbf{r})^*G = 0 \]

\[ E = E(\xi) \]
(\( \xi = \) longitudinal momentum measured from the fermi surface) gives the fermionic dispersion law

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For the FF condensate we take the direction of the Cooper pair total momentum $2q$ along the $z$--axis. The gap

$$\Delta(r) = \Delta e^{2iqz} \quad (8)$$

is therefore a complex number. If we take $G = \hat{G}(r)e^{iqr}$, $F = \hat{F}(r)e^{-iqr}$, we get from Eqs.(7)

$$[E - qv_x + iv \cdot \nabla] \hat{G}(r) = +i\Delta \hat{F}(r),$$
$$[E - qv_x - iv \cdot \nabla] \hat{F}(r) = -i\Delta \hat{G}(r). \quad (9)$$

**Dispersion law**

$$E_{\nu,d} = \pm \delta \mu \mp q \cdot v + \sqrt{\xi^2 + \Delta^2}$$
Strip (P=2), Cube (P=8)

Both the strip and the cubic structure have real $\Delta(r)$. The solutions of Eqs. (7) are Bloch functions

$$G = u(r)e^{ikr}, \quad F = w(r)e^{ikr},$$

(12)

with $u(r)$ and $w(r)$ periodic functions and $k$ in the first Brillouin zone. They satisfy

$$[E - \xi + i\mathbf{v} \cdot \nabla]u(r) = +i\Delta(r)w(r),$$

$$[E + \xi - i\mathbf{v} \cdot \nabla]w(r) = -i\Delta(r)u(r).$$

(13)

The corresponding quasi-particles are gapless [10]. In fact, for $E = 0$ and $\xi = 0$, the system (13) has two solutions. We have $w_\pm = \pm u_\pm$, with $u_\pm$ solutions of

$$\mathbf{v} \cdot \nabla u_\pm(r) = \pm\Delta(r)u_\pm(r),$$

(14)

and given by

$$u_\pm(r) = \exp\left[\pm \int \Delta(r') \frac{d(r' \cdot \mathbf{v})}{v^2}\right].$$

(15)

$$E^2 = \frac{\xi^2}{A_+ A_-},$$

Dispersion law

$$A_\pm = \frac{1}{V_c} \int_{\text{cell}} d\mathbf{r} \exp\left[\mp 2 \int \Delta(r') \frac{d(r' \cdot \mathbf{v})}{v^2}\right]$$

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\[ \Delta(r) = \Delta \cos 2qz \]

and one gets

\[ A_{\pm}^{(s)} \equiv A^{(s)} = I_0 \left( \frac{\Delta}{qv \cos \theta} \right), \]

where \( I_0(z) \) is the modified Bessel function of the zeroth order

\[ \Delta(r) = \Delta \cos 2qx \cos 2qy \cos 2qz. \]

If we define

\[ B = \frac{v}{4} \left( \frac{\sin 2q(x+y+z)}{u_x + u_y + u_z} + \frac{\sin 2q(x+y-z)}{u_x + u_y - u_z} + \frac{\sin 2q(x-y+z)}{u_x - u_y + u_z} \right), \]

one obtains for the cube

\[ A_{\pm}^{(fcc)} = \left( \frac{q}{\pi} \right)^3 \int_{\text{cell}} dV \exp \left\{ \pm \frac{\Delta}{qv} B \right\}, \]
Specific Heats
with R. Casalbuoni, R. Gatto, M. Mannarelli, M. Ruggieri, S. Stramaglia

\[ c_v = \rho \int \frac{d\Omega}{4\pi} \int d\xi \frac{dn(E,T)}{dT} \]

\[ \rho = 4\mu^2 / \pi^2 \]

For ungapped quarks:
\[ c_v = \mu^2 T / 3 \]

FF state
\[ c_v^{(FF)} = \frac{\rho T \pi^2}{3} \sqrt{1 - \frac{\Delta^2}{q^2}} \]

Strip, cube
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Effective Gap equation for large $\Delta$

Rewriting the gap equation for FF state at $T=0$

$$\Delta = \frac{g_p}{2} \int \frac{d\nu}{4\pi} \int_0^\beta \frac{d\xi}{2\pi} \int d\epsilon_0 \frac{\Delta_{eff}}{\xi^2 - \epsilon^2 - \Delta_{eff}^2} \Rightarrow \Delta = \frac{g_p}{2} \int \frac{d\nu}{4\pi} \int_0^\beta \frac{d\xi}{\sqrt{\xi^2 + \Delta_{eff}^2}}$$

$$\Delta_{eff} = \Delta_{eff}(v \cdot q, \xi) \Delta_{eff} = \Delta \theta(E_u) \theta(E_d) = \begin{cases} \Delta & \text{for } (\xi, v) \in PR \\ 0 & \text{elsewhere} \end{cases}$$

$$E_{u,d} = \pm \delta \mu \mp q \cdot v + \sqrt{\xi^2 + \Delta^2}$$

$E_u, E_d > 0 : $Pairing region

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Averaging procedure for the general case (several plane waves)

\( P \) plane waves

\[ \Delta(r) = \sum_{m=1}^{P} \Delta \phi^{2\psi n_m} r \]

We perform a weighted average of the Lagrangian over a region of the size of the lattice cell with weight function

\[ g(r) = \prod_{k=1}^{3} \sin \left( \frac{\pi \xi_k}{\xi} \right) \]

Averaging \( \Delta(r) \) produces a function \( \Delta_{\text{av}} \) vanishing outside an appropriate Pairing region.
A consistent approximation is found for $\Delta$ not too small

The approximation holds far off second order phase transition

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Pairing regions for the strip

$$\Delta(r) = \Delta \cos 2qz$$

Exact

<table>
<thead>
<tr>
<th>PR</th>
<th>BR</th>
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<tbody>
<tr>
<td>PR</td>
<td>BR</td>
</tr>
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</table>

Approximate

<table>
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<tr>
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<th>BR</th>
</tr>
</thead>
<tbody>
<tr>
<td>PR</td>
<td>BR</td>
</tr>
</tbody>
</table>

BR=[E<0]

PR=[E>0]

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Interpretation on terms of quasiparticles

\[ P_k = \{(v, \xi) \mid \Delta_E(v, \epsilon) = k\Delta\} \]

\[ P\Delta \ln \frac{2\delta}{\Delta_0} = \sum_{k=1}^{P} \int_{P_k} \frac{dv}{4\pi} \frac{d\xi}{\sqrt{\xi^2 + \Delta_E^2(v, \epsilon)}} = \sum_{k=1}^{P} \int_{P_k} \frac{dv}{4\pi} \frac{d\xi}{\sqrt{\xi^2 + k^2\Delta^2}} \]

The dispersion relation for the quasi-particles has several branches corresponding to the values \( k\Delta, k = 1, \cdots, P \). Each term in the sum corresponds to one branch of the dispersion law, i.e., to the propagation of a gapped quasi-particle with gap \( k\Delta \). The corresponding region is \( P_k \).
Two plane waves (strip)

\[
2\Delta \ln \frac{2\delta}{\Delta_0} = \int \int_{P_1} \frac{d\nu}{4\pi} \frac{d\xi}{\sqrt{\xi^2 + \Delta_{E\nu}^2(\nu, \xi)}} \Delta_E(\nu, \xi) + \int \int_{P_2} \frac{d\nu}{4\pi} \frac{d\xi}{\sqrt{\xi^2 + \Delta_{E\nu}^2(\nu, \xi)}} \Delta_E(\nu, \xi)
\]

Here \(\int_{P_1}\) represents a region where the two \(\Delta_{\alpha\beta}\) appearing in \(\Delta_E\) have no overlap. In this region

\[
\int \int_{P_1} \frac{d\nu}{4\pi} \frac{d\xi}{\sqrt{\xi^2 + \Delta_{E\nu}^2(\nu, \xi)}} = \Delta \int \int \frac{d\nu}{4\pi} \frac{d\xi}{\sqrt{\xi^2 + \Delta^2}} \left\{ \frac{\theta(E_\nu^1)\theta(E_\nu^2) + \theta(E_\nu^3)\theta(E_\nu^4) - 2\theta(E_\nu^1)\theta(E_\nu^2)\theta(E_\nu^3)\theta(E_\nu^4)}{\sqrt{\xi^2 + \Delta^2}} \right\}_{\nu, \xi}
\]

\(P_2\) represents the region where the two \(\Delta_{\alpha\beta}\) appearing in \(\Delta_E\) do overlap. In this region \(\Delta_{E\nu}^2(\nu, \xi) = 4\Delta^2\)

\[
\int \int_{P_2} \frac{d\nu}{4\pi} \frac{d\xi}{\sqrt{\xi^2 + \Delta_{E\nu}^2(\nu, \xi)}} = \Delta \int \int \frac{d\nu}{4\pi} \frac{d\xi}{\sqrt{\xi^2 + 4\Delta^2}} \left\{ 2\theta(E_\nu^1)\theta(E_\nu^2)\theta(E_\nu^3)\theta(E_\nu^4) \right\}_{\nu, \xi}
\]

where the subscript \(2\Delta\) on the r.h.s. means that in the dispersion laws \(E_{\nu, \xi}^{1,2}\) one has to use Eqns. (??) with \(\Delta \to 2\Delta\).

Test: the gap equation obtained in this way reproduces in the case of \(\mathbf{q}_1 = \mathbf{q}_2\) the FF result.

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Section of different pairing regions for the octahedron (body-centered cube)
FIG. 1: The pairing regions $P_1, \ldots, P_9$ (from left to right and from top to bottom) for the face centered cube in the $(z, \xi)$ plane for a fixed value of the polar angle $\varphi = 0.5$ rad. In each figure the grey area corresponds to the pairing region. Here $\delta \mu = \delta \mu_1$, $\Delta = 0.21 \Delta_0$, $z_1 = 0.9$. Note that for this particular value of the parameters the $P_8$ region covers all the Fermi surface.
\( \delta \mu = \delta \mu_1 \)

<table>
<thead>
<tr>
<th>( P )</th>
<th>( \delta \mu_2 / \Delta_0 )</th>
<th>Order</th>
<th>( z_q )</th>
<th>( \Delta / \Delta_0 )</th>
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</thead>
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<td>0.754</td>
<td>II</td>
<td>0.83</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0.83</td>
<td>I</td>
<td>1.0</td>
<td>0.81</td>
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<td>6</td>
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<td>I</td>
<td>0.95</td>
<td>0.43</td>
</tr>
<tr>
<td>8</td>
<td>1.32</td>
<td>I</td>
<td>0.9</td>
<td>0.35</td>
</tr>
</tbody>
</table>

From \( \delta \mu = \delta \mu_1 \) up to \( \delta \mu = 0.95\Delta_0 \) favored structure bcc (octahedron)

From \( \delta \mu = 0.95\Delta_0 \) up to \( \delta \mu = 1.32\Delta_0 \) favored structure cube fcc
Two-dimensional case (Shimahara 1998)

Analysis close to the critical line

\[ \Omega_a - \Omega_N = -\frac{1}{2}\rho b_z \left( \frac{T_c - T}{T_c} \right)^2 \]

\[ \Delta_a(\vec{r}) = \Delta_{FF} e^{iq \cdot \vec{r}} \]

\[ \Delta_a(\vec{r}) = 2\Delta_{FFLO} \cos(\vec{q} \cdot \vec{r}) \]

\[ \Delta_a(\vec{r}) = 2\Delta_{sq}[\cos(q_x) + \cos(q_y)] \]

\[ \Delta_a(\vec{r}) = \Delta_{tr}[e^{iq_1 \cdot \vec{r}} + e^{iq_2 \cdot \vec{r}} + e^{iq_3 \cdot \vec{r}}] \]

\[ \Delta_a(\vec{r}) = 2\Delta_{hex}[\cos(q_1 \cdot \vec{r}) + \cos(q_2 \cdot \vec{r}) + \cos(q_3 \cdot \vec{r})] \]

\[ \vec{q}_1 + \vec{q}_2 + \vec{q}_3 = 0 \]
Conclusions and outlook

• Color superconductive matter predicted by QCD can exist in different phases, most notably CFL and LOFF phase.
• Probably the only laboratory to detect it are pulsars or very compact stars. Here the LOFF phase of QCD might produce glitches.
• Gap equation: At the 2nd order phase transition GL expansion

• Approximation scheme for large $\Delta$ near the Clogston limit points to the bcc (octahedron) as favored structure

• Interpretation in terms of quasiparticles of gap $k\Delta$ existing in different regions of the phase space

• LOFF phase searched also in non conventional superconductors (high $T_c$, organic, alkali gas)