Aspects of the
QCD Phase Diagram from the Lattice

From Lattices to Stars
INT Seattle - April 29 2004

Prologue:

QCD at nonzero baryon density is a twenty years old, and difficult problem... ..In the last four years a few lattice techniques proven successful for \( \mu_B/T < 1 \)... This talk is about one such techique, and the results Massimo D’Elia and I obtained in the four flavor model...

Massimo D’Elia and Maria-Paola Lombardo
hep-lat/0309114/0209146/0205022 + new results
Importance Sampling and The Positivity Issue

\[ \mathcal{Z}(T, \mu, U) = \int dU e^{-(S_{YM}(U) - \log(\det M))} \]

\[ \det M > 0 \rightarrow \text{Importance Sampling} \]

\[ M^\dagger(\mu_B) = -M(-\mu_B) \]

\[ \mu = 0 \rightarrow \det M \text{ is real} \]

Particles-antiparticles \underline{symmetry}

Imaginary \( \mu \neq 0 \rightarrow \det M \text{ is real} \)

(Real) Particles-antiparticles \underline{symmetry}

Real \( \mu \neq 0 \) Particles-antiparticles \underline{asymmetry} \rightarrow \det M \text{ is complex in QCD}

**QCD with a real baryon chemical potential:**  
use information from the accessible region  

\[ \text{Real} \mu = 0, \text{Im} \mu \leq 0 \]
QCD and a Complex $\mu_B$

Analytic continuation along one arc in the complex $\mu$ plane:

Complex $\mu$ Plane

\[ Z(\mu) = Z(-\mu) \]

Can be mapped onto the complex $\mu^2$ plane

Complex $\mu^2$ Plane

To define $Z(\mu^2)$ which is real valued for real $\mu^2$

Analog with statistical models in external fields
Region accessible to simulations: $\mu^2$ real $\leq 0$.

Multiparameter Reweighting:
Fodor, Katz, Csikor, Egri, Szabo, Toth

Derivatives: Gupta, Gavai; MILC; QCD-Taro

Expanded Reweighting: Bielefeld-Swansea

Im $\mu$: de Forcrand, Philipsen; D’Elia, MpL
**The Roberge and Weiss analysis**

\[ Z(\nu) = \text{Tr} e^{-\beta H + i\beta \nu N} = e^{-\beta H + i\theta N} \]

1. \( Z(\theta) \) has a periodicity \( 2\pi \) anyway.

2. If only color singlet are allowed, then \( N = 0 \mod (N_c) \) and periodicity becomes \( 2\pi/N_c \)

However (Roberge Weiss (1986))

\( Z(\theta) \) has always period \( 2\pi/N_c \)

The imaginary chemical potential changes the preferred vacuum for the Polyakov loop from \( \phi_P = 0 \) to one of its \( Z_3 \) images

The strong coupling analysis shows that periodicity is smooth at low temperature, and p.t. theory suggests that it is sharp at high \( T \)
Analytic continuation from an imaginary $\mu$

QCD Results for

The critical line
2,3,2+1 flavor : Ph. de Forcrand O. Philipsen
4 Flavor: M. D'Elia, MpL *

Order parameter, Pressure, Baryon Density
4 Flavor: M. d'Elia, MpL *

QCD related models
Strong Coupling MpL
Dimensionally reduced model Hart, Laine, Philipsen
Two colors P. Giudice, A. Papa

* This Talk

Outline

I-QCD in the $T, \mu^2$ plane

II-The critical line

III-The hadronic phase

IV-The QGP phase
\[ S_{QCD} = S_{YM} + S_F \rightarrow g \rightarrow \infty = S_F \]

\[ Z = \left( \int V_{eff}(\langle \bar{\psi}\psi \rangle d \langle \bar{\psi}\psi \rangle) \right) V^s \]

\[ V_{eff}(\langle \bar{\psi}\psi \rangle, \mu) = 2\cosh(rN_tN_C\mu) + \sinh[(N_t + 1)N_C < \bar{\psi}\psi>] / \sinh(N_t < \bar{\psi}\psi>) \]

\[ V_{eff}(\langle \bar{\psi}\psi \rangle, i\mu) = 2\cos(rN_tN_C\mu) + \sinh[(N_t + 1)N_C < \bar{\psi}\psi>] / \sinh(N_t < \bar{\psi}\psi>) \]

\[ \langle \bar{\psi}\psi \rangle \text{ as a function of real and imm } \mu, \text{ for } T \simeq 0 \text{ and } T \simeq T_C \]
μ Imm.: Two Color QCD

P. Giudice and A. Papa, 2004

Lattice: 8x8x8x4
β = 2.0
m = 0.07

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Analytic continuation of the critical line from an imaginary $\mu$:
Ph. de Forcrand and O. Philpseen

Consideration of the $T, \mu^2$ plane helps the analysis:

Model analysis suggests this parametrization confirmed by numerical results:

$$(T + aT_c)(T - T_c) + k\mu^2 = 0, \ k > 0$$

*Gross Neveu Model*

The critical line:

$$1 - \mu/\Sigma_0 = 2T/\Sigma_0 ln(1 + e^{-\mu/T})$$

Reduces to:

$$T(T - T_c) + \mu^2/(8ln2) = 0$$

Second order approximation good up to $\mu \simeq T_c$
From O. Philipsen and E. Laermann


\[ T/\text{MeV} \]
\[ \mu_B/\text{MeV} \]

- \( N_f=2, [3] \)
- \( N_f=2+1, [1] \)
- \( N_f=2, [2] \)
- \( N_f=4, [4] \)

<table>
<thead>
<tr>
<th>Method</th>
<th>( N_f )</th>
<th>( m_q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>reweighting [1]</td>
<td>2+1</td>
<td>( am = 0.025, m_s = 8m_u, (m_\pi \approx 300 \text{MeV}) )</td>
</tr>
<tr>
<td>rew. +Taylor [2]</td>
<td>2</td>
<td>( am = 0.1, (m_\pi \approx 600) \text{ MeV} )</td>
</tr>
<tr>
<td>imag. ( \mu ) [3]</td>
<td>2</td>
<td>( am = 0.025, (m_\pi \approx 300) \text{ MeV} )</td>
</tr>
<tr>
<td>imag. ( \mu ) [4]</td>
<td>4</td>
<td>( am = 0.05 )</td>
</tr>
</tbody>
</table>

Scale invariant plot for the QCD critical line

Note studies of $N_f$ corrections in Gross-Neveu [D. Ebert et al. 2002]:

$$V_{eff} (\sigma, T, \mu) = V_0 - 2N_f(N_c - 2)V_1(T, \mu)$$
\[ n(T, i\mu) = \frac{\partial \log(Z(T, i\mu))}{V \partial \mu} \]

\[ \chi_q(\mu = 0) = \left. \frac{i \partial n(i\mu)}{\partial \mu} \right|_{\mu=0} \]
Pressure: integral method

\[
\frac{P(T, i\mu) - P(T, \mu=0)}{T^4} = N_t^4 \int d\mu n(i\mu)
\]
Hadronic Phase: $T < T_c$

Observables are smooth, analytic continuation in the $\mu^2 > 0$ half plane possible, but interesting only when $\chi_q(\mu = 0, T) > 0$

*Analytic continuation is valid till $\mu < \mu_c(T)$*

*Even and odd observables*

For observables which are even/odd in the chemical potential $O_e/O_o$ we consider two different parametrizations

- A Fourier serie

$$O_e = a_{e_F} + \sum b_{e_F} \cos(N_C N t \mu)$$

$$O_o = a_{o_F} + \sum b_{o_F} \sin(N_C N t \mu)$$

- A Taylor serie - useful to compare with $\mu = 0$ results:
  MILC; Bielefeld-Swansea; R. Gavai and S. Gupta.

$$O_e = a_{e_T} + b_{e_T} \mu^2 + c_{e_T} \mu^4$$

$$O_o = a_{o_T} + b_{o_T} \mu + c_{o_T} \mu^3$$

Even/odd observables purely real/imaginary at imaginary chemical potential
**Number density**

\[ n(i\mu) = a_1 \sin(i\mu N_c N_T) + b_1 \sin(i2\mu N_c N_T) \nu_1 \]

**Analytic continuation up to \( \mu = \mu_c(T) \):**

\[ a_1 \sin(i\mu N_c N_T) + a_2 \sin(i2\mu N_c N_T) \]

\[ \rightarrow a_1 \sinh(\mu N_c N_T) + a_2 \sinh(i2\mu N_c N_T) \]

**Critical density at** \( T = .985 \ T_c \ n_c(\mu_c)/T^3 \approx 0.5 \)

The errors from one and two Fourier coefficient fits are shown.
Mass dependence

From derivatives: \( \partial < \bar{\psi} \psi > / \partial \mu = \partial n(\mu) / \partial m \)

When: \( \bar{\psi} \psi(\mu, m_q) = a_C \cosh(3\mu N_T) + b_C \) and 
\( n(\mu, m_q) = a_n \sinh(3\mu N_T) \)

\[
\frac{n(\mu, m_q+\Delta m_q) - n(\mu, m_q)}{n(\mu, m_q)} = 3 \times N_T a_C / a_n \Delta m
\]

\[
\frac{\Delta n(\mu, m_q)}{n(\mu, m_q)} \approx 2.53 \Delta m_q / T
\]

\[
\frac{\Delta n(\mu, m_q)}{n(\mu, m_q)} \approx 4.03 \Delta m_q / T
\]
The chiral condensate $\langle \bar{\psi} \psi \rangle$ ($\mu_c$) ≠ 0 : first order transition for four flavor QCD, possibly weakening a bit with temperature.
The metastable branch

The analytic continuation is insensitive to a discontinuous phase transition since it lives on the metastable branch; it follows the secondary minimum and determines the spinodal point.

\[ \langle \bar{\psi} \psi \rangle = A(\mu - \mu^*)^\beta \]

The discontinuity can be related to \( \mu - \mu^* \).

Both shrinks to zero at the endpoint of a first order transition
**Hadron Resonance Gas Model**

In general, when one Fourier component suffices

\[ \frac{\partial \langle \bar{\psi} \psi \rangle(\mu, m, T)}{\partial \mu} = \frac{\partial (n(\mu, m, T))}{\partial m} = -kn(\mu, m, T) \]

\[ n(\mu, m, T) \propto Ae^{-m/T}\sinh(3\mu/T) \]

The results can be contrasted with an Hadron Resonance Gas model

\[ \ln Z(T, \mu) = \sum_{\text{mesons}} \ln Z^M(T, \mu) + \sum_{\text{baryons}} \ln Z^B(T, \mu) \]

\[ (m_N - \mu_B) > T \rightarrow \ln Z(T, \mu) \simeq \frac{p_B}{T^4} \simeq F(T, m) \cosh(3\mu/T) \]

F. Karsch, K. Redlich and A. Tawfik (2003): Hadron Resonance gas model from expanded reweighting

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Monitoring the approach to the SB (Lattice) behaviour:

analytic continuation from real to imaginary $\mu_B$ of the SB lattice result

Mass dependence from the derivative of the chiral condensate
Corrections to Free Field

A. Vuorinen 2004:

\[ \Delta P(\mu) = A T^2 \mu^2 + B \mu^4 + \ldots \]

Alternatively (Rafelski, Letessier 2003, Quasiparticle models)

\[ \Delta P = f(\mu)(A T^2 \mu + B \mu^3) \]

Trivial possibility: \( f(\mu) \) : constant effective number of flavors

\[ \Delta P = N_{\text{eff}}^f (A T^2 \mu + B \mu^3) \]

Effective number of active flavors as estimated from the ratio of the lattice results to the lattice free field: appear to be constant for \( T \geq 1.5 T_C \)
$T_c < T \lesssim 1.1T_c$

Interplay of thermodynamics and critical behaviour in the RW regime $T_C < T < T_E$

$log P(\mu, T) \propto (\mu - \mu_c)^\eta$

Incompatible with a free field for continuous transitions, and for first order transitions of finite strength
Correlation between $< \bar{\psi} \psi >$ and Polyakov loop at $\mu_I = 0.15$, demonstrating the chiral and deconfining nature of the transition at nonzero real baryon density.

$\beta_c(i\mu) - \beta_d(i\mu) = 0 \rightarrow \beta_c(\mu) - \beta_d(\mu) = 0$
Summary

Strength of the method: not limited by volume; gives access to critical values of observables.

Results for Four flavor QCD thermodynamics for $0.985T_C < T < 3.5T_C$

1. The critical line is of first order or a very sharp crossover:

$$\frac{T}{T_C^2} = 1 - 0.0021(2)(\mu/T)^2,$$

where $\mu < 500 \text{ MeV}$.

2. Chiral and “deconfining” transition remain correlated at nonzero baryon density: $T_C^{\text{chiral}}(\mu) = T_C^{\text{screening}}(\mu)$

3. In the Hadronic Phase $\Delta P \propto \cosh(\mu_B/T)$

4. $n(\mu_C, T = 0.985T_C, m_q = 0.05)/T^3 \simeq 0.5$, and the mass corrections $\Delta n = -4.03\Delta m_q/T$.

5. For $T \geq 1.5$ the results are compatible with lattice Stefan Boltzmann with an active fixed number of flavor 0.92 for $T = 3.5T_C$ and 0.89 for $T = 2.5T_C$.

6. For $T \approx 1.1T_C$ there is room for non trivial deviations for free field, possibly connected with the chiral transition at $\mu^2 < 0$

Future possibilities for Im $\mu$ calculations

**Hybrid Methods**: combining the Im $\mu$ approach with derivatives/rewinding

Assessing the critical behaviour (tricritical point, endpoint) by monitoring the discontinuities at the critical point