Ginzburg-Landau approach to color superconductivity

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References
**Introduction**

*QCD phase diagram*

**Crucial features of color superconductors**

**Superfluidity**
— presence of a condensate of quark Cooper pairs and superfluid baryon density ($n_s$)

**Color Meissner effects**
— transverse color fields screened in a spatial scale of order the London penetration depth $\sim (\mu / g^2 n_s)^{1/2}$
What happens if quark matter occurs in nature, e.g., in stars

Possible presence of a diquark condensate

— at lower densities ($\mu \sim 1$-2 GeV)
  
  **Strong coupling regime** vs. weak coupling regime where one-gluon exchange induces Cooper pairing in color antitriplet channel

— color neutral
  
  Differences in the chemical potential between colors

— subject to stellar magnetic fields and rotation
  
  Supercurrents and vortices

— at finite temperatures ($T \sim 100$ MeV)
  
  Thermal fluctuations in gauge and diquark fields

— affected by finite strange quark mass and electric neutrality
  
  Splitting of the critical temperature

Focus of this work

General Ginzburg-Landau analysis near $T_c$
— allowing us to systematically examine the above features
**Homogeneous superfluid phases**


**System**

massless quarks of *uds* flavors and *RGB* colors

temperature \( T \), baryon chemical potential \( \mu \)

restored chiral symmetry

Fermi momenta common to all colors and flavors

**Cooper pairing**

Even parity, same chirality, \( J=0 \), quark-quark pairing

antisymmetric in color and flavor space

**Homogeneous Ginzburg-Landau free energy near \( T_c \)**

\[
\Delta \Omega \equiv \Omega_s - \Omega_n = \alpha^+ \text{Tr} \left( \phi_+^\dagger \phi_+ \right) + \beta_1^+ \left[ \text{Tr} \left( \phi_+^\dagger \phi_+ \right) \right]^2 + \beta_2^+ \text{Tr} \left( \phi_+^\dagger \phi_+ \right)^2
\]

\[
= \bar{\alpha} \sum_a |d_a|^2 + \beta_1 \left( \sum_a |d_a|^2 \right)^2 + \beta_2 \sum_{ab} |d_a^* \cdot d_b|^2
\]

with \( \bar{\alpha} = 4\alpha^+ \), \( \beta_1 = 16\beta_1^+ + 2\beta_2^+ \), \( \beta_2 = 2\beta_2^+ \)
Homogeneous superfluid phases (contd.)

Candidates of energetically favorable pairing states


Two optimal states determined from energy minimization

1. 2-flavor color superconducting (2SC) state

   \[ d_R \parallel d_G \parallel d_B \]

   • Gapped quarks: two colors and two flavors ("anisotropic")

   • Gluo-electromagnetic properties:

     \[ d_R^* \cdot d_G = d_G^* \cdot d_B = d_B^* \cdot d_R = 0, \quad |d_R| = |d_G| = |d_B| \]

     Long wavelength gluons → 2SC condensate at \( T = 0 \) → 3 types: propagating

     5 types: screened

2. Color-flavor locked (CFL) state

   \[ d_R^* \cdot d_G = d_G^* \cdot d_B = d_B^* \cdot d_R = 0, \quad |d_R| = |d_G| = |d_B| \]

   • Gapped quarks: three colors and three flavors ("isotropic")

   • Gluo-electromagnetic properties:

     \[ d_R^* \cdot d_G = d_G^* \cdot d_B = d_B^* \cdot d_R = 0, \quad |d_R| = |d_G| = |d_B| \]

     Long wavelength gluons → CFL condensate at \( T = 0 \) → 8 types: screened

Cf. The CFL state is more favorable in the weak coupling limit.
Homogeneous superfluid phases (contd.)

In weak coupling

\[ \beta_1 = \beta_2 = \frac{7 \xi(3)}{8(\pi T_c)^2} N\left(\frac{\mu}{3}\right) \]

\[ \bar{\alpha} = 4 N\left(\frac{\mu}{3}\right) \frac{T - T_c}{T_c} \]

with \( N\left(\frac{\mu}{3}\right) = \frac{1}{2\pi^2} \left(\frac{\mu}{3}\right)^2 \)
Homogeneous superfluid phases (contd.)

In an overall color singlet state

Generally, chemical potential differences, \( \bar{\mu}_{ab} \), from \( \delta_{ab} \mu / 3 \) arise

such that \( \bar{\mu}_{aa} = \mu_a - \frac{\mu}{3} , \sum_a \bar{\mu}_{aa} = 0 \).

Then, the Ginzburg-Landau free energy reads

\[
\Delta \Omega = \Omega_0 + \Omega_{CN}
\]

\[
\Omega_0 = \bar{\alpha} \sum_a |d_a|^2 + \beta_1 \left( \sum_a |d_a|^2 \right)^2 + \beta_2 \sum_{ab} |d^*_a \cdot d_b|^2
\]

\[
\Omega_{CN} = 3\sigma \sum_{ab} |\tilde{\mu}_{ab}|^2 - 2\chi \sum_{ab} \left( d^*_a \cdot d_b \right) \tilde{\mu}_{ab} .
\]

\[
\tilde{\mu}_{ab} = \frac{\chi}{9\sigma} \left[ 3 (d_a \cdot d^*_b) - \delta_{ab} \sum_c |d_c|^2 \right]
\]

Color neutrality condition

\[
\frac{\partial \Delta \Omega}{\partial \tilde{\mu}_{RR}} = \frac{\partial \Delta \Omega}{\partial \tilde{\mu}_{GG}} = \frac{\partial \Delta \Omega}{\partial \tilde{\mu}_{BB}} ,
\]

\[
\frac{\partial \Delta \Omega}{\partial \tilde{\mu}_{ab}} = 0 \text{ for } a \neq b .
\]

When the order parameter is anisotropic in color space as in 2SC, the color having a larger gap has a smaller chemical potential, leading to smaller condensation energy.
Homogeneous superfluid phases (contd.)

\[
\Gamma = -\frac{1}{\sigma} \left( \frac{x}{3} \right)^2
\]
Responses to magnetic fields and rotation


**Gradient energy in the absence of real photon and gluon fields**

In inhomogeneous states of wavelengths larger than $T_c^{-1}$

— $\phi_+$ depends on the center-of-mass coordinate $r$ of the pair.

— To second order in $\nabla \phi_+$, the gradient energy reads

$$\Omega_g = \frac{1}{2} K_T \text{Tr}\left( \partial_i \phi_+ \partial_i \phi_+^\dagger \right).$$

**Superfluid density and superfluid momentum density**


Superfluid mass (baryon) density $\rho_s (n_s)$:

$$\rho_s = \mu n_s = \frac{4}{9} \mu^2 K_T \text{Tr}\left( \phi_+ \phi_+^\dagger \right)$$

Superfluid momentum (baryon current) density $g_s (j_s)$:

$$g_s = \mu j_s = -\frac{i}{3} K_T \mu \text{Tr}\left( \phi_+ \nabla \phi_+^\dagger - \phi_+^\dagger \nabla \phi_+ \right)$$

**Gradient energy in the presence of real photon and gluon fields**

Gradient energy invariant under color $SU(3)$ and electromagnetic $U(1)$ local gauge transformations of the quark spinors $\psi_{ai}$:

$$\Omega_g = \frac{1}{2} K_T \text{Tr}\left[ D_i \phi_+ (D_i \phi)^\dagger \right].$$

with covariant derivative $D_i \phi_+ \equiv \partial_i \phi_+ + \frac{ig}{2} \left[ \left( \lambda^\alpha \right)^* \phi_+ + \phi_+ \lambda^\alpha \right] A_i^\alpha + i e Q \phi_+ A_i$, $Q_{abij} = \delta_{ab}(q_i + q_j)$
Responses to magnetic fields and rotation (contd.)

**Field equations and charged supercurrents**

From extremization of

\[ \int d^3 r \left( \Omega_0 + \Omega_g + \frac{1}{4} G_{lm}^{\alpha} G_{lm}^{\alpha} + \frac{1}{4} F_{lm} F_{lm} \right), \]

- Gap equation

\[ -\frac{1}{2} K_T D_l (D_l \phi_+) + \alpha^+ \phi_+ + 2 \beta_1^+ \left[ \text{Tr} \left( \phi_+^\dagger \phi_+ \right) \right] \phi_+ + 2 \beta_2^+ \phi_+ \phi_+^\dagger \phi_+ = 0 \]

- The color Maxwell equation

\[
\partial_m G_m^\alpha + gf_{\alpha \beta \gamma} A_m^\beta G_m^\gamma = -\frac{1}{2} K_T g \text{Im} \left\{ \text{Tr} \left[ \left( \lambda^\alpha \right)^* \phi_+ + \phi_+ \lambda^\alpha \right]^\dagger \partial_i \phi_+ \right\} \\
- \frac{1}{4} K_T g^2 A_l^\beta \text{Re} \left\{ \text{Tr} \left[ \left( \lambda^\alpha \right)^* \phi_+ + \phi_+ \lambda^\alpha \right] \left( \lambda^\beta \right)^* \phi_+ + \phi_+ \lambda^\beta \right]\right\} \\
- \frac{1}{2} K_T g e A_l \text{Re} \left\{ \text{Tr} \left[ \left( \lambda^\alpha \right)^* \phi_+ + \phi_+ \lambda^\alpha \right] Q \phi_+^\dagger \right\} \equiv J_l^\alpha
\]

- The Maxwell equation

\[
\partial_m F_{ml} = -K_T e \text{Im} \left\{ \text{Tr} \left( Q \phi_+^\dagger \partial_i \phi_+ \right) \right\} - K_T e^2 A_l \text{Tr} \left( Q \phi_+ Q \phi_+^\dagger \right) \\
- \frac{1}{2} K_T g e A_l^\alpha \text{Re} \left\{ \text{Tr} \left[ Q \phi_+ \left( \lambda^\alpha \right)^* \phi_+ + \phi_+ \lambda^\alpha \right]^\dagger \right\} \equiv J_l
\]
Responses of the color-flavor locked (CFL) condensate

**Pairing gap**

\[
(d_a)_i = U_{ai} \kappa_A, \quad \kappa_A = e^{i\varphi} |\kappa_A|
\]

**Supercurrents associated with \(U(1)_B\) phase and photon fields**

In the London limit (spatial variation larger than \(\xi = \sqrt{K_T/2|\alpha^+|}\))

Baryonic: \(j_s = -8K_T|\kappa_A|^2 \nabla \varphi\) — \(U(1)_B\) phase induced

Color: \(J^8 = -K_T g|\kappa_A|^2 \left( \nabla \varphi_8 + 2gA^8 + \frac{4e}{\sqrt{3}} A \right)\)

Electric: \(J = \frac{2e}{\sqrt{3}g} J^8\)

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**Photon-gluon mixed fields**


\[
A = \frac{\sqrt{3}gA - 2eA^8}{3g_8}, \quad A^8 = \frac{\sqrt{3}gA^8 + 2eA}{3g_8}, \quad g_8 = \frac{1}{3} \sqrt{3g^2 + 4e^2}
\]

Corresponding supercurrents:

\[
J = 0
\]

\[
J^8 = -K_T g_8 |\kappa_A|^2 \left( \sqrt{3} \nabla \varphi_8 + 6g_8 A^8 \right)
\] — free propagation

\[SU(3)_{c+f}\) phase — mixed Meissner
Responses of the CFL condensate (contd.)

Response to magnetic fields

Low fields: imperfect diamagnetism

Uniformly applied weak $H_{\text{ext}}$

- The most part is propagating in the form of $B = \nabla \times A$
- A fraction of $H_{\text{ext}}$ included in $B^8 = \nabla \times A^8$ is screened in a length scale $\lambda_{\text{CFL}} = \left(\sqrt{6K_T g_8 |\kappa_A|}\right)^{-1}$

High fields: possible $SU(3)_{c+f}$ vortices

- Vortices can appear such that $\int dl \cdot (A^8 + \lambda_{\text{CFL}}^2 J^8) = 2\pi/g_8$
  when the system is Type II ($\kappa_{\text{CFL}} \equiv \frac{\lambda_{\text{CFL}}}{\xi} > 1/\sqrt{2} \times O(1)$).
- Complicated magnetic and supercurrent structure of a vortex
- Critical field $H_c = \frac{3}{2e\xi \lambda_{\text{CFL}}}$

Weak coupling expressions:

- $\xi \approx 0.26 \left(\frac{100 \text{ MeV}}{T_c}\right) \left(1 - \frac{T}{T_c}\right)^{-1/2}$ fm
- $\lambda_{\text{CFL}} \approx 1.7 \left(\frac{\sqrt{3}}{g_8}\right) \left(\frac{300 \text{ MeV}}{\mu/3}\right) \left(1 - \frac{T}{T_c}\right)^{-1/2}$ fm
- $H_c \approx 2.2 \times 10^{19} \left(\frac{g_8}{\sqrt{3}}\right) \left(\frac{T_c}{100 \text{ MeV}}\right) \left(\frac{\mu/3}{300 \text{ MeV}}\right) \left(1 - \frac{T}{T_c}\right) G \gg 10^{12} G$
Responses of the CFL condensate (contd.)

**Response to rotation**

Transformation to the rotating frame

— under constant angular velocity \( \omega \),

\[
\Omega_s \rightarrow \Omega_s - L_s \cdot \omega, \quad L_s = \mathbf{r} \times \mathbf{g}_s
\]

\[
\Omega_g = 6K_T \left[ \left( \nabla + \frac{2i\mu}{3} \mathbf{a} \times \mathbf{r} \right) \kappa_A \right]^2 + \frac{1}{2} g_8^2 |\kappa_A|^2 |A^8|^2 \right] - \frac{1}{2} \rho_s |\omega \times \mathbf{r}|^2
\]

A triangular lattice of singly quantized vortices

— superflow pattern simulating corotation of the condensate

Quantization condition on vorticity: \( \oint dl \cdot \mathbf{v}_s = 2\pi \frac{3}{2\mu} \)

Number of vortices (calculated from net circulation)

\[
N_v = \frac{2}{3} \mu R^2 \omega
\]

\[
\approx 6.4 \times 10^{18} \left( \frac{1 \text{ms}}{P_{\text{rot}}} \right) \left( \frac{\mu/3}{300 \text{ MeV}} \right) \left( \frac{R}{10 \text{ km}} \right)^2, \quad P_{\text{rot}} = \frac{2\pi}{\omega}
\]

— just like rotating superfluid helium and Bose-Einstein condensates of alkali atoms
Responses of the isoscalar (2SC) condensate

**Pairing gap**

\[
(d_a)_i = e^{i\varphi} |d| \delta_{aB} \delta_{is}
\]

**Supercurrents associated with** \(U(1)_B\) **phase and photon fields**

In the London limit (spatial variation larger than \(\xi = \sqrt{K_T/2|\alpha^+|}\))

**Baryonic:**

\[
\mathbf{j}_s = -\frac{8}{3} K_T |d|^2 \left( \nabla \varphi + \frac{g}{\sqrt{3}} A^8 + \frac{e}{3} \mathbf{A} \right)
\]

— \(U(1)_B\) phase induced

**Color:**

\[
\mathbf{J}^8 = -\frac{4}{\sqrt{3}} K_T g |d|^2 \left( \nabla \varphi + \frac{g}{\sqrt{3}} A^8 + \frac{e}{3} \mathbf{A} \right)
\]

**Electric:**

\[
\mathbf{J} = \frac{e}{\sqrt{3}g} \mathbf{J}^8
\]

---

**Photon-gluon mixed fields**  

\[
A \equiv \frac{\sqrt{3}gA - eA^8}{3g_8}, \quad A^8 \equiv \frac{\sqrt{3}gA^8 + eA}{3g_8}, \quad g_8 = \frac{1}{3} \sqrt{3g^2 + e^2}
\]

Corresponding supercurrents:

\[
\mathbf{J} = 0
\]

\[
\mathbf{J}^8 = -4K_T g_8 |d|^2 \left( \nabla \varphi + g_8 A^8 \right)
\]

— free propagation  
— mixed Meissner  

\(U(1)_{em}\) phase
Responses of the 2SC condensate (contd.)

Response to magnetic fields

Low fields: imperfect diamagnetism

Uniformly applied weak $H_{\text{ext}}$

- The most part is propagating in the form of $\mathbf{B} = \nabla \times \mathbf{A}$
- A fraction of $H_{\text{ext}}$ included in $\mathbf{B}^8 = \nabla \times \mathbf{A}^8$ is screened in a length scale $\lambda_{2\text{SC}} = \left(2\sqrt{|K_T g_8|d}\right)^{-1}$

High fields: possible $U(1)_{\text{em}}$ vortices — just like ordinary SC

- Vortices can appear such that $\oint dl \cdot (\mathbf{A}^8 + \lambda_{2\text{SC}}^2 J^8) = 2\pi / g_8$
  when the system is Type II ($\kappa_{2\text{SC}} \equiv \lambda_{2\text{SC}} / \xi > 1/\sqrt{2}$).

- Critical fields: $H_c = \frac{3}{\sqrt{2} e \xi \lambda_{2\text{SC}}} \ , \ H_{c1} = \frac{1}{\sqrt{2} \kappa_{2\text{SC}}} \ H_c \ln \kappa_{2\text{SC}} \ , \ H_{c2} = \sqrt{2} \kappa_{2\text{SC}} H_c$

Weak coupling expressions:

\[ \xi \approx 0.26 \left( \frac{100 \text{ MeV}}{T_c} \right) \left( 1 - \frac{T}{T_c} \right)^{-1/2} \text{ fm} \]

\[ \lambda_{2\text{SC}} \approx 1.5 \left( \frac{\sqrt{3}}{g_8} \right) \left( \frac{300 \text{ MeV}}{\mu/3} \right) \left( 1 - \frac{T}{T_c} \right)^{-1/2} \text{ fm} \]

\[ H_c \approx 3.6 \times 10^{19} \left( \frac{g_8}{\sqrt{3}} \right) \left( \frac{T_c}{100 \text{ MeV}} \right) \left( \frac{\mu/3}{300 \text{ MeV}} \right) \left( 1 - \frac{T}{T_c} \right) \text{ G} \gg 10^{12} \text{ G} \]
Responses of the 2SC condensate (contd.)

Response to rotation

Transformation to the rotating frame
— under constant angular velocity $\omega$,
\[
\Omega_s \rightarrow \Omega_s - L_s \cdot \omega, \quad L_s = r \times g_s
\]

\[
\Omega_g = -\frac{1}{2} \rho_s |\omega \times r|^2 + 2K_T \left\{ \left[ \partial_I - ig_8 A_I^8 \right]^* \frac{2i\mu}{3} (\omega \times r)_I \right\} d^* \right\} \left\{ \left[ \partial_I + ig_8 A_I^8 \right] + \frac{2i\mu}{3} (\omega \times r)_I \right\} d \right\}
\]

Absent from CFL

London magnetic field generation — just like ordinary SC

Superflow pattern corotating with the vessel:
\[
J^8 = \frac{3}{2} g_8 n_s \omega \times r
\]

Quantization condition on the vorticity:
\[
\oint dl \cdot (A^8 + \lambda_{2SC}^2 J^8) = 0
\]

London field

\[
B^8 = -\frac{4\mu}{3g_8} \omega, \quad A^8 = -\frac{2\mu}{3g_8} \omega \times r
\]

— small: $|B^8| \approx 0.15 \left( \frac{\sqrt{3}}{g_8} \right) \left( \frac{1\text{ms}}{P_{\text{rot}}} \right) \left( \frac{\mu/3}{300\text{MeV}} \right) G \ll 10^{12} G$

— color dominant: $|B^8| = \frac{\sqrt{3}g}{e} |B| \gg |B|$ (← $B = 0$)
**Responses to magnetic fields and rotation (contd.)**

**Summary**

Responses of homogeneous superfluid quark matter near $T_c$ to magnetic fields and rotation

<table>
<thead>
<tr>
<th>Phase</th>
<th>Magnetic fields</th>
<th>Rotation</th>
</tr>
</thead>
<tbody>
<tr>
<td>CFL</td>
<td>Partial screening</td>
<td>$U(1)_B$ vortices</td>
</tr>
<tr>
<td></td>
<td>Possible $SU(3)_{c+f}$ vortices</td>
<td></td>
</tr>
<tr>
<td>2SC</td>
<td>Partial screening</td>
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</tr>
<tr>
<td></td>
<td>Possible $U(1)_{em}$ vortices</td>
<td></td>
</tr>
</tbody>
</table>

**Ginzburg-Landau coherence length:** $\xi$

\[ \lambda_{2SC}^2 |J^8| + |A^8| = 1/g_8 r \]

**London penetration depth**

\[ 0 \leq \xi \leq \lambda_{2SC} \]

**Rotational vortex in the CFL phase**

\[ |j_s| \leq \frac{3n_s}{2\mu r} \]
Mean-square thermal fluctuations in the Gaussian approximation
where interactions between fluctuations can be ignored:

\[
\langle A^\alpha A^\beta \rangle = \delta_{\alpha\beta} 2T \int_{|k|<\Lambda} \frac{d^3k}{(2\pi)^3} \frac{1}{k^2 + (m_A)^2} \]

\[
= \delta_{\alpha\beta} T \left[ \frac{T_c}{\pi^2} - \frac{(m_A)^2}{2\pi} + \frac{(m_A)^2}{\pi^2 T_c} + \ldots \right]
\]

\[
\langle (\delta d)^\rho (\delta d)^\sigma \rangle = \delta_{\rho\sigma} \frac{T}{2K_T} \int_{|k|<\Lambda} \frac{d^3k}{(2\pi)^3} \frac{1}{k^2 + (m_d)^2} \]

\[
= \delta_{\rho\sigma} \frac{T}{4K_T} \left[ \frac{T_c}{\pi^2} - \frac{(m_d)^2}{2\pi} + \frac{(m_d)^2}{\pi^2 T_c} + \ldots \right]
\]

<table>
<thead>
<tr>
<th></th>
<th>2SC</th>
<th>CFL</th>
<th>( T&gt;T_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_d^2 )</td>
<td>( 2(\beta_1+\beta_2)</td>
<td>d</td>
<td>^2/K_T ) (1)</td>
</tr>
<tr>
<td></td>
<td>( -\beta_2</td>
<td>d</td>
<td>^2/K_T ) (8), 0 (9)</td>
</tr>
<tr>
<td>( m_A^2 )</td>
<td>( 4K_T g^2</td>
<td>d</td>
<td>^2/3 ) (1)</td>
</tr>
<tr>
<td></td>
<td>( K_T g^2</td>
<td>d</td>
<td>^2 ) (4), 0 (3)</td>
</tr>
</tbody>
</table>
Fluctuation-induced first order transition (contd.)

The region in which the Gaussian approximation is valid

1. Weak coupling regime

\[
\frac{|\beta_i|}{32\pi^2 K^2_T} \ll 1 \quad \text{and} \quad \frac{2\alpha_s}{\pi} \ll 1, \quad \text{with} \quad \alpha_s = \frac{g^2}{4\pi}
\]

Then, \( m_A \gg m_d \). “Type I”

2. Outside critical regions


\[
\frac{|T - T_c|}{T_c} \gg \epsilon_d \approx 10^{-2} \left( \frac{T_c}{100 \text{ MeV}} \right)^4 \left( \frac{1 \text{ GeV}}{\mu} \right)^4 \quad \text{for diquark fields}
\]

\[
\frac{|T - T_c|}{T_c} \gg \epsilon_A \approx \alpha_s \left( \frac{T_c}{100 \text{ MeV}} \right)^2 \left( \frac{1 \text{ GeV}}{\mu} \right)^2 \quad \text{for gluon fields}
\]

The valid temperature region in weak coupling:

The fluctuations in gluon fields are more important.

Cf. Fluctuation effects on the phase transition in ordinary superconductors are hard to probe experimentally since the critical region for the order parameter is very small.
Fluctuation-induced first order transition (contd.)

Effects of fluctuations in gluon fields —— the weak coupling CFL condensate ——

Free energy

\[ \Delta \Omega = 3 \alpha |\kappa_A|^2 + 3(3 \beta_1 + \beta_2)|\kappa_A|^4 + \frac{1}{2} \sum \int_0^{m^2_{\Lambda \alpha \alpha}} d(m^2_{\Lambda \alpha \alpha}) \langle A^\alpha A^\alpha \rangle \]

\[ = \left(3 \alpha + \frac{32}{\pi} T T_c K_T \alpha_s \right) |\kappa_A|^2 - \frac{8 \sqrt{2}}{3 \pi} T (4 \pi K_T \alpha_s)^{3/2} |\kappa_A|^3 + \left[3(3 \beta_1 + \beta_2) + 128 \left( \frac{T}{T_c} \right) K_T^2 \alpha_s^2 \right] |\kappa_A|^4 \]

Transition temperature

Weak first order transition

\[ T_c' < T_c < T_c^* \quad , \quad \frac{T_c^* - T_c'}{T_c} \sim O(\alpha_s) \]
Fluctuation-induced first order transition (contd.)

Comparison with other works—— the weak coupling regime——

Short-wavelength fluctuations beyond the Ginzburg-Landau framework can be important in strong Type-I superconductors (Ren’s talk).

<table>
<thead>
<tr>
<th>Approach</th>
<th>$m_A(k=0)$ at $T = T_{c}^*$</th>
<th>$(T_{c}^* - T_{c}')/T_{c}'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bailin &amp; Love</td>
<td>$m_A(k=0) \gg T_{c}$</td>
<td>$O\left(\alpha_s^3 \mu^2 T_{c'}^{-2}\right) \gg 1$</td>
</tr>
<tr>
<td>Present</td>
<td>$m_A(k=0) \sim T_{c}$</td>
<td>$O\left(\alpha_s\right)$</td>
</tr>
<tr>
<td>Giannakis et al.</td>
<td>$m_A(k=0) \gg T_{c}$</td>
<td>$O\left(\alpha_s^{1/2}\right)$</td>
</tr>
<tr>
<td>(Ren’s talk)</td>
<td></td>
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</tbody>
</table>

The table above compares different approaches to the weak coupling regime, focusing on the behavior of the order parameter $m_A(k=0)$ near the critical temperature $T_{c}^*$ and how it diverges from the Ginzburg-Landau prediction $m_A(k=0) \sim T_{c}$ with a correction term $O(\alpha_s)$. The figure illustrates the transition temperatures $T_{c}'$, $T_{c}$, and $T_{c}^*$, highlighting the first order transition at $T_{c}^*$. The $O(\alpha_s)$ notation indicates the order of magnitude of the correction term, with $\alpha_s$ being a dimensionless parameter that characterizes the system's anisotropy.
Melting pattern of diquark condensates in neutral quark matter

*What are the color superconducting phases like in a more realistic situation?*

- Color and electric neutrality
- Weak equilibrium with an electron gas
- Nonzero quark masses (especially strange quark mass)
- Instantons ...

$T=0$

- in weak coupling: **CFL with possible “$\eta$” condensation**  
  Ref. Kryjevski et al., hep-ph/0312363
- in strong coupling: many other possibilities
  - 2SC, gapless 2SC, gapless CFL, CFL with “$K$” condensation, Sarma, deformed Fermi surface, phase separation/mixed phase, LOFF, 1-color pairing, uSC, ...

Near $T_c$

- in weak coupling (**Tachibana’s talk**)  
  Ref. Iida et al., hep-ph/0312363
- in strong coupling (**Kouvaris’ talk**)  
  Ref. Rüster et al., hep-ph/0405170

Cf. The physics looks simpler near $T_c$ than at $T=0$. 

![Diagram showing phase transitions](image-url)
Melting pattern of diquark condensates in neutral quark matter (contd.)

Corrections to the Ginzburg-Landau free energy in the weak coupling and massless limit

Strange quark mass

\[ \delta k^s_F = -\frac{3m_s^2}{2\mu} \]

Electric neutrality and weak equilibrium

\[ \delta \mu_i = -q_i \mu_e, \quad \mu_e = \frac{3m_s^2}{4\mu} \]

Corrections up to \( O(m_s^2) \)

\[ 8N(\mu/3)\sigma \sum_a |d_a|^2 - |(d_a)_s|^2 \]

\[ 4N(\mu/3)\sigma \sum_a \frac{1}{3} |d_a|^2 - |(d_a)_u|^2 \]

with

\[ \sigma \equiv \frac{3\pi^2}{8\sqrt{2}} \frac{m_s^2}{g_s} \left( \frac{3}{\mu} \right)^2 \]
Melting pattern of diquark condensates in neutral quark matter (contd.)

Splitting of the critical temperature in weak coupling

Pairing ansatz \( \langle d_a \rangle_i = \begin{pmatrix} \Delta_1 & 0 & 0 \\ 0 & \Delta_2 & 0 \\ 0 & 0 & \Delta_3 \end{pmatrix} \)

Cf. Similar splitting of the critical temperature is also seen in superfluid \(^3\)He under magnetic fields:

A phase → normal at \( H=0 \) vs. \( A_2 \) phase → \( A_1 \) phase → normal at \( H \neq 0 \).
Conclusion

Ginzburg-Landau approach to color superconductivity helps us examine

- Phase diagrams near $T_c$
- Responses to external magnetic fields and rotation
- Effects of the thermally fluctuating color magnetic fields on the phase transition of weak-coupling Type I color superconductors
- Effects of non-zero strange quark mass and electric charge neutrality on melting pattern of diquark condensates

Many questions remain

- $\mu$ dependence of the parameters ($\alpha^+, \beta_1, \beta_2, K_T, g$) in the free energy at lower densities?
- Competition with other possible order parameters?