Aspects of High Density Effective Theory in QCD

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QCD and Dense Matter:
From Lattices to Stars (INT-04-1)

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1. Introduction

- **QCD describes the strong interaction:**
  1. The prediction on high energy hadron interaction is confirmed.
  2. The low energy hadron dynamics is in good agreement with $\chi$SB of QCD.

- **QCD should tell us how matter behaves at extreme environments:**
  heavy ion collision, early universe, compact stars

- **QCD predicts phase transitions;** $T_C, \mu_C \sim \Lambda_{QCD}$

![QCD Phases Diagram]

- **QCD Phases Diagram**
  - $T$ (Temperature)
  - $\mu$ (Chemical Potential)
  - Key Points:
    - **Hadron Matter**
    - **Quark-Gluon Plasma**
    - **Color Superconductor**
    - **RHIC**
    - **E**
Confirmed partially by Lattice QCD at zero density:

\[
\begin{align*}
Z(\mu) &= \int dA \det (M) e^{-S(A)} \\
M &= \gamma^\mu E D^\mu_E + \mu \gamma^A_E \neq P^{-1} M^\dagger P.
\end{align*}
\]

- Lattice QCD at finite density has a notorious sign problem

Figure 1: Karsch et. al
Recent progress by reweighting: Fodor and Katz, nucl-th/0201071.

\[
Z(\alpha) = \int d\phi \, \det(M, \alpha) \, e^{-S(\phi, \alpha)} = \int d\phi \, \det(M, \alpha_0) e^{-S(\phi, \alpha_0)} W(\phi, \alpha)
\]

• However, the complexity is due to fast modes \((\omega \gtrsim \mu)\): DKH+Hsu, PRD 02, 03.

• The sign problem is mild or absent for physics near the Fermi surface \((\omega \lesssim \mu)\).
2. High Density Effective Theory


- Fermi surface phenomena are determined by modes near F.S.
- At $\mu \gg \Lambda_{QCD}$, $(\vec{\alpha} \cdot \vec{p} - \mu) \psi_\pm = E_\pm \psi_\pm$.
- At energy $E \ll 2\mu$, the states near F.S. ($|\vec{p}| \sim p_F$) are easily excited.
• Introduce patches that cover FS only once.

![Diagram showing Fermi Surface and patches]

• Pick a quark near F. S. and decompose the quark momentum as

\[ p_\mu = \mu v_\mu + l_\mu, \quad |l_\mu| < \Lambda, \ \Lambda_\perp (\ll \mu). \]

• In the leading order in \(1/\mu\) expansion, the energy is independent of \(l_\perp\);

\[
\sum_{\text{patches}} \int_{\Lambda_\perp} d^2 l_\perp = 4\pi p_F^2.
\]
• Some degrees of freedom in QCD are irrelevant in cold QM;

\[
\Psi(x) = \sum_{\vec{v}_F} e^{-i\mu \vec{x} \cdot \vec{v}_F} [\psi_+(\vec{v}_F, x) + \psi_-(\vec{v}_F, x)], \quad \frac{1 \pm \vec{\alpha} \cdot \vec{v}_F}{2} \psi = \psi_{\pm} \tag{1}
\]

• The relevant modes are soft gluons and quasi-quarks near the F.S.

\[
\psi_+(\vec{v}_F, x) = \frac{1 + \vec{\alpha} \cdot \vec{v}_F}{2} e^{-i\mu \vec{v}_F \cdot \vec{x}} \psi(x)
\]

since at low energy \(E < \mu\) the Fermi velocity does not change.

• The quark Lagrangian becomes, \(P_{\pm} = (1 \pm \vec{\alpha} \cdot \vec{v}_F)/2\),

\[
\mathcal{L} \ni \bar{\psi} (iD + \mu \gamma^0) \Psi = \sum_{\vec{v}_F} \bar{\psi}(\vec{v}_F, x) (P_+ + P_-) (\mu N + \gamma) (P_+ + P_-) \psi(\vec{v}_F, x)
\]

\[
= \bar{\psi}_+ iD_\parallel \psi_+ + \bar{\psi}_- (2\mu \gamma^0 + iD_\parallel) \psi_- + [\bar{\psi}_- iD_\bot \psi_+ + \text{h.c.}]
\]

• Propagators:

\[
S_+^F = P_+ \frac{i}{\slashed{\mu}} \quad S_-^F = P_- \frac{i\gamma^0}{2\mu} \left[1 - \frac{i\gamma^0 \slashed{N}_\parallel}{2\mu} + \ldots\right] \tag{2}
\]
• By integrating out $\psi_-$ and hard gluons, we obtain the high density effective theory.

• Tree level matching: We eliminate the irrelevant modes by EOM:

$$
\psi_-(v_F, x) = -\frac{i\gamma^0}{2\mu + i D_\parallel} D_\perp \psi_+(v_F, x) = -\frac{i\gamma^0}{2\mu} \sum_{n=0}^{\infty} \left( -\frac{i D_\parallel}{2\mu} \right)^n D_\perp \psi_+(v_F, x)
$$

$$
\bar{\psi_+} i D_\perp \psi_-(v_F, x) \bar{\psi_-} i D_\perp \psi_+(v_F, y) = \includegraphics{tree-level-matching.png} = \includegraphics{tree-level-matching.png}
$$

Figure 2: tree-level matching

• New marginal operators for Cooper pairs at one-loop matching:

$$
\includegraphics{one-loop-matching.png} = \includegraphics{loop-diagram.png} + \includegraphics{cross-diagram.png}
$$

Figure 3: One-loop matching
• HDET has a systematic expansion in $1/\mu$ and $\alpha_s$:

$$\mathcal{L}_{\text{HDET}} = b_1 \bar{\psi}_+ i\gamma^\mu D_\mu \psi_+ - \frac{c_1}{2\mu} \bar{\psi}_+ \gamma^0 (\not{D}_\perp)^2 \psi_+ + \cdots,$$

$$b_1 = 1 + O(\alpha_s), \quad c_1 = 1 + O(\alpha_s), \cdots \quad (3)$$

• Power counting in HDET:

$$\left( \frac{D_\parallel}{\mu} \right)^n \cdot \left( \frac{D_\perp}{\mu} \right)^m \cdot \psi_+ \sim \left( \frac{\Lambda}{\mu} \right)^{n+m} \Lambda^{3l/2}. \quad (4)$$

• To be consistent with the power counting, we impose in loop integration

$$\int_{\Lambda_\perp} d^2 l_\perp l_\perp^n = 0 \quad \text{for} \quad n > 0. \quad (5)$$
3. More on Matching

- Current conservation: In the effective theory, the currents are given in terms of particles and holes (but with no antiparticles) as

$$J^\mu = \sum_{\vec{v}_F} \bar{\psi}(\vec{v}_F, x) \gamma^\mu_{\parallel} \psi(\vec{v}_F, x) - \frac{1}{2\mu} \psi^\dagger(\vec{v}_F, x) \left[ \gamma^\mu_{\perp}, i \vec{D}_{\perp} \right] \psi(\vec{v}_F, x) + \cdots$$

- The HDET current is not conserved unless one adds a counter term,

$$\langle J^\mu(x) J^\nu(y) \rangle = \frac{\delta^2 \Gamma_{\text{eff}}}{\delta A_\mu(x) \delta A_\nu(y)} = \int_p e^{-ip \cdot (x-y)} \Pi^{\mu\nu}(p)$$

$$\Pi^{\mu\nu}_{ab}(p) = -\frac{iM^2}{2} \delta_{ab} \int \frac{d\Omega_{\vec{v}_F}}{4\pi} \left( \frac{-2\vec{p} \cdot \vec{v}_F V^\mu V^\nu}{p \cdot V + i\epsilon \vec{p} \cdot \vec{v}_F} \right)$$

which is not transversal, \( p_\mu \Pi^{\mu\nu}_{ab}(p) \neq 0 \).
For the current conservation, we need to add DeBye mass term due to $\psi_-$. 

$$\Gamma^{\text{eff}} \mapsto \tilde{\Gamma}^{\text{eff}} = \Gamma^{\text{eff}} - \int_x \frac{M^2}{2} \sum \frac{A_\mu A_\nu g_{\mu\nu}^\perp}{\vec{v}_F}.$$  \hspace{1cm} (6)

$$\Pi^{\mu\nu}(p) \mapsto \tilde{\Pi}^{\mu\nu}(p) = \Pi^{\mu\nu} - \frac{i}{2} \sum \frac{g_{\mu\nu}^\perp M^2}{\vec{v}_F}, \quad p_\mu \tilde{\Pi}^{\mu\nu} = 0.$$  \hspace{1cm} (7)

![Two-point functions](image)

Figure 4: Matching two-point functions
• Axial anomaly in dense QCD is independent of $\mu$ (DKH+Hur+Son+Park, to appear)

$$\langle \partial_\mu J^\mu_5 \rangle = \frac{e^2}{8\pi^2} \tilde{F}_{\mu\alpha} F^{\mu\alpha} + \Delta^{\alpha\beta}(\mu) A_\alpha A_\beta, \quad \Delta^{\alpha\beta} = 0. \quad (8)$$

• Axial anomaly due to modes near F.S. is given as

$$\sum_{\vec{v}_F} \int_{x,y} e^{ik_1 \cdot x + ik_2 \cdot y} \left\langle \partial_\mu J^\mu_5(\vec{v}_F, 0) J^\alpha(\vec{v}_F, x) J^\beta(\vec{v}_F, y) \right\rangle = \Delta_{\text{eff}}^{\alpha\beta},$$

$$\Delta_{\text{eff}}^{0i}(k_1, k_2) = -\frac{e^2}{2\pi^2} \cdot \frac{1}{3} \left( \vec{k}_1 \times \vec{k}_2 \right)^i, \quad \Delta_{\text{eff}}^{ij} = \frac{e^2}{2\pi^2} \frac{2}{3} \epsilon^{ijkl} (k_{10}k_{2l} - k_{1l}k_{20}) $$

![Diagram showing axial anomaly calculations](image-url)
4. Color superconductivity in dense QCD

- At $\mu > \Lambda_{\text{QCD}}$, matter becomes quark matter due to asymptotic freedom.
- Cooper instability of Fermi surface

\[
\langle \psi_i(p) \psi_j(-p) \rangle \neq 0 \quad \text{or} \quad \langle \bar{\psi}_i(-p) \psi_j(p) \rangle \neq 0
\]

- Color exchange interaction is attractive for $\bar{3}$ or $8$.

- For $N_c = 3$, BCS is preferred to Overhauser:
  
• 2CS phase at an intermediate density \((\mu \Delta_0 < m_s^2)\):
\[
\langle \psi^a_{L_i} (\vec{p}) \psi^b_{L_j} (-\vec{p}) \rangle = -\langle \psi^a_{R_i} (\vec{p}) \psi^b_{R_j} (-\vec{p}) \rangle = \epsilon_{ab} \epsilon^{ij3} \Delta
\]

• Color-Flavor Locking (CFL) at \(\mu \Delta_0 > m_s^2\):
\[
\langle \psi^a_{L_i} (\vec{p}) \psi^b_{L_j} (-\vec{p}) \rangle = -\langle \psi^a_{R_i} (\vec{p}) \psi^b_{R_j} (-\vec{p}) \rangle = k_1 \delta^a_i \delta^b_j + k_2 \delta^a_j \delta^b_i,
\]
Alford, Rajagopal, Wilczek ‘98

• LOFF when \(\mu_e (= \delta \mu) > \Delta_0\)
\[
\langle \psi^a_{u} (\vec{p}_u) \psi^b_{d} (\vec{p}_d) \rangle = \epsilon^{ab3} \Delta (\vec{q}), \quad \vec{p}_u + \vec{p}_d = 2\vec{q}
\]
Larkin+Ovchinnikov ’64, Fulde+Ferrel ’64; Alford+Bowers+Rajagopal ’01
At the intermediate density gluons are screened:

$$\mathcal{L}_{QCD}^{\text{eff}} \ni \frac{G}{2} \bar{\psi} \psi \bar{\psi} \psi + \cdots,$$

(9)

BCS gap equation is given as

$$0 = \frac{\partial V_{\text{BCS}}(\Delta)}{\partial \Delta} = \frac{\Delta}{G} - 4 \int \frac{d^4k}{(2\pi)^4} \frac{\Delta}{k_0^2 + (\mathbf{k} \cdot \mathbf{v}_F)^2 + \Delta^2}$$

Pole structure guarantees solution.

$$k_0 = \pm \sqrt{(\mathbf{k} \cdot \mathbf{v}_F)^2 + \Delta^2} \mp i \epsilon$$

(10)

The BCS gap

$$\Delta_0 = 2\bar{\mu} \exp \left( -\frac{\pi^2}{2G\bar{\mu}^2} \right).$$

(11)

is estimated to be $10 \sim 100$ MeV at the intermediate density.
• The Cooper-pair gap equation for up and strange quark under stress.

\[
\Delta(p) = \int_\mathbb{R} \frac{i\Delta(l) K(p-l)}{(1+i\epsilon)l_0 - \vec{l} \cdot \vec{v}_u + \delta \mu^u} \frac{(1+i\epsilon)l_0 + \vec{l} \cdot \vec{v}_s - \delta \mu^u}{-\Delta^2},
\]

where \( \delta \mu^u = \mu - \bar{p} \) and \( \delta \mu^s = \mu - \sqrt{\bar{p}^2 + M_s^2} \). (\( \bar{p} \) is the pairing momentum and \( K \) is kernel.) Gap is biggest if \( \delta \mu^u = -\delta \mu^s \) or \( \bar{p} = \mu - \frac{M_s^2}{4\mu} \).

• Pole structure changes and gap closes if \(-\delta \mu^u \delta \mu^s > \Delta^2 / 4\) or \( \Delta < M_s^2 / (2\mu) \).
5. Higher Order Corrections

- At high density magnetic gluons are not screened. The long-range pairing force leads to the Eliashberg gap equation:

\[
\Delta(p_0) = \frac{g_s^2}{36\pi^2} \int_{-\mu}^{\mu} dq_0 \frac{\Delta(q_0)}{\sqrt{q_0^2 + \Delta^2}} \ln \left( \frac{\bar{\Lambda}}{|p_0 - q_0|} \right). 
\]

\[
\bar{\Lambda} = 4\mu/\pi \cdot (\mu/M)^5 e^{3/2} / (\xi). 
\]

Son '98, DKH '99, Miransky, Shovkovy, Wijewardhana '99

Schäfer, Wilczek '99, Pisarski, Rischke '99

- Cooper pair gap at high density is

\[
\Delta_0 = \frac{27\pi^4}{N_f^{5/2}} e^{3\xi/2 + 1} \cdot \frac{\mu}{g_s^5} \exp \left( -\frac{3\pi^2}{\sqrt{2}g_s} \right). 
\]
• The gap equation in dense QCD takes a following form;
\[
\Delta(p_0) = \frac{g_s^2}{c^2} \int dq_0 \frac{\Delta(q_0)}{\sqrt{q_0^2 + \Delta^2}} \left[ (1 + \eta) \ln \left( \frac{\mu}{|p_0 - q_0|} \right) + \ln b + \zeta \right]. \tag{15}
\]

• We use
\[
T^a \ a(p) \ [(p_0 + \mu)\gamma^0 + b(p) \ \vec{p}] - a(p') \ [(p'_0 + \mu)\gamma^0 + b(p') \ \vec{p}'] \ T^a \\
= (p - p')_\mu \Lambda^\mu(p, p') T^a + \Gamma^a(p, p'; -p - p'). \tag{16}
\]
where
\[
\Gamma^a(p, p'; k)\delta(k + p + p') = \int_{z, x, y} e^{i(z \cdot k + x \cdot p + y \cdot p')} \langle \partial^\mu j^a_\mu(z) \psi(x) \bar{\psi}(y) \rangle. \tag{17}
\]

• The one-loop vertex correction has two parts;
Figure 7: The solid line denotes quarks and the curly lines gluons.

While second part negligible, (Fig. 7a) is related to the correction to the wavefunction renormalization constant as

\[(p - p')_\mu \Lambda^{(a)\mu} = a(p) [(p_0 + \mu)\gamma^0 + b(p) \vec{p}] - a(p') [(p_0' + \mu)\gamma^0 + b(p') \vec{p}].\]

The nonlocal gauge, where \(a(p) = 1 = b(p)\), is found to be

\[\xi \approx \frac{2}{3} \ln \frac{(2\mu)^3}{M_0^2 \Delta + \pi M^2 |p_4 - q_4|/2} \approx \frac{1}{3}.\]
6. Positivity of HDET and Vafa-Witten Theorem

Simple example in (1+1) dimensions

- Euclidean (1+1) action of non-relativistic fermions interacting with a gauge field $A$

$$S = \int d\tau dx \psi^* [(-\partial_\tau + i\phi + \epsilon_F) - \epsilon(-i\partial_x + A)] \psi$$  \hspace{1cm} (19)

where $\epsilon(p)$ is the energy as a function of momentum, $\epsilon(p) \approx \frac{p^2}{2m} + \cdots$.

- Dispersion relation with chemical potential: $E(p) = \epsilon(p) - \epsilon_F$. Low energy modes have momentum near $\pm p_F$ ($\epsilon(\pm p_F) = \epsilon_F$).

- Near the Fermi points, the energy as a function of momentum,

$$E(p \pm p_F) \approx \pm v_F p, \quad v_F = \frac{\partial E}{\partial p}|_{p_F}$$  \hspace{1cm} (20)
• Action (19) not obviously positive. Operator in brackets \([ \cdots ]\) has complex eigenvalues.

• Assume gauge field has small amplitude and is slowly varying relative to scale \(p_F\).

Extract the slowly varying component of the fermion \(\to\) low energy effective theory involving quasiparticles and gauge fields with positive, semi-definite determinant.

• Extract quasiparticle modes \(\psi_{L,R}\):

\[
\psi(x, \tau) = \psi_L e^{+ip_F x} + \psi_R e^{-ip_F x}, \tag{21}
\]

Using \(e^{\pm ip_F x} E(-i\partial_x + A) e^{\mp ip_F x} \psi(x) \approx \pm v_F (-i\partial_x + A)\psi(x)\), to obtain

\[
S_{\text{eff}} = \int d\tau dx [\bar{\psi}_L (-\partial_\tau + i\phi + i\partial_x - A)\psi_L \\
+ \bar{\psi}_R (-\partial_\tau + i\phi - i\partial_x + A)\psi_R]. \tag{22}
\]

• Introducing the Euclidean (1+1) gamma matrices \(\gamma_{0,1,2}\) and \(\psi_{L,R} = \frac{1}{2} (1 \pm \gamma_2)\psi\) we obtain a positive action:

\[
S_{\text{eff}} = \int d\tau dx \bar{\psi} \gamma^\mu (\partial_\mu + iA_\mu)\psi \equiv \int d\tau dx \bar{\psi} D/\psi. \tag{23}
\]
• Since \( (\partial_\mu + iA_\mu) \) is anti-Hermitian, the operator \( \mathcal{D} \) in (23) has purely imaginary eigenvalues. Since \( \gamma_2 \) anticommutes with \( \mathcal{D} \), the eigenvalues come in conjugate pairs: given \( \mathcal{D} \phi = \lambda \phi \), we have

\[
\mathcal{D} (\gamma_2 \phi) = -\gamma_2 \mathcal{D} \phi = -\gamma_2 \lambda \phi = -\lambda (\gamma_2 \phi_n)
\]

Hence the determinant \( \det \mathcal{D} = \prod \lambda^* \lambda \) is real and positive semi-definite.

• Since the gamma matrices are Hermitian, and the operator \( (\partial_\mu + iA_\mu) \) is anti-Hermitian, the operator \( \mathcal{D} \) in (23) has purely imaginary eigenvalues. Since \( \gamma_2 \) anticommutes with \( \mathcal{D} \), the eigenvalues come in conjugate pairs: given \( \mathcal{D} \phi = \lambda \phi \), we have

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\]

Hence the determinant \( \det \mathcal{D} = \prod \lambda^* \lambda \) is real and positive semi-definite.

• By considering only the low-energy modes near the Fermi points, we obtain an effective theory with desirable positivity properties.

• Note: interactions (background gauge field \( A \)) must not strongly couple the low-energy modes to fast modes which are far from the Fermi points – reasonable approximation in many situations: interactions among quasiparticles of primary interest.
• **RECIPE:** Near Fermi surface, modes have low energy and are slowly varying. Coupling these modes to slowly varying background field $A$ leads to a positive effective theory.

• **Slowly varying** = relative to Fermi momentum $p_F$.

• **QCD:** strong coupling dynamics at scales $\sim \Lambda_{\text{QCD}}$. By taking

$$\mu \sim p_F \gg \Lambda_{\text{QCD}}$$

we ensure that quark quasiparticles couple only to slowly varying, small amplitude background fields $A$. 
Degenerate free NR fermions

- Consider an electron system, described by

\[ \mathcal{L} = \psi^\dagger [i \partial_t - \epsilon(\vec{p})] \psi + \mu \psi^\dagger \psi, \]

(24)

- The system suffer “sign problem”, since the Euclidean determinant has complex eigenvalues,

\[ M = -\partial_\tau - \epsilon(\vec{p}) + \mu. \]

(25)

- For modes near the Fermi surface, however, the sign problem becomes mild: The determinant becomes, if we integrate the fast modes,

\[ M_{\text{EFT}} = -\partial_\tau - \vec{v}_F \cdot \vec{l}, \quad M_{\text{EFT}}(\vec{v}_F)M_{\text{EFT}}(-\vec{v}_F) \leq 0, \]

(26)

\[ \text{if } \epsilon(\vec{p}) = \epsilon(-\vec{p}). \]
Positivity of QCD at asymptotic density

- The quasiparticles are described by

\[
\mathcal{L}_{\text{eff}} = \bar{\psi}\gamma_\parallel \cdot D\psi + (\vec{v}_F, x) - \psi^\dagger \frac{(D_\perp)^2}{2\mu} \psi + (\vec{v}_F, x) + \cdots
\]

- The leading term has a positive determinant:

\[
M_{\text{eft}} = \gamma_\parallel \cdot D(A) = \gamma_5 M_{\text{eft}}^\dagger \gamma_5 \quad (27)
\]

- Anomaly

\[
i \not{\sigma}_\parallel P \psi = P_- i \not{\sigma}_\parallel \psi\, . \quad (28)
\]

The divergence of the quark current at one loop is

\[
\langle \partial_\mu J_\mu^a (\vec{v}_F, x) \rangle = g_s \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-i p \cdot x}}{p^\mu \Pi_\mu_\nu (p) A_\parallel^\nu (-p)} \neq 0. \quad (29)
\]

- However, if we include \( \psi_+ (\vec{v}_F, x) \) and \( \psi_+ (-\vec{v}_F, x) \) the current is conserved:

\[
\langle \partial_\mu J_\mu^a (\vec{v}_F, x) + \partial_\mu J_\mu^a (-\vec{v}_F, x) \rangle = 0. \quad (30)
\]
• Under a gauge transformation, $U(x) = e^{i\vec{q} \cdot \vec{x}}$, the energy level shifts

$$E = \vec{l} \cdot \vec{v}_F \iff E = \vec{l} \cdot \vec{v}_F + \vec{q} \cdot \vec{v}_F.$$  

(31)

Figure 8: Spectral Flow
Vafa-Witten Theorem at High Density: CFL is exact.

- Cooper theorem says pairing at color anti-triplet channel. For three light flavors,
  \[
  \left\langle \psi_{L_{i\alpha}}^a (\vec{v}_F, x) \psi_{L_{j\beta}}^b (-\vec{v}_F, x) \right\rangle = - \left\langle \psi_{R_{i\alpha}}^a \psi_{R_{j\beta}}^b \right\rangle = \epsilon_{ij} \epsilon^{abc} \epsilon_{\alpha\beta\gamma} K^\gamma_c (p_F)
  \]

- By the global color and flavor symmetry, \( K^\gamma_c = \delta^\gamma_c K^\gamma \).

- The vacuum energy in the HDL approximation is given as
  \[
  V(\Delta) = -\text{Tr} \ln S^{-1} + \text{Tr} \ln \phi + \text{Tr} (S^{-1} - \phi) S + (2\text{PI diagrams})
  = \frac{\mu^2}{4\pi} \sum_{i=1}^{9} \int \frac{d^2 l_\parallel}{(2\pi)^2} \left[ \ln \left( \frac{l_\parallel^2}{l_\parallel^2 + \Delta_i^2 (l_\parallel)} \right) + \frac{1}{2} \cdot \frac{\Delta_i^2 (l_\parallel)}{l_\parallel^2 + \Delta_i^2 (l_\parallel)} \right] + h.o.
  \]
The gluon energy is subleading, \( V_g(\Delta) \sim M^2 \Delta^2 \ln(\Delta/\mu) \sim g_s \mu^2 \Delta^2 \),

\[
V(\Delta) \simeq -0.43 \frac{\mu^2}{4\pi^2} \sum_i |\Delta_i(0)|^2
\]
Since $\Delta_{ab}^{\alpha\beta} = \epsilon_{\alpha\beta\gamma} \epsilon^{abc} \Delta_\gamma \delta_c^\gamma$,

$$V(\Delta) \simeq -0.43 \frac{\mu^2}{4\pi^2} |\Delta_u|^2 f(x, y),$$

where $\Delta_d/\Delta_u = x$ and $\Delta_s/\Delta_u = y$. $f(x, y) \leq 13.4$ has a maximum at $x = 1 = y$.

$K_{\alpha} = K\delta_{\alpha}^\gamma$ and $SU(3)_V$ is unbroken

$$\Delta_i = \Delta_u \cdot (1, 1, 1, -1, 1, -1, 1, -1, -2).$$
• CFL is exact at asymptotic density:
  Vector current correlators fall off exponentially, if all quarks are gapped.
  
  \[
  \left\langle J^A_\mu (\vec{v}_F, x) J^B_\nu (\vec{v}_F, y) \right\rangle^A = -\text{Tr} \gamma_\mu T^A S^A (x, y; \Delta) \gamma_\nu T^B S^A (y, x; \Delta),
  \]
  with \( J^A_\mu (\vec{v}_F, x) = \bar{\psi}_+ (\vec{v}_F, x) \gamma_\mu T^A \psi_+ (\vec{v}_F, x) \).

• The (anomalous) propagator with \( SU(3)_V \)-invariant IR regulator \( \Delta \) is given as
  
  \[
  \langle x | \frac{1}{M} | y \rangle = \int_0^\infty d\tau \langle x | e^{-i\tau (-iM)} | y \rangle
  \]
  where with \( D = \partial + iA \)
  
  \[
  M = \gamma_0 \begin{pmatrix} D \cdot V & \Delta \\ \Delta & D \cdot \bar{V} \end{pmatrix}
  \]
  Since the eigenvalues of \( M \) are bound from below by \( \Delta \), we have
  
  \[
  \left| \langle x | \frac{1}{M} | y \rangle \right| \leq \int_R^\infty d\tau e^{-\Delta \tau} \sqrt{\langle x | x \rangle} \sqrt{\langle y | y \rangle} = \frac{e^{-\Delta R}}{\Delta} \sqrt{\langle x | x \rangle} \sqrt{\langle y | y \rangle}.
  \]
• The current correlators fall off rapidly as $R \equiv |x - y| \to \infty$;

$$\left| \int dA_+ \det M_{\text{eff}}(A) e^{-S_{\text{eff}}} \left\langle J^A_{\mu}(\bar{v}_F, x) J^B_{\nu}(\bar{v}_F, y) \right\rangle^{A+} \right|$$

$$\leq \int_{A+} \left| \left\langle J^A_{\mu}(x) J^B_{\nu}(y) \right\rangle^{A+} \right| \leq \frac{e^{-2\Delta R}}{\Delta^2} \int_{A+} \sqrt{\langle x|x \rangle} \sqrt{\langle y|y \rangle}$$

• The (IR regulated) vector currents do not create a massless mode out of vacuum or Fermi sea.

• No Goldstone mode along the $SU(3)_V$ channel!

• For three light flavors $SU(3)_V$ has to be unbroken as in CFL.

• When $m \neq 0$, one has to take the $\mu \to \infty$ and $m_q \to 0$ limit carefully. If $m_s$ goes to zero too slowly, Kaon condenses to break isospin.
Operator formalism

- We introduce an operator formalism for lattice

\[ \vec{v} = \frac{-i}{\sqrt{-\nabla^2}} \frac{\partial}{\partial \vec{x}} , \]  

(32)

- The quasi-quarks near F.S. are defined as

\[ \psi = \exp (+i \mu x \cdot v \alpha \cdot v) \psi_+ , \]  

(33)

- Now, the Lagrangian becomes with \( X = \mu x \cdot v \alpha \cdot v \)

\[ \mathcal{L}_+ = \bar{\psi}_+ \gamma_\parallel^\mu \left( \partial^\mu + i A_+^\mu \right) \psi_+ , \quad (A_+^\mu = e^{-iX} A^\mu e^{+iX}) \]  

(34)

- Using \( v \cdot \partial \bar{v} \cdot \gamma = \partial \cdot \gamma \), we get

\[ \gamma_\parallel^\mu \partial^\mu = \gamma^\mu \partial^\mu \]  

(35)
Integrating out the fast modes ($\psi$ and hard gluons), the slow modes have a positive measure in the leading order with an effective action

$$S_{\text{eff}}(A) \approx \int d^4 x_E \left( \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{M^2}{16\pi} \sum_{v_F} A_{\perp \mu}^a A_{\perp \mu}^a \right) \geq 0,$$

where $A_{\perp} = A - A_{\parallel}$ and the Debye mass is $M = \sqrt{N_f/(2\pi^2)} g_s \mu$.

QCD partition function now becomes

$$Z = \int dA d\bar{\psi} d\psi e^{-S(A,\psi,\bar{\psi})} = \int dA_+ \det (M_{\text{eff}}) e^{-S_{\text{eff}}(A_+)}$$

$$\det M_{\text{eff}} \geq 0, \quad S_{\text{eff}}(A_+) = \text{positive} \cdot [1 + O(1/\mu)].$$
7. Lattice Simulation

- Back to the original (not HDET) QCD partition function:

\[
Z(\mu) = \int dA_\mu \det (M) e^{-S(A_\mu)}.
\]

\(M\) is the Dirac operator at finite density, \(A\) the usual gauge field.

- It is easy to show that at \(A = 0\) (zero background gauge field), the Dirac determinant is real even at finite density.

- Now, consider small amplitude, slowly varying background gauge fields \(A\) whose magnitude and derivatives \(\partial A\) are small relative to \(\mu\). (e.g. \(\mu \gg \Lambda_{QCD} \sim A, \partial A, \text{ or } F_{\mu\nu}, D_\mu\).)

- Expand about the FS. Integrate out heavy modes (antiquarks, quarks far from the FS). These modes contribute to \(\det (M)\), but their contribution is suppressed by \(1/\mu\). Find

\[
\det (M) = \text{[real, positive]} \left( 1 + \mathcal{O}\left( \frac{F}{\mu^2} \right) \right).
\]
• How to enforce small amplitude, slowly varying gauge field $A$?

• Use two lattices with different spacings $a_{\text{det}}, a_{\text{gauge}}$. Compute determinant on lattice with spacing $a_{\text{det}} \sim \mu^{-1} << a_{\text{gauge}}$.

• Determinant is a function of plaquettes $\{U_{x\mu}\}$ which are obtained by interpolation from the plaquettes on the coarser $a_{\text{gauge}}$ lattice.

• Interpolation: link variables $U_{x\mu} \in SU(3)$. Connect any two points $g_1, g_2$ on the group manifold:

$$ g(t) = g_1 + t(g_2 - g_1), \ 0 \leq t \leq 1 $$
• Use leading real, positive part of determinant for importance sampling.

• Nontrivial check on analytic results at asymptotic density. Extrapolate to intermediate density?
8. Conclusion

- We have constructed an effective theory (HDET) for dense QCD.
  1. It deals with relevant modes only and is effectively (1+1) dim’nal.
  2. Marginal Cooper-pair operator, Screening mass, · · · naturally arise at one-loop.
  3. Systematic expansion in $1/\mu$ and $\alpha_s$.
  4. It has a consistent power counting rule.
- With this effective theory, we calculate the properties of various phases in dense QCD: Cooper-pair gap, Critical temperature and density ...
  1. At high density, the superconducting gap takes at two-loop

$$
\Delta_0 = \frac{2^7 \pi^4}{N_f^{5/2}} e^{1.5} \cdot \frac{\mu}{g_s^5} \exp \left( -\frac{3\pi^2}{\sqrt{2}g_s} \right).
$$

  2. Critical phenomena at $T_C = 0.57\Delta_0$ or $\mu_C \simeq 0.22\text{GeV}$. 


• Quark matter under stress.
  1. Crystalline superconductor if $\Delta_0/\sqrt{2} < (\mu_d - \mu_u)/2 < 0.74\Delta_0$
  2. Kaon condensation if $m_s > m^{1/3} \Delta_0^{2/3}$.

• The HDET has a positive measure.
  1. Therefore, lattice QCD is positive at asymptotic density. Lattice simulation should be possible for dense QCD
  2. Vafa-Witten theorem applies: Vector symmetries except $U(1)_B$ are unbroken in QCD at asymptotic density: CFL is exact!

• The lattice simulation is possible with HDET or two-lattice QCD.