Lattice Simulations at the Fermi Surface

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- The QCD Phase Diagram & Color Superconductivity
- Friedel Oscillations
- In-Medium Effects in the Gross-Neveu Model
- Fermi Liquid Theory
- Mesons & Zero Sound

SJH, J.B. Kogut, C.G. Strouthos, T.N. Tran PRD68(2003)016005
SJH, B. Lucini & S.E. Morrison, PRD65(2002)036004
SJH & D.N. Walters hep-lat/0401018
The QCD Phase Diagram

- **quark–gluon plasma**
- **hadronic fluid**
- **nuclear matter**
- **color superconductor**
- **compact stars**
- **critical endpoint**
- **crossover**
- **(μ_E, T_F)**
- **GSI?**

Axes:
- \( T \) \((\text{MeV})\)
- \( \mu \) \((\text{MeV})\)
Color Superconductivity

In the asymptotic limit $\mu \to \infty$, $g(\mu) \to 0$, the ground state of QCD is the color-flavor locked (CFL) state characterised by a BCS instability, ie. diquark pairs at the Fermi surface condense via

$$\langle q_i^\alpha(p) C \gamma_5 q_j^\beta(-p) \rangle \sim \varepsilon^{A\alpha\beta} \varepsilon_{Aij} \times \text{const.}$$

breaking $\text{SU}(3)_c \otimes \text{SU}(3)_L \otimes \text{SU}(3)_R \otimes \text{U}(1)_B \otimes \text{U}(1)_Q$ \rightarrow $\text{SU}(3)_\Delta \otimes \text{U}(1)_{\tilde{Q}}$

The ground state is simultaneously superconducting (8 gapped gluons), superfluid (1 Goldstone), and transparent (all quasiparticles with $\tilde{Q} \neq 0$ gapped).

What can we say at smaller densities $\mu \sim O(1 \text{ GeV})$ where weak coupling methods can’t be trusted? Lattice QCD simulations can’t help because the Euclidean path integral measure $\det M(\mu)$ is not positive definite.

In condensed matter theory there are two tractable limits:

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Both model QFT’s can be studied with $\mu \neq 0$ using lattice simulations.

High-$T_c$ superconducting compounds, and perhaps QCD, are difficult problems because they belong to neither limit.
Gross-Neveu model in $2 + 1$ dimensions...

\[ \mathcal{L} = \sum_{i=1}^{N_f} \bar{\psi}_i (\partial \psi_i + m) - \frac{g^2}{2N_f}(\bar{\psi}_i \psi_i)^2, \]

...just about the simplest QFT with fermions
Can also write in terms of an auxiliary scalar $\sigma$:

\[ \mathcal{L} = \bar{\psi}_i (\partial \psi_i + m + \frac{g}{\sqrt{N_f}} \sigma) + \frac{1}{2} \sigma^2. \]

For $g^2 > g_c^2 \sim O(\Lambda^{-1})$ the ground state has a dynamically-generated fermion mass $\Sigma_0 = \frac{g}{\sqrt{N_f}} \langle \sigma \rangle \neq 0$
given in the $N_f \to \infty$ limit by the chiral Gap Equation

\[ \Sigma_0 = g^2 \text{tr} \int p \frac{1}{i\not{\psi} + \Sigma_0} \]
In same limit $\sigma$ acquires non-trivial dynamics:

$$D_{\sigma}^{-1}(k^2) = 1 - \Pi(k^2) \propto \begin{cases} 
  k^2 + 4\Sigma_0^2 & k \ll \Sigma_0 \\
  \sqrt{k^2} & k \gg \Sigma_0
\end{cases}$$

⇒ The model is unexpectedly renormalisable

(1/$N_f$ expansion)

The $\text{GN}_{2+1}$ model has an UV-stable renormalisation group fixed point and an interacting continuum limit as $g \to g_c$.

Can also formulate NJL model with $\text{SU}(2)_L \otimes \text{SU}(2)_R$ chiral symmetry: auxiliary fields are now $\sigma$ and $\vec{\pi}$. 
GN Thermodynamics

The large-$N_f$ approach can also to be applied to $T, \mu \neq 0$ and predicts a chiral symmetry restoring phase transition:

$$T_c|_{\mu=0} = \frac{\Sigma_0}{2 \ln 2}; \quad \mu_c|_{T=0} = \Sigma_0$$

Remarkably, lattice Monte Carlo simulations can be applied to $N_f < \infty$ even for $\mu \neq 0$.

There is even evidence for a tricritical point at small $\frac{T}{\mu}$!

[J.B. Kogut and C.G. Strouthos PRD63(2001)054502]
Friedel Oscillations in Dense Matter

Consider $q\bar{q}$ “jawbone” diagram

\[ C(\bar{y}, x_0) = \sum_{\bar{x}} \text{tr} \int_p \int_q \Gamma \frac{e^{ipx}}{i\bar{\phi} + \mu \gamma_0 + M} \frac{e^{-iqx} e^{-i\bar{q} \cdot \bar{y}}}{i\bar{\phi} + \mu \gamma_0 + M} \]

$\mu < \mu_c$:
\[ C \propto \int_0^\infty p dp J_0(py) e^{-2x_0 \sqrt{p^2 + M^2}} \sim \frac{M}{x_0} e^{-2Mx_0} \exp \left(-\frac{|\bar{y}|^2 M}{4x_0}\right) \]
Gaussian width $O(\sqrt{x_0})$

$\mu > \mu_c$:
\[ C \propto \int_\mu^\infty p dp J_0(py) e^{-2px_0} \sim \frac{\mu}{x_0} e^{-2\mu x_0} J_0(\mu |\bar{y}|) \propto J_0(k_F y) \]
Oscillatory profile; shape constant as $x_0 \rightarrow 0$
Oscillations develop as $\mu \rightarrow$ 
Graphic evidence for existence of a sharp Fermi surface
Why does free-field theory prediction work so well?
\[ D_{\sigma}^{-1} = 1 - \Pi(k; \mu) = 1 - \Phi \]

For \( T, \mu \neq 0 \) fermion propagator is
\[ S_{F}^{-1}(k_0, \vec{k}) = i(k_0 - i\mu)\gamma_0 + i\vec{k} \cdot \vec{\gamma} \] with \( k_0 \in \omega_n \equiv (2n - 1)\pi T \).

\[ \Rightarrow \Rightarrow \quad D_{\sigma}^{-1}(k_0, \vec{k}) = 4g^2T \sum_n \int \frac{d^2\vec{q}}{(2\pi)^2} \times \]
\[
\left\{ \begin{array}{ccc}
1 & \frac{1}{(\omega_n - i\mu)^2 + \vec{q}^2} & - \frac{k_0(\omega_n - k_0 - i\mu) + \vec{k} \cdot (\vec{q} - \vec{k})}{[(\omega_n - i\mu)^2 + \vec{q}^2][((\omega_n - k_0 - i\mu)^2 + (\vec{q} - \vec{k})^2)}
\end{array} \right\}
\]

\( I \quad \quad II \quad \quad III \)

\( I \) is the “1” from the gap equation at \( T = \mu = 0 \)
\[ \lim_{T \to 0} \{ I + II \} = \frac{g^2}{\pi}(\mu - \mu_c) \] with \( \mu_c = \Sigma_0 \).
Term \( III \) describes all momentum dependence

Perform \( \sum_n \) using standard formulæ and take \( T \to 0 \):

\[
III = g^2 \int \frac{d^2 \tilde{q}}{(2\pi)^2} \frac{1}{E_1 E_2} \times \\
\left\{ -k_0^2 + \tilde{k}.(\tilde{q} - \frac{1}{2} \tilde{k}) - iE_1 k_0 \right\} \left( \frac{\theta(E_1-\mu)}{ik_0 - E_1 - E_2} + \frac{\theta(E_2-\mu) - \theta(E_1-\mu)}{ik_0 - E_1 + E_2} \right) \tag{i} \\
- \left[ -k_0^2 + \tilde{k}.(\tilde{q} - \frac{1}{2} \tilde{k}) + iE_1 k_0 \right] \frac{\theta(E_2-\mu)}{ik_0 + E_1 + E_2} \right\} \tag{ii} \\
\tag{iii}
\]

where \( E_{1,2} = |\tilde{q} \pm \frac{1}{2} \tilde{k}| \)

Now expand in powers of \( \frac{k}{\mu} \):

**Hard Dense Loop approximation**
\[ I + II + III \Rightarrow \]

\[
D_{\sigma}^{-1}(k_0, \vec{k}) = \frac{g^2}{4\pi\mu} \times \left\{ 4\mu(\mu - \mu_c) + k_0^2 + \frac{k_0|\vec{k}|^2}{k_0 + \sqrt{k_0^2 + |\vec{k}|^2}} \right. \\
\left. + i \frac{k_0|\vec{k}|^2}{4\mu} \left( 1 + \frac{|k|^2}{(k_0 + \sqrt{k_0^2 + |\vec{k}|^2})^2} \right) + O\left(\frac{k^4}{\mu^2}\right) \right\}
\]

**Static Limit** \( k_0 = 0 \):

Complete screening for \( r > 0 \) \( \iff \) Debye mass \( M_D = \infty \)

Explains free-field Friedel oscillations

**Zero Momentum Limit** \( \vec{k} = \vec{0} \):

\[
D_{\sigma}^{-1} = \frac{g^2}{\pi} (\mu - \mu_c)
\]

Conventional boson of mass \( M_\sigma = 2\sqrt{\mu(\mu - \mu_c)} \)

Stable because decay into \( q\bar{q} \) requires energy \( 2\mu \)

and is Pauli-blocked. \( \iff \) Plasma frequency \( \omega_P = M_\sigma \)
For states in motion must consider *retarded* propagator, yielding the dispersion relation

\[ E(\vec{k}) \simeq M_\sigma + \frac{|\vec{k}|^2}{4} \left( \frac{1}{M_\sigma} + \frac{1}{2\mu} \right) \]

⇒ non-relativistic particle of mass $2\mu$ as $\mu \to \infty$. 
Numerical Results with $N_f = 4$

In contrast with behaviour in the chirally-symmetric bulk phase, in quark matter the $\sigma$ exhibits a sharply-defined pole at $M_\sigma(\mu)$ consistent with $O(1/N_f)$ corrections to the leading order result $M_\sigma = 2\sqrt{\mu(\mu - \mu_c)}$ with $\mu_c a \approx 0.16$

Note $\sigma$ tightly bound for $\frac{\mu - \mu_c}{\mu} \ll 1$
The $\sigma$ dispersion relation $E(|\vec{k}|)$ is also modified as $\mu \nearrow$ in qualitative agreement with the large-$N_f$ result.

Discretisation artifacts near the zone edge?
Fermi Liquid Theory

A semi-quantitative description of interacting degenerate matter first given by Landau (1958), and given a relativistic generalisation by Baym & Chin (1976).

Basic idea: dominant low energy excitations are quasiparticles carrying same quantum numbers as fundamental particles.

Quasiparticle energy: \( \varepsilon_k \)

Width: \( \sim (\varepsilon_k - \mu)^2 \)

Equilibrium distribution: \( n_k = \left( \exp\left(\frac{\varepsilon_k - \mu}{T}\right) + 1 \right)^{-1} \)

For \( T \to 0 \)

\[
\varepsilon_k \approx \mu + \beta_F \left( |k| - k_F \right)
\]

- Fermi energy
- Fermi velocity
- Fermi momentum
The heart of Landau’s approach is the variation of $\varepsilon_{\vec{k}}$ under small departures from equilibrium:

$$\delta \varepsilon_{\vec{k}} = \int \frac{d^2 \vec{k}'}{(2\pi)^2} \mathcal{F}_{\vec{k},\vec{k}'} \delta n_{\vec{k}'}$$

where the *Fermi Liquid Interaction* is related to the 2-particle forward scattering amplitude:

$$\mathcal{F}_{\vec{k},\sigma,\vec{k}',\sigma'} = -\mathcal{M}_{\vec{k},\sigma,\vec{k}',\sigma'}$$
Direct
vanishes in chiral limit

Exchange
naturally $O(1/N_f)$

\[ \mathcal{F}_{\vec{k},\vec{k}'} = \frac{g^2}{4N_f} \left[ 1 - \frac{\vec{k}.\vec{k}'}{\varepsilon_{\vec{k}}\varepsilon_{\vec{k}'}} \right] D_\sigma(\varepsilon_{\vec{k}} - \varepsilon_{\vec{k}'}, \vec{k} - \vec{k}') \]
\[ = \frac{\pi \mu}{N_f M_\sigma^2(\mu)} (1 - \cos \theta) \]

Since at Fermi surface $\varepsilon_{\vec{k}} - \varepsilon_{\vec{k}'}, \varepsilon_{\vec{k}} \sim 0$
we can take the static limit of $D_\sigma$...
Relations between $\mu$, $k_F$ & $\beta_F$ follow:

*eg.* Consistency under Lorentz boosts $\Rightarrow$

$$\varepsilon_{\vec{k}} \vec{\nabla}_{\vec{k}} \varepsilon_{\vec{k}} = \vec{k} + \int \frac{d^2 k'}{(2\pi)^2} F_{\vec{k}', \vec{k}'} \varepsilon_{\vec{k}'} \vec{\nabla}_{\vec{k}'} n_{\vec{k}'}$$

$T \rightarrow 0$, $n_{\vec{k}'} = \theta(\mu - \varepsilon_{\vec{k}'})$, $\beta_F \delta(\varepsilon_{\vec{k}} - \mu) = \delta(|\vec{k}| - k_F)$ $\Rightarrow$

$$\beta_F = \frac{k_F}{\mu} + g \frac{k_F \mu}{4 N_f M^2_\sigma(\mu)}$$

where $g$ is the degeneracy factor of each momentum state

$g = 2$ for the GN model (*ie.* spin $\uparrow, \downarrow$)

$g = 4$ for the NJL model (*ie.* $u^\uparrow$, $u^\downarrow$, $d^\uparrow$, $d^\downarrow$)
Compressibility:

\[
\frac{\partial \mu}{\partial n} = \frac{\partial \varepsilon_k}{\partial k_F} \frac{\partial k_F}{\partial n} + \mathcal{F} = \beta_F \frac{2\pi}{g k_F} + \frac{\pi \mu}{N_f M^2_\sigma(\mu)}
\]

Integrate: \( \int_{k_{Fc}}^{k_F} k_F dk_F = \int_{\mu_c}^{\mu} \frac{\mu d\mu}{1 + \frac{3g}{16 N_f} \frac{\mu}{\mu - \mu_c}} \)

\( \Rightarrow \) for \( \mu \gg \mu_c \) we have

\[
\frac{k_F}{\mu} = 1 - \frac{3g}{32 N_f}; \quad \beta_F = 1 - \frac{g}{32 N_f} < 1
\]

Consistent with causality in this limit.
Causality violated for \( \mu - \mu_c \approx O(\mu_c) \)

\( \Rightarrow \) breakdown of HDL approach
The fermion dispersion relation is fitted with

\[ E(\sqrt{k^2}) = -E_0 + D \sinh^{-1}(\sin k_0) \]

yielding the Fermi liquid parameters

\[ K_F = \frac{E_0}{D}; \quad \beta_F = D \frac{\cosh E_0}{\cosh K_F} \]

Results for GN model consistent with \( g = 2, N_f = 4 \)

\[ 0.2 \lesssim \frac{\mu - \mu_c}{\mu} \lesssim 0.8 \] in this range
Simulations of NJL$_{2+1}$ at stronger couplings, however, yield Fermi liquid parameters inconsistent with $g = 4, N_f = 4$

$0.13 \lesssim \frac{\mu - \mu_c}{\mu} \lesssim 0.22$ in this range

Not clear whether due to discretisation or tightly bound $\sigma$
Finally, simulations of NJL$_{3+1}$, reveal a genuine gap at the Fermi surface consistent with an orthodox BCS mechanism as a result of diquark pair condensation.

The ratio $\Delta / \Sigma_0 \approx 0.15 \Rightarrow \Delta \approx 60 \text{MeV}$

This result is comparable with self-consistent approaches.
Meson Correlation Functions

\[ \sum_{\bar{x}} \bar{\psi}(0) \cdot \bar{\psi}(x) \exp(i\vec{k} \cdot \bar{x}) \]

For \( \vec{k} \neq 0 \), one can always excite a particle-hole pair with almost zero energy \( \Rightarrow \) algebraic decay of correlation functions

\[
\begin{align*}
|\vec{k}| & \ll \mu & \Rightarrow C \sim \frac{1}{x_0^2} \\
|\vec{k}| & = 2\mu & \Rightarrow C \sim \frac{1}{x_0^{3/2}} \\
|\vec{k}| & > 2\mu & \Rightarrow C \sim \frac{e^{-(|\vec{k}| - 2\mu)x_0}}{x_0^{3/2}}
\end{align*}
\]
Plots of $C_{\gamma_5}(\vec{k}, x_0)$ show special behaviour for $|\vec{k}| \approx 2\mu$

- $\mu=0.2$
- $\mu=0.4$
- $\mu=0.6$
- $\mu=0.8$
In the spin-1 channel, $C_{\gamma_{\perp}}$ also looks algebraic, but $C_{\gamma_{\parallel}}$ shows evidence of exponential decay.

The lattice operator corresponding to this excitation is

$$\bar{\chi}(n)\eta_x(n)\chi(n + \hat{x})(-1)^{x+t}$$

In terms of continuum fermions this is

$$\bar{q}(\gamma_0 \otimes \tau_2)q$$
Dispersion relation $E(|\vec{k}|)$ extracted from $C_{\gamma\parallel}$

A massless vector excitation?
Sounds Unfamiliar?

Light vector states in medium are of great interest: Brown-Rho scaling, vector condensation...

In the Fermi liquid framework a possible explanation is a collective excitation thought to become important as $T \to 0$: Zero Sound

Ordinary FIRST sound is a breathing mode of the Fermi surface: velocity $\beta_1 \approx \frac{1}{\sqrt{2}} \frac{k_F}{\mu}$

ZERO sound is a propagating distortion of the Fermi surface: velocity $\beta_0$ must be determined self-consistently
Boltzmann equation in collisionless limit:

\[
\frac{s - \cos \theta}{\cos \theta} \nu(\theta) = \frac{\mu g}{4\pi^2} \int_{\theta'} \mathcal{F}_{\theta, \theta'} \nu(\theta') = G \int \frac{d\theta'}{2\pi} [R - \cos(\theta - \theta')] \nu(\theta')
\]

for GN model with \( \mu \gg \mu_c \) \( G = \frac{g}{8N_f} \), \( R = 1 + \frac{m^2}{\mu^2} \), \( s \equiv \frac{\beta_0}{\beta_F} \).

\[ \nu(\theta) \propto \frac{\cos \theta + x \cos^2 \theta}{s - \cos \theta}, \]

with \( s = \frac{1}{2}(a + a^{-1}) > 1 \); \( x = \frac{a^2 - 1 - 2GR}{2GRs} < 0 \);

\[
2a^2 = 1 + (2R - \frac{3}{2})G - G^2 R + \sqrt{(1 + (2R - \frac{3}{2})G - G^2 R)^2 - 2G}.
\]

Solutions exist only for \( R > R_c = \frac{32}{31} \) in large-\( \mu \) limit.
Empirically, $\omega(k) \simeq \beta_0(k - ak^2)$ with $\beta_0 \lesssim \beta_F$ once discretisation artifacts taken into account.

Results from simulations on $48^3$
Summary
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