The QCD equation of state at finite $T$ and $\mu$: lattice result

Ferenc Csikor (Eötvös University, Budapest)


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Introduction, equation of state (EOS)

QCD at finite $T$ and/or $\mu$ is of fundamental importance describes: early universe, neutron stars, heavy ion collisions present/future heavy ion collisions at CERN, Brookhaven, GSI to detect and study experimentally the quark-gluon plasma

- failure of perturbative approaches $\Rightarrow$ lattice methods at $\mu=0$

line of constant physics (LCP): change “$a$” but not the physics

pure gauge theory: standard, improved actions, continuum limit analysis is automatically on the LCP [Boyd et al 1995, CP-PACS 1999]

staggered QCD: standard and improved actions, $N_t=4,6$

fixed m.a approach: non-LCP [MILC 1997, Karsch et al 2000]

QCD with Wilson quarks: determination of LCPs, $\beta$ functions

EOS along LCPs for $m_\pi/m_\rho=0.65-0.95, N_t=4,6$ [CP-PACS 2001]

no known simulation technique at finite $\mu$
Equation of state along the line of constant physics (LCP)

energy density ($\epsilon$) and pressure ($p$) from partition function:

$$
\epsilon(T) = \frac{T^2}{V} \frac{\partial (\log Z)}{\partial T}, \quad p(T) = T \frac{\partial (\log Z)}{\partial V}.
$$

$T$, $V$ are varied by $a$, take $\frac{\partial \beta}{\partial a}$ and $\frac{\partial (am_i)}{\partial a} = m_i$ on the LCP

$$
\frac{\epsilon - 3p}{T^4} = -\frac{L_t^3}{L_s^3} a \frac{d(\log Z)}{da} = -L_t^4 \left[ a \frac{\partial \beta}{\partial a} < P > + a \frac{\partial (am_u)}{\partial a} < \bar{uu} > + a \frac{\partial (am_s)}{\partial a} < \bar{ss} > \right]
$$

the pressure ($p \propto \log[Z]$) along the LCP by the integral method:

$$
\frac{p}{T^4} = L_t^4 \int d(\beta, m \cdot a) \left( \frac{\partial (\log Z)}{\partial \beta}, \frac{\partial (\log Z)}{\partial (m \cdot a)} \right) = L_t^4 \left[ < P > + m_u \frac{\partial a}{\partial \beta} < \bar{uu} > + m_s \frac{\partial a}{\partial \beta} < \bar{ss} > \right]
$$
Lines of constant physics in the $m \cdot a - \beta$ plane

two LCP-s are defined by
$m_u/T_c = 0.48$ and $m_s/m_u = 2.08$ (LCP1)
$m_u/T_c = 0.384$ and $m_s/m_u = 2.08$ (LCP2).

the critical $\beta_c$ values at $N_t = 4, 6, 8, 10$ were determined

blue diamonds: for non-LCP approach and path-independence

solid line: LCP* defined by $T = 0$ observables (masses, $R_0$)
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<th>$N_t$</th>
<th>$\beta$</th>
<th>$m a$</th>
<th>$\sqrt{\sigma R_0}$</th>
<th>$m_{\rho} R_0$</th>
<th>$\sqrt{\sigma R_1}$</th>
<th>$m_\pi / m_\rho$</th>
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Different dimensionless quantities along
- $\text{LCP}_2$ (upper part),
- $\text{LCP}_2^*$ (middle part) and
- non-$\text{LCP}$ (lower part)

In the non-$\text{LCP}$ approach $m_\pi / m_\rho$ increases almost 60%
Equation of state at vanishing chemical potential

differences in $p, \epsilon - 3p$ values following the LCP or using the "non-LCP approach" (fixed m·a)

the value of the pressure is independent of the integration path

$p \approx 20 - 25\%$ below the Stefan-Boltzmann limit at $T \approx 3T_c$
Overlap improving multi-parameter reweighting

[Z. Fodor, S.D. Katz, 2001]

- generic system with $\psi$ fermions and $\phi$ bosons

fermionic Lagrangian: $\bar{\psi}M(\phi)\psi \Rightarrow$ after Grassmann integration

$$Z(\alpha) = \int \mathcal{D}\phi \exp[-S_{bos}(\alpha, \phi)] \det M(\phi, \alpha)$$

$\alpha$: parameter set (gauge coupling, mass, chemical potential)

$$Z(\alpha) = \int \mathcal{D}\phi \exp[-S_{bos}(\alpha_0, \phi)] \det M(\phi, \alpha_0) \left\{ \exp[-S_{bos}(\alpha, \phi) + S_{bos}(\alpha_0, \phi)] \det M(\phi, \alpha) / \det M(\phi, \alpha_0) \right\}$$

first line: measure; curly bracket: observable (will be measured)
simultaneously changing several parameters: better overlap
e.g. transition configurations are mapped to transition ones
• weight of a single configuration (any $\beta, m$) can be kept 1:

$$w = \exp(V \Delta \beta P + \log \det M(\mu \neq 0) - \log \det M(\mu = 0))$$

for an ensemble: minimizing the spread of these weights, $\log(w)$

similarly to the transition line: other best weight lines
$\beta$ decreases $\Rightarrow$ we need two LCPs to keep us on an LCP
(another option is three-parameter reweighting: $\beta, \mu, m$)
Reliability of reweighting

(see also [Ejiri, 2004] for other possibilities)

Idea:
From the original configurations $\Phi_i$ generate a new set $\Phi'_i$ with a Metropolis-like accept reject:

$$
\begin{align*}
\Phi'_1 &= \Phi_1 \\
\Phi'_i &= \Phi_i \text{ with probability } \min(1, w_i / w'_{i-1}) \\
\Phi'_i &= \Phi'_{i-1} \text{ otherwise}
\end{align*}
$$

$\Phi'$ has the desired distribution $\rightarrow$ errors can be computed

Problems:
- $\Phi'$ may not be thermalized
- what to do with complex weights (use $|w_i|$ or $|\text{Re } w_i|$)

extreme reweighting: $\Phi'$ only changes when $w_i > w'_{i-1}$.
after the largest weight all $\Phi'_i$ the same $\rightarrow$ zero information
$\rightarrow$ use always $\Phi'_i$-s after the largest weight only
• direct test: chiral condensate of $n_f=4$ dynamical QCD
  
  $m_qa=0.05$ staggered fermions, imaginary $\mu$ on $4 \cdot 6^3$ lattices

Glasgow-method: based on an ensemble of the high-$T$ phase
  
  high-$\mu$ phase does not overlap with the states of interest

error estimate: based on Metropolis technique using the weights
  
  naturally works for “continuously connected” samples
  
  for first order transitions the method needs all sectors (e.g. $Z_3$)
  
  realistic only for huge samples (no inherent definition)
• goodness of the $\mu$ reweighting

reweighting is accurate only for “very high statistics”
ensured by important but rare fluctuations $\implies$ overlap problem

Overlap measure:

$\alpha$: the fraction of independent $\Phi_i'$ configurations

approximate scaling of the half-width: $\mu_{1/2} \propto V^{-\gamma}$ with $\gamma \approx 1/3$
Equation of state at finite $\mu$

- analogous equations for $p$ and $\epsilon - 3p$

Pressure can be determined by the integral method

$$\frac{p}{T^4} = L_t^4 \int d(\beta, m \cdot a, \mu a) \left( \frac{\partial (\log Z)}{\partial \beta}, \frac{\partial (\log Z)}{\partial (m \cdot a)}, \frac{\partial (\log Z)}{\partial (\mu a)} \right) = L_t^4 \int d\beta \left[ <P> + m_u \frac{\partial a}{\partial \beta} <\bar{u}u> + m_s \frac{\partial a}{\partial \beta} <\bar{s}s> + \mu \frac{\partial a}{\partial \beta} \frac{\partial (\log \det M)}{\partial (\mu a)} \right]$$

The observables ($<P>$, $<\bar{\psi}\psi>$) are reweighted to finite $\mu$

The best weight lines are used $\Rightarrow \beta$ gets smaller

Two LCPs are needed to keep the system still on an LCP

(Another option is three-parameter reweighting: $\beta, \mu, m$)
integral method for the pressure

integral is performed in two steps (path independence)

a. integral over $\beta = \beta_0...\beta_T$ at $\mu = 0$ (same as in the $\mu = 0$ case)
b. integral over $(\beta_T, \mu = 0)...(\beta, \mu \neq 0)$ along the weight line

mixing steps a,b one can successfully check path independence
• equation of state at finite chemical potential up to $\mu_q \approx T_c$

difference between the $\mu=0$ and $\mu \neq 0$ pressure values

chemical potential for $\mu_B \approx 100, 210, 330, 410, 530$ MeV

normalize with the leading order $\mu$ dependence: $\Delta p^{SB}$

almost universal temperature dependence for $\Delta p/\Delta p^{SB}$
Other results

Bielefeld-Swansea group, 2002:
- Taylor expansion in $\mu$.
- similar results

Gavai, Gupta, 2003:
- Taylor expansion on quenched configurations

D’Elia, Lombardo, 2003:
- Taylor expansion from imaginary $\mu$ in the high $T$ phase
- quantitative agreement in the first coefficient
Summary, conclusions

• lattice QCD at finite $\mu$ is an old, unsolved problem
  new method: overlap improving multi-parameter reweighting
  presumably good enough to locate the endpoint, give EOS

• overlap improving multi-parameter reweighting:
  standard importance sampling with reweighting in $\beta$ and $\mu$
  maps transition configurations to transition ones
  and hadronic/QGP configurations to hadronic/QGP ones

• reweighting works for volumes upto at least $4 \cdot 12^3$
  for temperatures well below and above $T_c$
  definitions of “best weight lines” and “overlap measure”
  the overlap’s half-width depends on $V$: $\mu_{1/2} \propto V^{-\gamma}$ with $\gamma \approx 1/3$

• equation of state: obtained on the line of constant physics
  at finite $T=0.8 \ldots 3\cdot T_c$ and $\mu_B=0\ldots500$ MeV