Freeze-Out Hyper-Surface Extraction
With Digital Image Processing Techniques

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The Knowledge of a Freeze-Out Hyper-Surface (FOHS) and its corresponding Field Quantities (e.g., from Relativistic Hydrodynamics) allows to calculate both, Single-Inclusive Momentum Spectra and Bose-Einstein Correlations.

**Single-Inclusive Momentum Spectra are given by (i.e., Particles of 1st Generation):**

\[
E \frac{dN_i}{dk_i} = \frac{g_i}{(2\pi)^3} \int \frac{k^\mu \, d\sigma_\mu}{\exp \left( \frac{k^\mu u_\mu - \mu_i}{T_f} \right) \pm 1}
\]

\[\mathbf{x}_\mu : \text{FOHS Space-Time Coordinates}\]

\[d\sigma_\mu : \text{FOHS 4-Normals}\]

\[k^\mu : \text{Particle 4-Momenta}\]

\[u_\mu : \text{4-Velocity of Fluid}\]

**Bose-Einstein Correlations (e.g., for identical Pion Pairs) are given by:**

\[
C_2(\vec{k}_1, \vec{k}_2) = 1 + \frac{\langle a^\dagger(\vec{k}_1)a(\vec{k}_2)\rangle \langle a^\dagger(\vec{k}_2)a(\vec{k}_1)\rangle}{\langle a^\dagger(\vec{k}_1)a(\vec{k}_1)\rangle \langle a^\dagger(\vec{k}_2)a(\vec{k}_2)\rangle}
\]

\[
\sqrt{E_1 E_2} \langle u^\dagger(\vec{k}_i)u(\vec{k}_j)\rangle = \frac{1}{(2\pi)^3} \int \frac{\frac{1}{2}(k_i^\mu + k_j^\mu) d\sigma_\mu(x_\mu)}{\exp \left[ \frac{1}{2}(k_i^\mu + k_j^\mu) u_\mu(x_\mu) - T_f(x_\mu) \right] - 1} \cdot \exp[ix_\mu(k_i^\mu - k_j^\mu)]
\]

\[\mu_i : \text{Chemical Potentials (e.g., for B,S)}\]

\[T_f : \text{Freeze-Out Temperature}\]

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Definition of FOHS 4-Normals


Freeze-Out Hyper-Surface Element 4-Normal: \( d\sigma^\mu = \sqrt{-g} \, dS^\mu, \quad \mu = 0, 1, 2, 3. \)

\( g = \text{det}(g^{ik}) \), with \( g^{ik} \), i.e., the “Metric Tensor”, and 
\( dS^i = -\frac{1}{6} \epsilon^{iklm} dS_{klm} \),

where \( \epsilon^{iklm} \) is the “Completely Antisymmetric Unit Tensor” of fourth rank.

Conversely, we also have \( dS_{klm} = \epsilon_{nklm} dS^n \).

The Projections of the Volume of a Parallelepiped are spanned by three 4-Vectors:

\( dx_1^\mu, dx_2^\mu, dx_3^\mu, \quad \mu = 0, 1, 2, 3. \)

They are given by the Determinants

\( dS^{ikl} = \begin{vmatrix} dx_1^i & dx_2^i & dx_3^i \\ dx_1^k & dx_2^k & dx_3^k \\ dx_1^l & dx_2^l & dx_3^l \end{vmatrix} \)

\( d\sigma_0 = \sqrt{-g} \, dS^{123} \)
\( d\sigma_1 = \sqrt{-g} \, dS^{032} \)
\( d\sigma_2 = \sqrt{-g} \, dS^{013} \)
\( d\sigma_3 = \sqrt{-g} \, dS^{021} \)

\( d\sigma_\mu \) points to the Exterior of an enclosed Space-Time Region.
1.) A 1+1 dimensional Freeze-Out Hyper-Surface:

\[ R_1^\mu = (t_1, z_1), \quad R_2^\mu = (t_2, z_2) \]
\[ dx_1^\mu = R_2^\mu - R_1^\mu \]
\[ dx_1^\nu = (t_2 - t_1, z_2 - z_1) \]
\[ d\sigma_\mu = (z_2 - z_1, t_1 - t_2) \]

2.) A 3+1 dimensional Freeze-Out Hyper-Surface with cylindrical Symmetry:

\[ R_1^\mu = (t_1, z_1, r_1, 0) \]
\[ R_2^\mu = (t_2, z_2, r_2, 0) \]
\[ R_3^\mu = (t_3, z_3, r_3, 0) \]
\[ R_4^\mu = (t_4, z_4, r_4, \Phi) \]
\[ dx_1^\mu = (t_2 - t_1, z_2 - z_1, r_2 - r_1, 0) = (dt_1, dz_1, dr_1, 0) \]
\[ dx_2^\mu = (t_3 - t_2, z_3 - z_2, r_3 - r_2, 0) = (dt_2, dz_2, dr_2, 0) \]
\[ dx_3^\mu = (0, 0, 0, d\Phi) \]
\[ d\sigma_\mu = r \, d\Phi (dz_1 \, dr_2 - dz_2 \, dr_1, \, dr_1 \, dt_2 - dr_2 \, dt_1, \, dt_1 \, dz_2 - dt_2 \, dz_1, \, 0) \]
The Enclosure of Space-Time Regions is numerically approximated best by:

- **Line-Elements in 2 D** → **Contours**
- **Triangles in 3 D** → **Surfaces**
- **Tetrahedrons in 4 D** → **Hyper-Surfaces**

Such Choices may lead to Size Correction Factors for \( d\sigma_\mu \).
In the Field of Digital Image Processing, Algorithms for Contour and/or Surface Extraction are widely used:

- **2 D Shape Characterization through Skeletonization**
- **3 D Shape Rendering in Medical CT Scan Data**

The Applications for Contour and Surface Extraction are numerous.
Grid Points of Hydrodynamic Lattices may be viewed as Centers of Pixels in 2D, or Centers of Voxels in 3D.

Each Center is characterized by Field Quantities such as Temperature, Energy Density, Baryon Density, Fluid Velocity, … etc.

Pixel = Picture Element (2D)

Voxel = VOlume piXEL (3D)
Many Papers have been written on Contour Extraction in Binary Images. Here is a selection:


In the Following, we shall discuss the Approach of Freeman [1 – 3] in more Detail.
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The Chain Code Algorithm


1. A Code Table is used for Boundary Coding.
2. “Active” Pixels are traced along the Shape Boundary and result in a Chain Code:

0710700011222344434444556666

Problem: The Contours of the Chain Code Algorithm may be self-intersecting and degenerated, i.e., NOT all enclosed Areas are > 0!
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DICONEX – Dilated CONtour EXtraction, and Further Choices

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1: Initial Distribution of “Active” Pixels.
2 / 3: Left / Right-Turning Vectors.
4 / 5: Dilated Contours.
6 / 7: Boundary Pixel Tracing Contours.
8: as in 6 / 7 with removed Singularities.

Fast 3-Step Algorithm: “Vectors – Connect – Shift”

Los Alamos Preprint LA-UR-02-3813.
1. Only four Pixel Neighbors have to be considered during DICONEX Contour Construction.

2. DICONEX Contours are never self-intersecting or degenerated!

Smoothing of DICONEX Contours:

Initially given Gray-Level Images provide additional Spectral Information, which allow the Dislocation of the Points, which support the DICONEX Contours:
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Example: Extraction of 1+1 D FOHS  (Part 1)

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Temperature Field
(Color Coded Gray-Level Image)

Objective: Provide FOHS for
\[ T_{\text{Pixel}} = T_{\text{Fluid}} \geq T_f \approx 139 \text{ MeV}. \]

Pixel Border Tracing Contours
with Edge Normal Vectors
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Example: Extraction of 1+1 D FOHS  (Part 2)

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Dilated Contours (DICONEX) with Point Normal Vectors

Dilated Contours (DICONEX) with Shifted Point Normal Vectors
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Example: Extraction of 1+1 D FOHS (Part 3)

Dilated Contours (DICONEX) Shifted to Origins of Shifted Point Normal Vectors

Boundary Pixel Tracing Cont.s

Dilated Contours (DICONEX) Shifted with Respect to Temperatures along Shifted Point Normal Vectors
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Example: Extraction of 1+1 D FOHS  (Part 4)

Shuffled DICONEX Contours with Edge Normal Vectors

Final 1+1 D Freeze-Out Hyper-Surface after Removal of unphysical Edges
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Contour Point Set Down-Sampling

Steps (a) – (c): Removal Of Points On Straight Line Between Endpoints.
Steps (d) – (k): Removal Of Points That Fit Into A Minimum Enclosing Rectangle Of Width $w_0$, And Which Don’t Support A Turn $> \pi/2$.

Time-Sequence of 2-Dim. Temperature Fields shown as Color-Encoded Gray-Level Images. ($t_1 < \ldots < t_{11}$)

The Gray-Level Images are superimposed with Blob-Pixels, where
\( T_{\text{Pixel}} = T_{\text{Fluid}} \geq T_f \approx 139 \text{ MeV.} \)

Note the Break-Up of ONE Blob into TWO Blobs between Images \( t_7 \) and \( t_8 \).
The Color-Encoded Gray-Level Images are superimposed with DICONEX Contours.

Note the Break-Up of ONE Contour into TWO Contours between Images $t_7$ and $t_8$. 
While Building Temporal Correlations between the Contours, one may encounter Correspondence Problems.

Discontinue Approach. STOP!

Consider a Temporal Stack of Blob-Pixels (→ Voxels). Use only Voxel Faces, which separate the Shapes Interior from its exterior Regions.

Rather Continue Here. GO!
Here, t-Contours generate Correspondence Problems.

However, for the given Example, r-Contours and z-Contours have no Correspondence Problems.

In general, one has to expect Correspondence Problems!

Used by HYLANDER in the past.
For a given Group of Voxels, one may perform Scans in each principal Direction. Dilated Contour Extraction yields all Vectors, which form local Surface-Cycles after their Superposition. The Surface-Cycles provide a complete Surface Coverage.

Similar Conclusions were made by the Authors of the 3D Marching Lines Algorithm, which was designed to extract Characteristic Curves from 3D Images.

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The Marching Cubes Algorithm

1. 2x2x2 Voxel Cubes are inspected for their Configurations of “active” Voxels (Volume Scan or Random Access). There are $256 = 2^8$ different Configurations of “active” Voxels in a 2x2x2 Cube.

2. However, only 15 principal Configurations remain. Surface Templates are provided by a Look-Up Table.


http://exaflop.org/docs/marchcubes
Problems with the Marching Cubes Algorithm:

1. The exact Location of the Surface is not always known from the Configuration of “active” Voxels.

2. As a Result, the Marching Cubes Algorithm is not guaranteed to produce a closed Surface:

Fixes for Marching Cubes Algorithm through:

- **Static Analysis** (Uniform Orientation, Face Adjacency, Simplex Decomposition)
- **Interpolation Analysis** (Closest Orientation, Interpolation, Subdivision, … etc.)


A Surface Construction Algorithm that requires Fixes is not really desirable!
Fast 3-Step Algorithm: “Faces – Connect – Substitute”

1: Initial Voxel Faces and Voxel Face Centers.
2: Vector Cycle, connecting Voxel Faces.
3: Reduced Vector Cycle, connecting Voxel Face Centers. (Only N-Cycles with N = 3, 4, 5, 6, 7 are possible.)
4: Rendered VESTA Surface for Single Voxel.

Voxels, which are in Contact with another Voxel only through one single Edge may be disconnected or connected.
Some N-Cycles are planar, some are non-planar!
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VESTA - Surface Cycle Break-Up

Such Break-Up allows for Point Dislocations!
While using the actual Temperature Values within a Voxel and one of its Neighbors, one may dislocate the Face Centers of the VESTA Surface with respect to $T_f$ (or $\varepsilon_f$, or $n_f$, … etc.). This usually leads to a Surface Smoothing.

Surface Tiles with all of their Components $t \leq 0$, or $r \leq 0$, have no meaning for 2+1 D Hydrodynamics. They are therefore removed.
are the Normal Vectors for each FOHS Triangle; their Lengths are equal to their corresponding Triangle Area.

are the Centers of Mass for each FOHS Triangle; all related Field Quantities are evaluated at these Points.
Here, all other Field Quantities, such as Energy Density $\varepsilon$, Speed of Sound $c_0^2$, etc., are constant, and therefore not shown.
Rotation of cylinder symmetric 2+1 D Image Data at fixed Times results in 3+1 D Image Data. This enables the Extraction of 3D FOHS Projections for fixed Times.

For all z-Positions:

1. Rotate r-Entries
2. Linear Interpolation
3. FOHS-Projection
4. xyz-Data

Such Rotations will aid the Validation Process for fully 3+1 D FOHS Extraction Algorithms and Computer Codes.
VESTA Rendering of FOHS in 2+1 D Hydrodynamics at fixed Times ($t_1 < \ldots < t_8$).
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Movie 1: Time-Sequence of FOHS Projections
In 3+1 D Hydrodynamical Calculations, VESTA is useful for the Graphical Rendering of Projections of FOHS. A Construction of a 4D FOHS requires a Generalization of VESTA into 4D.

VESTA Rendering of FOHS in 3+1 D Hydrodynamics at fixed Times ($t_1 < \ldots < t_{14}$).

3+1 D Hydrodynamic Density Data, courtesy D. Strottman, Theoretical Division, Los Alamos National Laboratory.
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Movie 2: Time-Sequence of FOHS Projections

3+1 D Hydrodynamic Density Data, courtesy D. Strottman, Theoretical Division, Los Alamos National Laboratory.
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4D Analogies with 2D and 3D (Part 1)

The Generalization
DICONEX → VESTA → “STEVE”
is quite straightforward.

(STEVE = Space-Time Enclosing Volume Extraction)
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4D Analogies with 2D and 3D (Part 2)

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2D: 4 Edges
3D: 6 Faces
4D: 8 Volumes
nD: 2^n Hyper-Faces

2D: 4 Line Segments
3D: 8 Triangles
4D: 16 Tetrahedrons
nD: 2^n Hyper-Triangles

Furthermore:

2D: 2 Pixels, which touch in only one **Point** may be connected or disconnected.
3D: 2 Voxels, which touch in only one **Edge** may be connected or disconnected.
4D: 2 Hyper-Voxels, which touch in only one **Face** may be connected or disconnected.
nD: 2 Hyper-Voxels, which touch in only one **Hyper-Edge** may be connected or disconnected.
Numerical Calculation of Freeze-Out Hyper-Surfaces:

1. The Field of Image Processing provides useful Tools for the numerical 1+1 D and 2+1 D FOHS Construction.
2. Both, the DICONEX and VESTA Algorithms
   - provide *high-resolution* Iso-Hyper-Surfaces.
   - have a parallel and totally linear Component, and are therefore *fast*.
   - always yield *perfect* Results (i.e., Contours or Surfaces).
3. A 4D Algorithm (*STEVE*) can be derived from DICONEX and VESTA, while using Analogies.

Ongoing Work:

- Definition of further Specifications for STEVE.
- Validation with rotated 2+1 D Hydrodynamic Data.
- Implementation into 3+1D Hydrodynamics Code.