Small-$x$ evolution in the high-density QCD

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Plan:

• Introduction: saturation in DIS

• High-energy QCD and Wilson lines

• Non-linear evolution at small $x$ in the saturation region

• Effective field theory for the small-$x$ evolution

• Nuclear structure functions: non-linear evolution $\Rightarrow$ saturation

• Conclusions.

• Outlook: effective field theory for the heavy-ion collisions.

• Outlook: Evolution and unitarization.
"Phase diagram" for the deep inelastic scattering

\[ \alpha_s \sim 1 \]
\[ \Lambda_{QCD}^2 \]

saturation region
(can be understood by small coupling methods)

non-perturbative region
(not much is known
coupling is large)

BFKL
DGLAP

Small-\(x\) evolution in the high-density QCD

3 years of heavy ions at RHIC

6 May 2003
3 years of heavy ions at RHIC

**BFKL evolution**

fast (\(p \gg q_z\)) hadron

\[ t \ (time) \]

Emission of partons \(\sim \rho \) (density)
Annihilation of partons \(\sim \frac{\alpha_s}{Q^2} \rho^2\)

(the amplitude of the annihilation of two partons in the cascade is \(\frac{\alpha_s}{Q^2}\))

\[ \Rightarrow \]

The equilibrium between emission and annihilation (saturation) should be described by simple non-linear equation

\[ \frac{d\rho}{d\ln(1/x)} = \frac{N_c\alpha_s}{\pi} (K_{BFKL} \otimes \rho - \text{const} \times \frac{\alpha_s}{Q^2} \times \rho^2) \]

**Nonlinear evolution**

\[ Q^2 = Q_s^2 \]

\[ \frac{1}{q_z} \]

\[ \text{photon} \]

Small-\(x\) evolution in the high-density QCD

-3- 6 May 2003
Small-$x$ DIS from the nucleus

\[ \gamma^* \]

Fast quark moves along the straight line $\Rightarrow$

The quark propagator reduces to the Wilson line collinear to quark’s velocity

\[ U(x_\perp, \eta) \equiv [\infty n_\eta + x_\perp, -\infty n_\eta + x_\perp] \]

\[ [x, y] \equiv \text{Pexp} \left\{ ig \int_0^1 dv (x - y)^\mu A_\mu(vx + (1 - v)y) \right\} \]

Small-$x$ evolution in the high-density QCD
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High-energy QCD and Wilson lines

Particles from clusters with different rapidity perceive each other as Wilson lines:

\[ U(x_\perp, n_A) \]

“Wilson line” - infinite gauge link

\[ U(x_\perp, n_B) \]

\[ \Rightarrow U(x_\perp, \eta) \ (\eta \equiv \text{slope}) \text{ is the relevant degree of freedom for high-energy scattering.} \]
High-energy scattering as a scattering of color dipoles (with A. Babansky)

In the first (nontrivial) order in pert. theory

= an analytic function of $\theta$ (a typical term is $\ln[1 + \frac{x_1^2}{b_1^2} \sin^2 \theta] \to \ln[1 + \frac{x_1^2}{b_1^2} \sinh^2 \eta]$)

$\theta =$ angle in the Euclidean space,

$i\theta = \eta =$ rapidity in Minkowski space

IR divergences cancel, $\text{UV} = \alpha_s(z_{\perp})$
At large $s$ it gives the first iteration of the BFKL kernel \( \bar{\alpha} \equiv \frac{\alpha_s N_c}{\pi} \)

\[
A(x, y, b; \eta) = \int d^2k d^2r \frac{e^{ik \cdot x + ir \cdot b}}{k^2(r - k)^2} \left( 1 + \frac{\bar{\alpha}}{4\pi} \eta \hat{K}_{\text{BFKL}} \right) e^{-iky}
\]

In the LLA \( \alpha_s \ll 1, \alpha_s \eta \sim 1 \)

\[
\int d^2k d^2r \frac{e^{ik \cdot x + ir \cdot b}}{k^2(r - k)^2} \sum \left( \frac{\bar{\alpha}}{4\pi} \eta \hat{K}_{\text{BFKL}} \right)^n e^{-iky}
\]

It looks like the operator evolution equation

but the operator equation for the small-$x$ evolution of the dipole is \textbf{non-linear}
Numerical estimates: For the scattering of equal-size dipoles BFKL asymptotics starts rather late, at $\eta \sim 5$:

$$\sigma(a, \eta) = 16\pi \alpha_s^2(a)a^2 \coth^2 \eta \left[1 + 6\alpha_s(a)\Phi(\eta)\right]$$

$$\sigma_{asy}(a; \eta) = 16\pi \alpha_s^2(a)a^2 \left[1 + 6\alpha_s(a)\Phi_{LLA+IF}(\eta)\right]$$

For the dipoles of different sizes (“DIS” from the dipole) - work in progress
At high energies, the amplitude of $\gamma^* A \rightarrow \gamma^* A$ scattering reduces to the matrix element of a two-Wilson-line operator (“color dipole”).

\[
A(s) = \int \frac{d^2 k_\perp}{4\pi^2} I(k_\perp) \\
\times \langle A | \text{Tr}\{U_{A}^{\eta A}(k_\perp)U_{\eta A}^\dagger(-k_\perp)\}|A\rangle + \ldots
\]

Energy dependence of the amplitude $A(s)$ is determined by the dependence of the Wilson lines on the rapidity $\eta_A$ defined by the slope of the line.
Technically, it is more convenient to use the spectator frame

High-speed nucleus shrinks to a “pancake” \( \Rightarrow \)

\[
\int d^4x d^4z \ e^{-ip_A \cdot x} \langle T\{j_A(x + z)j'_A(z)\}\rangle_A
\]

\[
= \int \frac{d^2k_\perp}{4\pi^2} I^A(k_\perp) \text{Tr}\{U^{\eta A}(k_\perp)U^{\eta A\dagger}(-k_\perp)\} + \ldots \Rightarrow
\]

\[A(s) = \int \frac{d^2k_\perp}{4\pi^2} I(k_\perp)
\times \langle A|\text{Tr}\{U^{\eta A}(k_\perp)U^{\eta A\dagger}(-k_\perp)\}|A\rangle + \ldots
\]
To get the evolution equation, consider the dipole with the slope $\parallel \eta_1$ and integrate over the gluons with rapidities $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with the slope corresponding to $\eta_2$).

In the frame $\parallel \eta_1$ the gluons with $\eta < \eta_2$ are seen as a pancake $\Rightarrow$
One-loop evolution

The structure is
(perperturbative propagator) x (instantaneous interaction with the shock wave described by $U^{ab}(z_\perp)$) x (perturbative propagator)

$$U^{ab}(z) = \text{Tr}\{t^aU_zt^bU_\dagger_z\} \Rightarrow$$

$$(U_xU_\dagger_y)^{\eta_1} = (U_xU_\dagger_y)^{\eta_1} + \alpha_s(\eta_1 - \eta_2)(U_xU_\dagger_zU_zU_\dagger_y)^{\eta_2}$$

$\Rightarrow$ non-linear evolution

Small-$x$ evolution in the high-density QCD
Non-linear evolution equation

\[
\frac{\partial}{\partial \eta} U(x_\perp, y_\perp) =
\]
\[- \frac{\bar{\alpha}}{4\pi} \int dz_\perp \left\{ U(x_\perp, z_\perp) + U(z_\perp, y_\perp) - U(x_\perp, y_\perp) \right. \\
+ \left. U(x, z)U(z, y) \right\} \frac{(\vec{x} - \vec{y})_\perp^2}{(\vec{x}_\perp - \vec{z}_\perp)^2(\vec{z}_\perp - \vec{y}_\perp)^2} \]

\[
U(x_\perp, y_\perp) \equiv \frac{1}{N_c} \left( \text{Tr}\{U(x_\perp)U^\dagger(y_\perp)\} - N_c \right)
\]

LLA for DIS in pQCD \(\Rightarrow\) BFKL
LLA for DIS in sQCD \(\Rightarrow\) NL eqn
(s for semiclassical)

Example - LLA for the structure functions of large nuclei: \(\alpha_s \ln \frac{1}{x} \sim 1, \ \alpha_s^2 A^{1/3} \sim 1\)
Non-linear equation sums up the “fan” diagrams

Example of the diagrams left behind by the NL eqn: pomeron loops
Timeline:

- Gribov, Levin, Ryskin (1983) - GLR eqn suggested
- Mueller, Qui (1986) - DLA limit of GLR eqn proved
- Balitsky (1996) - the above eqn derived
- Kovchegov (1999) - the above eqn red-erived (in the dipole model) and used for the large nuclei
- Braun (M.A.) (2000) - NL = GLR + 3-pomeron vertex from Bartels et. al.
- McLerran & Iancu, Weigert (2000) - obtained from the RG eqn for Color Glass Condensate
Small-\(x\) evolution in the NLO

(in collaboration with A. Belitsky)

\[
\begin{align*}
\frac{\partial}{\partial \eta} U(x_\perp, y_\perp) &= \frac{\bar{\alpha}}{4\pi} \left( K_{\text{BFKL}} + \alpha_s K_{\text{BFKL}}^{\text{NLO}} \right) \\
&+ \frac{\bar{\alpha}}{4\pi} \int d^2 z \left[ \frac{-(x-y)^2}{(x-z)^2(z-y)^2} + \bar{\alpha} K_2^{\text{NLO}} \right] U(x, z) U(z, y) \\
&+ \frac{\bar{\alpha}}{4\pi} \int d^2 z d^2 z' K_3(x, y; z, z') U(x, z) U(z, z') U(z', y)
\end{align*}
\]

\(K_3\) - calculated

Typical contribution:

\[
K_3(x, y; z, z') = \frac{(X, Y) + (X', Y')}{(z - z')^2(X'^2 Y'^2 - X^2 Y^2)} \ln \frac{X^2 Y'^2}{X'^2 Y^2}
\]

where \(X \equiv x - z, X' \equiv x - z'\)

\(K_{\text{BFKL}}^{\text{NLO}}\) and \(K_2^{\text{NLO}}\) - work in progress
Formal solution of the NL eqn: the effective field theory

\[ U^\eta_A(x_\perp) \otimes U^{\dagger\eta_A}(y_\perp) \]

\[ = \int D\pi_1(z, \eta) D\pi_2(z, \eta) D\Omega_1(z, \eta) D\Omega_2(z, \eta) \times (\Omega_1^{\dagger\eta_A} U^{\eta_0} \Omega_2^{\eta_A})_x \otimes (\Omega_2^{\dagger\eta_A} U^{\eta_0} \Omega_1^{\eta_A})_y \times \exp \left\{ \int_{\eta_0}^{\eta_A} d\eta \int d^2z \left[ \frac{1}{g} \partial^2 \pi_1^a (\Omega_1^{\dagger} \frac{\partial}{\partial\eta} \Omega_1)^a \right. \right. \]

\[ + \left. \left. \frac{1}{g} \partial^2 \pi_2^a (\Omega_2^{\dagger} \frac{\partial}{\partial\eta} \Omega_2)^a - \frac{1}{4\pi} \pi_1^a \partial^2 (\Omega_1^{\dagger} U^{\eta_0} \Omega_2)^{ab} \pi_2^b \right] \right\} \]

\[ \Omega_1(z, \eta), \Omega_2(z, \eta) \in SU(3) - \text{dynamical Wilson-line variables.} \]

\[ \pi_i(z, \eta) - \text{canonical momenta} \]

The action is local.

Perturbation theory

\[ \Omega_1(z, \eta) = e^{-ig\phi_1(z, \eta)}, \quad \Omega_2(z, \eta) = e^{-ig\phi_2(z, \eta)} \]

Propagators:

\[ \phi_i^a(x_\perp, \eta) \pi_j^b(y_\perp, \eta') = -i\delta_{ij} \delta^{ab} \theta(\eta - \eta') \langle x_\perp | \frac{1}{\partial^2_\perp} | y_\perp \rangle, \]

\[ \phi_i^a(x_\perp, \eta) \phi_j^b(y_\perp, \eta') = 0, \quad \pi_i^a(x_\perp, \eta) \pi_j^b(y_\perp, \eta') = 0 \]

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Small-$x$ evolution: QCD diagrams versus effective field theory diagrams.

Small-$x$ evolution in the high-density QCD

-18- 6 May 2003
Initial conditions for the small-$x$ evolution

BFKL evolution in terms of Wilson lines:

\[
\langle N| \text{Tr}\{U(z_\perp, \eta_0)U^\dagger(0, \eta_0)\}|N\rangle \quad \text{- color dipole with the energy } s_0 = s\omega_0
\]

Initial point of the evolution $s_0$:

- small from the high-energy viewpoint:
  \[
  \alpha_s \ln \frac{s_0}{m^2} \ll 1
  \]

- large from the viewpoint of the low-energy physics:
  \[
  \frac{s_0}{m^2} \gg 1
  \]

For small dipoles (size $\ll 1$fm)

\[
\langle N| \text{Tr}\{U(z_\perp, \eta_0)U^\dagger(0, \eta_0)\}|N\rangle \approx z_\perp^2 \omega_0 G(\omega_0, \mu^2 = \frac{1}{z_\perp^2})
\]
Nuclear structure functions

In the case of large nuclei the initial conditions can be taken from the Glauber formula.

\[
N_\eta(x_\perp, b_\perp) \equiv \frac{1}{N_c} \langle A | \text{Tr} U(x_\perp + z) U^\dagger(z) | A \rangle
\]

Saturation scale: \( Q_s^2 = 1/x_{s\perp}^2 \) such that

\[
S_\eta(x_{s\perp}, b_\perp) = 1 - N(x_{s\perp}, b_\perp) \sim 1
\]
Theoretical estimates (Y. Kovchegov, A.H. Mueller, L. McLerran & E. Iancu) give

\[ Q_s(\eta) \sim Q_0 e^{\lambda x_B} \]

\( x_B \) decreases \( \Rightarrow \) \( \alpha_s(Q_s) \rightarrow 0 \).

Numerical results (E. Levin, from THERA book)

Both numerical and theoretical estimates show that even if we start from small target fields (e.g. \( \gamma^* \)) with \( S_\eta(x_\perp) \ll 1 \)

\[ x_B \rightarrow 0 \xrightarrow{\text{BFKL}} \quad S_\eta(x_\perp) \text{ increases} \xrightarrow{\text{NL}} \text{saturation} \]
Conclusions:

- The small-\(x\) evolution in the high-density QCD is described by a non-linear equation.
- Numerical and theoretical estimates show that this non-linear evolution leads to the saturation of parton density at high energies.
3 years of heavy ions at RHIC

**Outlook: A search for the 2+1 effective theory for the heavy-ion collisions**

![Diagram showing heavy ion collisions with shock waves]

**Wanted:**

\[
\langle e^{i\rho^A_i U_i(\eta_A)} e^{i\rho^B_i U_i(\eta_B)} \rangle_{\text{QCD}} \xrightarrow{s \to \infty} \\
\int \mathcal{D}U(z_\perp, \eta) \ e^{i\rho^A_i U_i(\eta_A)} e^{i\rho^B_i U_i(\eta_B)} \exp \left\{ i \int_{\eta_B}^{\eta_A} d\eta \int d^2 z_\perp \mathcal{L}(U) \right\}
\]

\[U_i \equiv U_i^\dagger \frac{i}{g} \partial_i U \quad \text{and} \quad e^{i\rho^A_i U_i(\eta_A)} \equiv e^{\int d^2 z \rho^A_i(z_\perp) U_i(z_\perp, \eta_A)}
\]

\[\rho^A_i(z_\perp) \quad \text{and} \quad \rho^B_i(z_\perp) \quad \text{- sources for the Wilson-line operators (color densities in the ions)}.
\]

**How to approach this goal?**

- pQCD
- sQCD (s for semiclassical)

Small-\(x\) evolution in the high-density QCD

6 May 2003
“Power counting” for sources in the LLA.

In pQCD, the parameters of the expansion are $g^2$, $g^2\eta$, and $g^2\rho^A\rho^B\eta$ ($\eta \equiv \ln s$).

LLA: $g^2 \ll 1$ $g^2\eta \sim 1$ (NLO is $\sim g^4\eta \ll 1$).

3 regimes:

- $\rho \sim 1$ ($\gamma^*\gamma^*$ scattering) $\xrightarrow{\text{LLA}}$ BFKL pomeron

- $\rho \gg 1$ ($\rho \sim A^{1/3}$ for the heavy-ion collisions) $g^2\rho^A \sim 1$, $g^2\rho^B \sim 1$ $\Rightarrow$ $g^2\rho^A\rho^B\eta \gg 1$ $\Rightarrow$ $g^4\rho^A\rho^B\eta \ll 1$ $\Rightarrow$ LLA is not enough.

  Best hope for this region is sQCD.

- $\rho_A \sim 1$, $\rho_B \gg 1$ (DIS from the heavy nuclei) $\Rightarrow$ $g^2\rho^A\rho^B\eta \gg 1$ but $g^4\rho^A\rho^B\eta \ll 1$ $\Rightarrow$ NLO is small $\Rightarrow$ LLA effective action for the small-$x$ DIS solves the problem:

$$\langle e^{i\rho^A_i U_i(\eta_A)} e^{i\rho^B_i U_i(\eta_B)} \rangle_{\text{LLA}}$$

$$= \int D\Omega_1(z, \eta) D\Omega_2(z, \eta) e^{i\rho^A_i (\Omega_2^\dagger \rho^B \Omega_1)_i}$$

$$\times \exp \left\{ \frac{1}{g^2} \int_{\eta_B}^{\eta_A} d\eta \int d^2z \, \Omega_1^\dagger \dot{\Omega}_1 \partial^2 \left[ \partial^2 (\Omega_2^\dagger \rho_B \Omega_1) \right]^{-1} \partial^2 \Omega_2^\dagger \dot{\Omega}_2 \right\}$$
\( \rho_A, \rho_B \gg 1 \): effective action = ?

For the DIS from a nucleus A

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\[ A \gg 1 \]
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“upside down” evolution

When a nucleus is a spectator and the \( \gamma^* \) is a target

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\[ B \gg 1 \]
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“bottom up” evolution

For the nucleus-nucleus scattering

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A \gg 1 \times B \gg 1 = A \gg 1, B \gg 1 + \text{pomeron loops}
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Small-\( x \) evolution in the high-density QCD -25- 6 May 2003
3 years of heavy ions at RHIC

Evolution and unitarization: a model for the dipole-dipole scattering beyond the LLA

\[ A(s) = \sum g^{2n} \left( \ln \frac{s}{m^2} + \text{const} \pm i\pi \right)^n \]
\[ \Rightarrow A(s_0) = \sum g^{2n} \left( \ln \frac{s_0}{m^2} + \text{const} \pm i\pi \right)^n \sim \sum (\pm i\pi g^2)^n \]

\[ \langle U_{x_1}^{\eta_0} \ldots U_{x_n}^{\eta_0} U_{y_1}^{\eta_B} U_{y_2}^{\eta_B} \rangle = \int D\Omega_1 D\Omega_2 D\Omega_1 D\Omega_2 \int d^2z V_i W_i \]
\[ \Rightarrow \langle U_{x_1}^{\eta_A} U_{x_2}^{\eta_A} U_{y_1}^{\eta_B} U_{y_2}^{\eta_B} \rangle = \int D\Omega_1 D\Omega_2 D\Omega_1 D\Omega_2 \int d^2z V_i W_i e^{iS_{\text{eff}}(\Omega_i, V)} \]
A symmetric version of this model gives the functional integral representation of Mueller’s dipole picture of unitarization

\[ \langle U^{\eta_A} U^\dagger y_1 U^{\eta_B} U^\dagger y_2 \rangle = \int DV DW e^{iV_i W_i} \int D\Omega_i D\Lambda_i (\Omega_1 V_2)_x (\Omega_2 V_1^\dagger \Omega_1)_y \]

\[ (\Lambda_1^\dagger W \Lambda_2)_y (\Lambda_2^\dagger W^\dagger \Lambda_1)_y e^{iS_{\text{eff}}^\eta_A,\eta_B} (\Omega_i, V) e^{iS_{\text{eff}}^\eta_A,\eta_B} (\Lambda_i, V) \]