How Can We Extract Spectral Functions from Lattice Data?

Masayuki Asakawa
Kyoto University (MELQCD Collab.)
PLAN

- Spectral Function
- Necessity of MEM (*Maximum Entropy Method*)
  - Outline (Review)
- Zero and Finite Temperature Result
**Spectral Function**

- **Definition of Spectral Function**

\[
A_{\eta\eta}(k_0, \vec{k}) = \frac{1}{(2\pi)^3} \sum_{n,m} \frac{e^{-E_n/T}}{Z} \langle n | J_{\eta}(0) | m \rangle \langle m | J_{\eta'}(0) | n \rangle (1 \mp e^{-p_{mn}^0/T}) \delta^4(k^\mu - P_m^\mu) \\
- (+): Boson (Fermion)
\]

- \( J_{\eta}(0) \): A Heisenberg Operator with some quantum #
- \( |n\rangle \): Eigenstate with 4-momentum \( P_n^\mu \)
- \( P_{mn}^\mu = P_m^\mu - P_n^\mu \)

Pretty important function to understand QCD

Dilepton production rate, etc.

\[
\frac{dN (e^+e^- \text{ production at } T)}{d^4x d^4q} = - \frac{\alpha^2}{3\pi^2 k^2} \frac{\Delta_{\alpha}^\mu (k_0, \vec{k})}{e^{k_0/T} - 1}
\]

holds regardless of states, either in Hadron phase or QGP

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Hadron Modification in HI Collisions?

Experimental Data

Comparison with Theory

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Why Theoretically Unsettled

Mass Shift (Partial Chiral Symmetry Restoration)

Spectrum Broadening (Collisional Broadening)

Observed Dileptons

Sum of All Contributions (Hot and Cooler Phases)
**Difficulty on Lattice**

What’s measured on Lattice is Correlation Function $D(\tau)$

\[
D(\tau) = \int \left\langle O(\tau, \vec{x}) O(0, \vec{0}) \right\rangle d^3x
\]

$D(\tau)$ and $A(\omega) \equiv A(\omega, \bar{0})$ are related by

\[
D(\tau) = \int_0^\infty K(\tau, \omega) A(\omega) d\omega
\]

However,

- Measured in Imaginary Time
- Measured at a **Finite Number** of discrete points
- Noisy Data \(\rightarrow\) Monte Carlo Method

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Thus, Inversion Problem

\[ D^A(\tau) = \int_0^\infty K(\tau, \omega)A(\omega) d\omega \]

\[ D^A(\tau) \Rightarrow A(\omega) \]

Typical ill-posed problem
Problem since Lattice QCD was born
Way out?

\( \chi^2 \)-fitting

- need to assume the form of \( A(\omega) \)
  (1-pole, 2-poles, 1-pole + continuum…etc.)
- many degrees of freedom \( \rightarrow \) many solutions
- resonance mass depends on \( T_{\text{min}} \)
Way out? (cont’d)

Example of $\chi^2$-fitting failure

QCDPAX, 1995

$\beta = 6.0$

$24^3 \times 54$ lattice
MEM

Maximum Entropy Method

successful in crystallography, astrophysics, …etc.

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**Principle of MEM**

- **MEM**
  a method to infer the most statistically probable image \( A(\omega) \) given data

  In MEM, Statistical Error can be given to the Obtained Image

- **Theoretical Basis:** Bayes’ Theorem

  \[
  P[X \mid Y] = \frac{P[Y \mid X]P[X]}{P[Y]}
  \]

  \( P[X \mid Y] \): Probability of \( X \) given \( Y \)

  *A Theorem that reverses the roles of Cause and Result*
Application of MEM to Lattice QCD

In Lattice QCD

D: Lattice Data (Average, Variance, Correlation...etc.)

H: All definitions and prior knowledge such as \( A(\omega) \geq 0 \)

Bayes Theorem

\[ P[A | DH] \propto P[D | AH] P[A | H] \]

Most Probable Spectral Function \( A(\omega) \)

\( A(\omega) \) that Maximizes Posterior Prob. \( P[A | DH] \)

In MEM, basically this Most Probable Spectral Function is calculated
Ingredients of MEM

- \( P[D | AH] = \chi^2\)-likelihood function
  \[ P[D | AH] = \exp(-L)/Z_L \]

- \( P[A | H] \) given by Shannon-Jaynes Entropy
  \[ P[A | H \alpha m] = \frac{\exp(\alpha S)}{Z_s} \]
  \[ S = \int \left[ A(\omega) - m(\omega) - A(\omega) \log \left( \frac{A(\omega)}{m(\omega)} \right) \right] d\omega \]
  \[ Z_s = \int e^{\alpha S} dA, \quad \alpha \in \mathbb{R} \]
  max at \( A(\omega) = m(\omega) \)

Default Model \( m(\omega) \in \mathbb{R} \): Prior knowledge about \( A(\omega) \)

such as semi-positivity, perturbative asymptotic value, ...etc.

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What to Maximize in MEM

Therefore, we obtain

\[ P[A|DH] \propto P[D|AH]P[A|H_\alpha m] \propto \exp(\alpha S - L) \]

The Maximum of \( \alpha S - L \)

*Unique if it exists!*

Mock Data Analysis

1. Take a test input image \( A_{\text{in}}(\omega) \equiv \omega^2 \rho_{\text{in}}(\omega) \)

2. Transform \( A_{\text{in}}(\omega) \) with an appropriate Kernel \( K(\tau, \omega) \)

\[
D_{\text{in}}(\tau) = \int K(\tau, \omega) A_{\text{in}}(\omega) d\omega, \quad K(\tau, \omega) = e^{-\omega \tau} \quad \text{Dirichlet Kernel}
\]

3. Make a mock data \( D_{\text{mock}}(\tau_i) \) by adding noise to \( D_{\text{in}}(\tau_i) \)

\[
\sigma / D_{\text{in}}(\tau_i) = b \times \tau_i / a, \quad a = \text{Lattice spacing, } b = \text{const.} \\
C_{ij} = \text{diagonal (for simplicity)}
\]

4. Apply MEM to \( D_{\text{mock}}(\tau_i) \) and construct the output image

\( A_{\text{out}}(\omega) \equiv \omega^2 \rho_{\text{out}}(\omega) \)

5. Compare \( \rho_{\text{out}}(\omega) \) with \( \rho_{\text{in}}(\omega) \)
Result of Mock Data Analysis (1)

N(# of data points)-b(noise level) dependence

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Result of Mock Data Analysis (2)

N(# of data points)-b(noise level) dependence
Error Analysis in MEM

MEM is based on Bayesian Probability Theory

In MEM, *Errors can be and must be assigned*

This procedure is *essential* in MEM Analysis

Error Bars can be put to

Average of Spectral Function in $[\omega_1, \omega_2]$,\[ \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \rho(\omega) d\omega \]

Decay Constants, e.g.,\[ f^2 = 4 \left( \frac{\kappa Z}{m_{\rho}} \right)^2 \int_{\text{pole}} \omega \rho_\nu(\omega) d\omega \quad \cdots \text{etc.} \]
SPF in V Channel \((T=0)\)

![Spectral Function in the Vector Channel](image)

- **Ground State**: \(\rho\)
- **Excited State**
- **Continuum State**

- **Parameters**:
  - \(\beta = 6.0\)
  - 160 configurations
  - Wilson fermion
  - Quenched Approx.
  - \(\kappa_c = 0.1571\)
  - \(20^3 \times 24\)

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SPF in PS Channel \((T=0)\)
$T=0$ Result in V Channel and Error Analysis

- Perturbative Continuum Value
- Renormalization of Composite Operator on Lattice
Finite $T$ Calculation (1)

How many points are needed in $\tau$ direction?

$40^3 \times 30$ lattice

$40^3 \times 30$ lattice

$\beta = 6.47$

isotropic lattice

$N_\tau \approx 30$ or larger: needed
Finite $T$ Calculation (2)

- This data suggest more than ~30 points are needed in $\tau$ direction at the highest $T$.

- The highest $T$ : set to $\sim 2.5 T_c$

  In order to have large enough $L_\sigma$ and $N_\tau$, we employ anisotropic lattice

  \[ \xi = \frac{a_\sigma}{a_\tau} = 4 \]
Number of Configurations

As of April 25, 2003

\( N_\sigma = 32, \quad \beta = 7.0, \quad \xi = 4.0 \)

<table>
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<tr>
<th>(N_\tau)</th>
<th>32</th>
<th>40</th>
<th>46</th>
<th>54</th>
<th>72</th>
<th>80</th>
<th>96</th>
</tr>
</thead>
<tbody>
<tr>
<td>(T / T_c)</td>
<td>~2.3</td>
<td>~1.9</td>
<td>~1.6</td>
<td>~1.4</td>
<td>~1.04</td>
<td>~0.93</td>
<td>~0.78</td>
</tr>
<tr>
<td># of Config.</td>
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<td>181</td>
<td>182</td>
<td>150</td>
<td>150</td>
<td>110</td>
<td>194</td>
</tr>
</tbody>
</table>

Fairly Large Statistics in Lattice Standard
Polyakov Loop and PL Susceptibility

Polyakov Loop

- $= 0$ in Confining Phase
- $0$ in Deconfined Phase

Confining Phase  Deconfined Phase

Polyakov Loop Susceptibility

- $\sim T_c$
- $\sim 2T_c$
- $\propto T$

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Result at $T \sim 1.9T_c$ (PS Channel)

\[ m \sim \rho_{\text{pert}}(\omega) \]
\[ = \frac{3}{8\pi^2} \left( 1 + \frac{11\alpha_s(\mu)}{3\pi} \right) \left( \frac{1}{2\sqrt{\kappa_s \kappa_c Z_{\text{PS}}(\mu a)}} \right)^2 \]
\[ = \mathcal{O}(1) \]
**Hadronic Correlations above $T_c$?**

- $T \sim 1.4T_c$
  - Mass Gap?
  - Massive Free Quark Gas

- $T \sim 1.9T_c$
  - $m_\pi/m_\rho \sim 0.7$ (at $T = 0$)
  - Landau Damping?
  - But statistically NOT significant

- Similar Result on smaller lattice

![Graphs showing rho(\omega) vs. omega for different temperatures]

- $64^3 \times 16$
- $T \sim 1.5T_c$ and $3T_c$
- Chiral limit

Karsch et al. (01)
Is the width statistically significant?

PS Channel

Narrow Lowest Peak ↔ Broad Lowest Peak

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Summary and Perspective

- Hadronic Spectral Functions in QGP Phase were obtained on large lattices at several T.

- It seems there are nontrivial modes in QGP.
- Sudden Qualitative Change between $1.4T_c$ and $1.9T_c$?
- Physics behind still unknown.

Further study needed for better understanding of QGP!