The idea: to combine the best shell model methods available:

- m-scheme spherical shell-model
- SU(3) symmetry based shell-model

Developers …
- Vesselin Gueorguev
- Jerry Draayer
- Erich Ormand
- Calvin Johnson

\[ H\Psi = E\Psi \]

m-scheme states

\[ (H - E_g)\Psi = 0 \]

SU(3) states
Problems that we understand well

Exactly solvable (symmetry at play)

\[ H = H_0 \]

\[ H = H_0 + V \]

\[ H = H_1(\square) + H_2(\square) + \ldots \]

What about more than one exactly solvable part beyond the perturbative regime?

Transition from phase one to phase two should occur.

Maybe two or more different sets of basis states could be employed to understand such problems …
Two-Mode Toy System

Particle in 1D box + Harmonic Oscillator = Particle in a 1D box subject to harmonic oscillator potential.

$E_c = \frac{1}{2} \hbar^2 a^2$
The Challenge in Nuclei...

Nuclei display unique characteristics:

- Single-particle Features
- Pairing Correlations
- Deformation/Rotations

Closed Shell

Pairing

Rotations

Vibrations

Closed Shell
Nuclear Shell-Model Hamiltonian

\[ H = \sum_i \mathcal{H}_i a_i^+ a_i + \sum_{i,j,k,l} V_{ijkl} a_i^+ a_j^+ a_k a_l = \sum_i \mathcal{N}_i + \mathcal{Q} \cdot \mathcal{Q} + U_{\text{residual}} \]

where \( a_i^+ \) and \( a_i \) are fermion creation and annihilation operators, \( \mathcal{H}_i \) and \( V_{ijkl} \) are real and \( V_{ijkl} = V_{klij} = \mathcal{V}_{jikl} = \mathcal{V}_{ijkl} \)

- Spherical shell-model basis states are eigenstates of the one-body part of the Hamiltonian - single-particle states.

- The two-body part of the Hamiltonian \( H \) is dominated by the quadrupole-quadrupole interaction \( \mathcal{Q} \cdot \mathcal{Q} \sim C_2 \) of SU(3).

- SU(3) basis states - collective states - are eigenstates of \( H \) for degenerate single particle energies \( \mathcal{N}_i \) and a pure \( \mathcal{Q} \cdot \mathcal{Q} \) interaction.
SU(3) Basics

The SU(3) SO(3) Reduction

- SU(3) generators as SO(3) tensors:

\[
\begin{align*}
&\left[ L_m, L_{m'} \right] = \sqrt{2} (1m1m' \mid 1m + m') L_{m + m'} \\
&\left[ Q_m, L_{m'} \right] = \sqrt{6} (2m1m' \mid 2m + m') Q_{m + m'} \\
&\left[ Q_m, Q_{m'} \right] = 3\sqrt{10} (2m2m' \mid 1m + m') L_{m + m'}
\end{align*}
\]

Algebraic quadruple operator:

\[
Q_{m}^{(a)} = \left( 4 \frac{\not{i}/5 \right)^{1/2} \left( r^2 Y_{2,m} (\vec{r}, \vec{r}) + b^4 p^2 Y_{2,m} (\vec{p}, \vec{p}) \right)
\]

\[
H_0 = r^2 + b^4 p^2, \quad \vec{L} = \vec{r} \not{i} \vec{p}
\]

- State labels: \((\square, \square, \square \mid m_l)\)
  - \((\square, \square)\) - SU(3) irrep labels
  - \(l\) - total orbital angular momentum
  - \(m_l\) - angular momentum projection (laboratory axis)
  - \(\square\) - angular momentum projection (body-fixed axis)
The reduction $SU(3) \to SU(2) \to U(1)$

- $SU(3)$ generators as $SU(2)$ tensors:
  - $\{Q_0; L_0, Q_{+2}, Q_{-2}\} \to U(1) \to SU(2)$
  - $\{L_{+1}, Q_{+1}, L_{-1}, Q_{-1}\} \to 2$ conjugate $[1/2]$ irreps of $SU(2)$ with $\ell = \pm 3$

- State labels: $|l(m)\square n \square m_i>$
  - $\Box$ - $SU(3)$ irrep labels
  - $\square$ - quadruple moment
  - $m_i$ - third projection of the angular momentum
  - $n_\square$ - number of oscillator quanta in (x,y) plane for $(\Box, 0)$ irreps

- Label’s values:
  - $\square = \square \Box 2 \Box, \square \Box 2 \Box +3, ... , 2 \Box + \square$
  - $n_\square = 0, 1, ... , \Box + \square$
  - $m_i = \square n_\square, \square n_\square + 2, ... , n_\square$
Basis States

Strong $SU(3)$ coupling:

$| N S (l,m) \rangle = \langle SU_p(3) SU_n(3) | SU(3) > <SU_{sp}(2) SU_{sn}(2) | SU_S(2) >$

$| N_p S (l,m) \rangle_\text{p} \quad | N_n S (l,m) \rangle_\text{n}$

- $SU(3)$
- $SU(4)$
- $SU(3)$
- $SU_S(2)$
- $SU_T(2)$ leading irreps

- $e = -2$
- $e = -1$
- $e = 0$
- $e = 1$
- $e = 4$

$m_l = -2$
$m_l = 0$
$m_l = 2$
$m_l = -1$
$m_l = 1$
$m_l = 0$

Similar, but much simpler construction of $m$-scheme basis states:

just configurations with same total $M_J$. 
The Shell-Model Hamiltonian

\[ H = \sum_i a_i^+ a_i + \sum_{i,j,k,l} V_{ijkl} a_i^+ a_j^+ a_k a_l, \]

Single-particle energies

Kuo-Brown-3 (KB3)

\((\varepsilon_p - \varepsilon_r = 0.096 \text{ MeV})\)
SU(3) Symmetry Breaking in the pf-shell nuclei

Realistic spin-orbit \((l\cdot s)\) single particle energy splitting!

Turn off the s.p.e. spin-orbit splitting!

Coherent state structure in \(^{48}\text{Cr}\) using Kuo-Brown-3 interaction.
Eigenvalue Problem in an Oblique Basis

Spherical basis states $e_i$  SU(3) basis states $E_a$

Overlap matrix $g$

\[ \hat{g} = \begin{pmatrix} \langle e_i | e_j \rangle \\ \langle E_a | e_j \rangle \end{pmatrix} = \begin{pmatrix} \langle e_i | E_b \rangle \\ \langle E_a | E_b \rangle \end{pmatrix} = \begin{pmatrix} 1 \\ \hat{g} \end{pmatrix} \]

The eigenvalue problem

\[ H = E \hat{g} \cdot \hat{g} \]
Example of an Oblique Basis Calculation: $^{24}\text{Mg}$

We use the **Wildenthal USD interaction** and denote the **spherical basis** by $\text{SM}(#)$ where $#$ is the number of nucleons outside the $d_{5/2}$ shell, the **SU(3) basis** consists of the leading irrep $(8,4)$ and the next to the leading irrep, $(9,2)$.

<table>
<thead>
<tr>
<th>Model Space</th>
<th>SU3 $(8,4)$</th>
<th>SU3+ $(8,4)$ &amp; $(9,2)$</th>
<th>GT100</th>
<th>SM(0)</th>
<th>SM(1)</th>
<th>SM(2)</th>
<th>SM(4)</th>
<th>Full</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Dimension</strong> (m-scheme)</td>
<td>23</td>
<td>128</td>
<td>500</td>
<td>29</td>
<td>449</td>
<td>2829</td>
<td>18290</td>
<td>28503</td>
</tr>
<tr>
<td>%</td>
<td>0.08</td>
<td>0.45</td>
<td>1.75</td>
<td>0.10</td>
<td>1.57</td>
<td>9.92</td>
<td>64.17</td>
<td>100</td>
</tr>
</tbody>
</table>

**Visualizing** the SU(3) space with respect to the SM space using the naturally induced basis in the SU(3) space.

[Insert portrait slides]
Better Dimensional Convergence!

Ground State Convergence for 24Mg

Ground State Convergence for $^{44}\text{Ti}$
Level Structure for $^{44}$Ti

Oblique Basis Results
Overlaps With The Exact Eigenvectors For 44Ti

Overlaps With The Exact Eigenvectors For 24Mg
Summary

- The spin-orbit interaction drives the breaking of the SU(3) symmetry in the lower pf-shell.
- The nuclear interaction has a clear two-mode structure: s.p.e. and SU(3) invariant two-body part…

Use of two different sets of states can enhance our understanding of complex systems.
- There is better dimensional convergence.
- Correct level order of the low-lying states.
- Significant overlap with the exact states.
  - 10% versus 64% for $^{24}$Mg (good SU(3) limit)
  - 50% versus 84% for $^{44}$Ti (poor SU(3) limit)