Conformality Lost

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Motivation: QCD at LARGE $N_c$ and $N_f$
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Define $x = N_f / N_c$, treat as a continuous variable

- asymptotic freedom
- conformal
- trivial

$\langle \bar{\psi} \psi \rangle \neq 0 \quad x_c \quad 11/2 \quad x$
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gauge coupling: $\alpha_*$

Banks-Zaks fixed point

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Colors  Flavors

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What is the nature of this transition?

How does the IR scale appear as conformality is lost?

 asymptotic freedom         conformal       trivial

$0 < \langle \bar{\psi} \psi \rangle \neq 0$  $x_c$  $11/2$  $x$

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Banks-Zaks fixed point

$\langle \bar{\psi} \psi \rangle \
eq 0$!


David B. Kaplan
I. A mechanism for vanishing conformal invariance

II. The Berezinskii-Kosterlitz-Thouless (BKT) transition

III. A quantum mechanics model: the $1/r^2$ potential

IV. AdS/CFT

V. Relativistic model: defect Yang-Mills

VI. QCD with many flavors? A partner theory QCD* with a nontrivial UV fixed point?
A theory with an infrared conformal fixed point at $g = g^*$ has a zero in the beta function:

$$\beta(g) = \mu \frac{\partial g}{\partial \mu} = \frac{\partial g}{\partial t}$$
A theory with an infrared conformal fixed point at $g=g_\ast$ has a zero in the beta function:

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Suppose the theory has another parameter $\kappa$ such that the fixed point at $g=g_\ast$ vanishes for $\kappa>\kappa_\ast$.
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**Example:** supersymmetric QCD is conformal for $3/2 \leq N_f/N_c \leq 3$
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“$\kappa$” = $N_f/N_c$, “$\kappa^*$” = $3/2, 3$

How is conformality lost?
Three ways to lose an infrared fixed point:

#1: Fixed point runs to zero:

\[
\beta(g; \kappa) \quad \quad \kappa < \kappa^* \quad \quad \quad \quad \quad \beta(g; \kappa) \quad \quad \kappa > \kappa^*
\]
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\[ \Rightarrow \text{Increasing flavors, leave conformal window. } \kappa = \frac{N_f}{N_c}, \kappa^* = 3 \]
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\( N_f/N_c \leq 3 \) ⇒ weak coupling Banks–Zaks conformal fixed point

\( N_f/N_c > 3 \) ⇒ trivial QED-like “free electric” theory
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\[ \Rightarrow \text{Increasing flavors, leave conformal window. } \kappa = N_f/N_c, \kappa^* = 3 \]

\[ N_f/N_c \lesssim 3 \Rightarrow \text{weak coupling Banks-Zaks conformal fixed point} \]

\[ N_f/N_c \gtrsim 3 \Rightarrow \text{trivial QED-like "free electric" theory} \]

\[ F_E \sim \frac{g^2}{r^2 \ln (r \Lambda_{UV})} \]
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Possible example? SQCD again  \( \Rightarrow \kappa = N_f/N_c, \kappa_* = 3/2 \)

For \( \kappa \leq \kappa_* \) get "free magnetic phase" [Seiberg]
#2: Fixed point runs off to infinity:

\[ F_E \sim \frac{g^2 \ln (r \Lambda_{UV})}{r^2} \quad F_M \sim \frac{g_M^2}{r^2 \ln (r \Lambda_{UV})} \quad g_M \sim \frac{1}{g} \]

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For \( \kappa \leq \kappa_* \) get “free magnetic phase” \[\text{[Seiberg]}\]

- electric theory dual to a QED-like magnetic theory:
#3: UV and IR fixed points annihilate:

A toy model:

\[ \beta(g; \kappa) = (\kappa - \kappa_*) - (g - g_*)^2 \]

\( \kappa \geq \kappa_* \) : \quad g_\pm = g_* \pm \sqrt{\kappa - \kappa_*} \quad \text{UV, IR fixed points} \n
\( \kappa = \kappa_* \) \quad \text{fixed points merge} \n
\( \kappa < \kappa_* \) \quad \text{conformality lost} \n
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\[ \beta(g; \kappa) \]

UV \quad IR

\[ g_* \]

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- Start: \( g = g_{\text{UV}} < g_* \) in the UV
- \( g \) grows, **stalling** near \( g_* \)
- \( g \) strong at scale \( \Lambda_{\text{IR}} \)
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i. Start: $g = g_{\text{UV}} < g_*$ in the UV

ii. $g$ grows, **stalling** near $g_*$

iii. $g$ strong at scale $\Lambda_{\text{IR}}$

(Not like 2$^{\text{nd}}$ order phase transition: $\Lambda_{\text{IR}} \approx \Lambda_{\text{UV}} \sqrt{|\kappa - \kappa_*|}$)
\[ \Lambda_{IR} \simeq \Lambda_{UV} e^{-\frac{7\pi}{\sqrt{|\kappa - \kappa^*|}}} \]

Scaling behavior of toy model is reminiscent of the Berezinskii-Kosterlitz-Thouless (BKT) transition (an “infinite order” phase transition)
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box size $R$, vortex core size $a$:
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E = E_0 \ln \frac{R}{a}, \quad S = 2 \ln \frac{R}{a}
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\[
F = E - TS = (E_0 - 2T) \ln \frac{R}{a}
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Vortices condense for $T > T_c = E_0/2$; can show correlation length forms:

$$\xi \sim a e^{b/\sqrt{T-T_c}}$$
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Z = \mathcal{N} \sum_{N_+, N_-} z^{N_+} z^{N_-} \frac{z_{N_+} z_{N_-}}{N_+! N_-!} \int \prod_{i=1}^{N_+} \prod_{j=1}^{N_-} d^2 x_i d^2 y_j \int D\phi e^{-\int d^2 x \frac{T}{2} \left(\nabla \phi\right)^2 + i \sum_{i,j} (\phi(x_i) - \phi(y_j))}
\]

- fugacity
- Sum over vortex positions/numbers
- Coulomb field
- vortices
- anti-vortices
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The XY model is equivalent to the Sine-Gordon model
Classical XY model BKT transition = zero temperature quantum transition in Sine-Gordon model:

\[ \mathcal{L} = \frac{T}{2} (\nabla \phi)^2 - 2z \cos \phi \]
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New variables:

\[ u = 1 - \frac{1}{8 \pi T} , \quad v = \frac{2z}{T \Lambda^2} \]

Perturbative \( \beta \)-functions:

\[ \beta_u = -2v^2 , \quad \beta_v = -2uv \]
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\[ \text{Dimensionful quantities in units of XY model interaction strength} \]

\[ T < T_c \]

• bound vortices
• trivially conformal
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Dimensionful quantities in units of XY model interaction strength

\( T < T_c \):
- bound vortices
- trivially conformal

\( T > T_c \):
- Coulomb gas
- screening length
\[ u = 1 - \frac{1}{8\pi T}, \quad v = \frac{2z}{T\Lambda^2} \]

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**Newer variables:**

\[ \tau = (u + v), \quad \kappa = (u^2 - v^2) \]

\[ \beta_\tau = \kappa - \tau^2, \quad \beta_\kappa = 0 \]
Correlation length in BKT transition:

For small negative $\kappa$, assume $\tau$ small & positive in UV

$\tau$ blows up in RG time

$$t = \int \frac{d\tau}{\beta(\tau)} = -\frac{\pi}{2\sqrt{-\kappa}}$$
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...giving rise to an IR scale (like $\Lambda_{QCD}$) which sets the scale for the finite correlation length for $\alpha<0$:

$$\xi_{\text{BKT}} \sim \frac{1}{\Lambda} e^{\frac{\pi}{2\sqrt{-\kappa}}}$$
So far:

- BKT transition = loss of conformality via fixed point merger
- Mechanism of fixed point merger in general gives rise to “BKT scaling”:

\[
\Lambda_{\text{IR}} \simeq \Lambda_{\text{UV}} e^{-\frac{\pi}{\sqrt{|\kappa - \kappa^*|}}}
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So far:

- BKT transition = loss of conformality via fixed point merger
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Next: other examples:

- QM with $1/r^2$ potential
- AdS/CFT
- Defect Yang-Mills
- QCD with many flavors
Example: QM in d-dimensions with $1/r^2$ potential

\[ -\nabla^2 + V(r) - k^2 \] \psi = 0 , \quad V(r) = \frac{\kappa}{r^2}
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$k=0$ solutions:

$$\psi = c_- r^{\nu_-} + c_+ r^{\nu_+}$$

$$\nu_\pm - \left( \frac{d - 2}{2} \right) \pm \sqrt{\kappa - \kappa_*} \quad \kappa_* = - \left( \frac{d - 2}{2} \right)^2$$
Example: QM in d-dimensions with $1/r^2$ potential

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\[\nu_\pm - \left( \frac{d-2}{2} \right) \pm \sqrt{\kappa - \kappa_\star} \quad \kappa_\star = - \left( \frac{d-2}{2} \right)^2\]

• valid for \(\kappa_\star < \kappa < (\kappa_\star + 1)\)
  • \(\kappa < \kappa_\star\): \(V_\pm\) complex, no ground state
  • \(\kappa = \kappa_\star\): \(V_+ = V_-\)
  • \(\kappa > (\kappa_\star + 1)\): \(r^{\nu_-}\) too singular to normalize
\[ [-\nabla^2 + V(r) - k^2] \psi = 0 , \quad V(r) = \frac{\kappa}{r^2} \]

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\]

• \( c_+ = 0 \text{ or } c_- = 0 \) are scale invariant solutions
• If \( c_+ \neq 0, \quad \psi \to c_+ r^{\nu_+} \) for large \( r \) (\( \nu_+ > \nu_- \))
• to make sense of BC at \( r=0 \), introduce \( \delta \)-function:
\[-\nabla^2 + V(r) - k^2\] \(\psi = 0\), \(V(r) = \frac{\kappa}{r^2}\)

\textit{k=0 solutions:} \(\psi = c_- r^{\nu_-} + c_+ r^{\nu_+}\)

\(\nu_{\pm} = \left(\frac{d-2}{2}\right) \pm \sqrt{\kappa - \kappa_*}\), \(\kappa_* = -\left(\frac{d-2}{2}\right)^2\)

• \(c_+ = 0\) or \(c_- = 0\) are scale invariant solutions
• If \(c_\neq 0\), \(\psi \rightarrow c_+ r^{\nu_+}\) for large \(r\) (\(\nu_+ > \nu_-\))
• to make sense of BC at \(r=0\), introduce \(\delta\)-function:

\[V(r) = \frac{\kappa}{r^2} - g\delta^{(d)}(r)\]
\[-\nabla^2 + V(r) - k^2\] \(\psi = 0\), \quad V(r) = \frac{\kappa}{r^2}

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\[V(r) = \frac{\kappa}{r^2} - g\delta^{(d)}(r)\]

• \(r^{\nu_+}\) dominates at large \(r\) -- corresponds to IR fixed point of \(g\)

• \(r^{\nu_-}\) dominates at small \(r\) -- corresponds to UV fixed point of \(g\)
1. Non-perturbative RG treatment of $1/r^2$ potential:

regulate with square well:

$$V(r) = \begin{cases} \frac{\kappa}{r^2} & r > r_0 \\ -\frac{g}{r_0^2} & r > r_0 \end{cases}$$

E=0 solution for $r > r_0$:

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I. Non-perturbative RG treatment of $1/r^2$ potential:

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Solve for $c_+/c_-$ (a physical dimensionful quantity) and require invariance: \( d(c_+/c_-)/dr_0 = 0 \):
I. Non-perturbative RG treatment of $1/r^2$ potential:

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E=0 solution for $r>r_0$: $\psi = c_- r^{\nu-} + c_+ r^{\nu+}$

Solve for $c+/c-$ (a physical dimensionful quantity) and require invariance: $d(c+/c-)/dr_0 = 0$:

Find exact $\beta$-function for $g$. Eg, for $d=3$:

$$\beta = \frac{2\sqrt{g} \left( \kappa + \sqrt{g} \cot \sqrt{g} - g \cot^2 \sqrt{g} \right) - \cot \sqrt{g} + \sqrt{g} \csc^2 \sqrt{g}}{\sqrt{g}}$$

$\kappa_* = -\frac{1}{4}$, $g_* \approx 1.36$
Aside: Even better to define a modified coupling constant

\[ \gamma = \left( \frac{\sqrt{g} J_{d/2}(\sqrt{g})}{J_{d/2-1}(\sqrt{g})} \right) \]

Condition \( d(c_+/c_-)/dr_0 \) yields exact \( \beta \)-function in \( d \)-dimensions:

\[ \beta_\gamma = \frac{\partial \gamma}{\partial t} = (\kappa - \kappa_*) - (\gamma - \gamma_*)^2, \quad \gamma_* = \frac{d - 2}{2} \]

- Toy model is exact!
- \( \gamma \) is a periodic function of \( g \), \( \gamma = \pm \infty \) equivalent
- Aside: Limit cycle behavior for \( \kappa < \kappa_* \): describes “Efimov states”
II. Perturbative RG treatment of $k/r^2$ potential:

$k_* \equiv -(d-2)^2/4$ so work in $d=2+\varepsilon$
II. Perturbative RG treatment of $k/r^2$ potential:

$\kappa_* \equiv -(d-2)^2/4$ so work in $d=2+\varepsilon$

\[
S = \int dt \, d^d x \left( i \psi^{\dagger} \partial_t \psi - \frac{\nabla \psi}{2m} \right) + \frac{g\pi}{4} \psi^{\dagger} \psi^{\dagger} \psi \psi \\
- \int dt \, d^d x \, d^d y \, \psi^{\dagger}(t, x) \psi^{\dagger}(t, y) \frac{\kappa}{|x - y|^2} \psi(t, y) \psi(t, x)
\]

\[\begin{align*}
\text{propagator:} & \quad \frac{i}{\omega - p^2/2m} \\
\text{contact vertex:} & \quad i\pi g\mu^{-\varepsilon} \\
\text{“meson exchange”:} & \quad \frac{2\pi i \kappa}{\varepsilon |q|^\varepsilon}
\end{align*}\]
II. Perturbative RG treatment of $\kappa/r^2$ potential:

$\kappa_\ast \equiv -(d-2)^2/4 \quad$ so work in $d=2+\varepsilon$

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$$- \int dt \int d^d x \int d^d y \psi^\dagger(t, x) \psi^\dagger(t, y) \frac{\kappa}{|x - y|^2} \psi(t, y) \psi(t, x)$$

**propagator:** \( \frac{i}{\omega - p^2/2m} \)

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**“meson exchange”:** \( \frac{2\pi i \kappa}{\varepsilon} \frac{1}{|q|^\varepsilon} \)

Find $g$ runs:

\[ \beta(g; \kappa) = \mu \frac{\partial g}{\partial \mu} = \left( \kappa + \frac{\varepsilon^2}{4} \right) - (g - \varepsilon)^2 \]
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Same as toy model! $\kappa_* = -\varepsilon^2/4, \ g_* = \varepsilon$

Exact, $\varepsilon=1$: $\kappa_* = -1/4, \ g_* = 1.36$
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- $\int dt \, d^d x \, d^d y \, \psi^\dagger(t, x) \psi^\dagger(t, y) \frac{\kappa}{|x-y^2|} \psi(t, y) \psi(t, x)$

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Same as toy model! $\kappa_* = -\varepsilon^2/4$, $g_* = \varepsilon$

Exact, $\varepsilon=1$: $\kappa_* = -1/4$, $g_* = 1.36$

$B \sim \left( \frac{\Lambda^2_{\text{IR}}}{m} \right) \sim \left( \frac{\Lambda^2_{\text{UV}}}{m} \right) e^{-2\pi/\sqrt{\kappa_* - \kappa}}$

bound state energy
Conformal phases: measure correlations, not $\beta$-functions!
Look at operator scaling dimensions:
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From Nishida & Son, 2007:

- Replace $V(r_1-r_2) \rightarrow V(r_1-r_2) + \frac{1}{2} \omega^2 |r_1^2 + r_2^2|$
- Compute 2-particle ground state energy $E_0$
- Operator dimension of $\psi\psi$ is $\Delta_{\psi\psi} = E_0 / \omega$
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2-particle wavefunction at $|r_1-r_2|=0$
Conformal phases: measure correlations, not $\beta$-functions! Look at operator scaling dimensions:

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As the two conformal theories merge when $\kappa \rightarrow \kappa_*$, operator dimensions in the two CFTs merge
Conformal phases: measure correlations, not $\beta$-functions! Look at operator scaling dimensions:

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As the two conformal theories merge when $\kappa \rightarrow \kappa_*$, operator dimensions in the two CFTs merge

For $1/r^2$ potential -- find for the two conformal theories:

$$[\psi\psi]: \quad \Delta_{\pm} = (d + \nu_{\pm}) = \left(\frac{d + 2}{2}\right) \pm \sqrt{\kappa - \kappa_*}$$

```
"+" = UV fixed point
"-" = IR fixed point
```

Note: $(\Delta_+ + \Delta_-) = (d+2)$: scaling dimension of nonrelativistic spacetime.
Analog in AdS/CFT:
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\[ ds^2 = \frac{1}{z^2} \left( dz^2 + \sum_{i=1}^{d} dx_i^2 \right) \]
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Massive scalar in the bulk

two solutions to eq. of motion, corresponding to two different CFT’s:

\[ \phi = c_+ z^{\Delta_+} + c_- z^{\Delta_-} \]

\[ \Delta_{\pm} = \frac{d}{2} \pm \sqrt{m^2 + \left( \frac{d}{2} \right)^2} = \frac{d}{2} \pm \sqrt{m^2 - m_0^2} \]

\( \Delta_{\pm} = \text{operator dim} \)
**Analog in AdS/CFT:**

**AdS:**
\[ ds^2 = \frac{1}{z^2} \left( dz^2 + \sum_{i=1}^{d} dx_i^2 \right) \]

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**AdS**

- \((\Delta_+ + \Delta_-) = d = \text{spacetime dim of CFT}\)

**QM**

- \((\Delta^+_{\psi\psi} + \Delta^-_{\psi\psi}) = (d+2) = \text{conformal wt. of nonrelativistic d-space+time}\)
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**AdS**

- \((\Delta_+ - \Delta_-) = d = \text{spacetime dim of CFT}\)
- when \(m^2 = m_*^2 = -d^2/4\), \(\Delta_\pm = d/2\)

**QM**

- \((\Delta^+_\psi \psi + \Delta^-_{\psi \psi}) = (d+2) = \text{conformal wt. of nonrelativistic d-space+time}\)
- \(K = k_* = -(d-2)^2/4 \Rightarrow \Delta_\pm = (d+2)/2\)
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**AdS**
- \((\Delta_+ + \Delta_-) = d\) = spacetime dim of CFT
- when \(m^2 = m_*^2 = -d^2/4\), \(\Delta_\pm = d/2\)
- Instability (no AdS or CFT) for \(m^2 < m_*^2\) (B-F bound)

**QM**
- \((\Delta^+_\psi + \Delta^-\psi) = (d+2)\) = conformal wt.
  of nonrelativistic \(d\)-space+time
- \(K = K_* = -(d-2)^2/4 \Rightarrow \Delta_\pm = (d+2)/2\)
- Conformality lost for \(K < K_*\)
AdS/CFT cont’d:

As with QM example, 2 different solutions ⇒ 2 different CFTs
AdS/CFT cont'd:

As with QM example, 2 different solutions $\Rightarrow$ 2 different CFTs

$$\varphi = \varphi_0 z^\Delta + : \quad Z\text{grav.}\left|_{\varphi \rightarrow \varphi_0 z^\Delta} \right. = Z_{\text{CFT}}[\varphi_0]$$
AdS/CFT cont’d:

As with QM example, 2 different solutions ⇒ 2 different CFTs

\[ \varphi = \varphi_0 z^\Delta^+ : \quad Z_{\text{grav.}} \bigg|_{\varphi \to 0} \varphi_0 z^\Delta^+ = Z_{\text{CFT}}[\varphi_0] \]

\[ S = S_{\text{CFT}} + \int d^d x \phi_0 \mathcal{O} \]
AdS/CFT cont’d:

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\[ \varphi = \varphi_0 z^\Delta^+ : \quad Z_{\text{grav.}} \bigg|_{\varphi \rightarrow \varphi_0 z^\Delta^+} = Z_{\text{CFT}}[\varphi_0] \]

\[ \varphi = J z^\Delta^- : \quad Z_{\text{grav.}} \bigg|_{\varphi \rightarrow J z^\Delta^-} = Z_{\text{CFT}}[J] \]

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\[ = \int D\varphi Z_{\text{CFT}}[\varphi] e^{i \int d^d x J \varphi} \]
AdS/CFT cont’d:

As with QM example, 2 different solutions $\Rightarrow$ 2 different CFTs

$\varphi = \varphi_0 z^\Delta^+ : \quad Z_{\text{grav.}} \left. \frac{\varphi}{z \to 0 } \right| \varphi_0 z^\Delta^+ = Z_{\text{CFT}}[\varphi_0]$

$\varphi = J z^\Delta^- : \quad Z_{\text{grav.}} \left. \frac{\varphi}{z \to 0 } \right| J z^\Delta^- = Z_{\text{CFT}}[J]$

$S = S_{\text{CFT}} + \int d^d x \phi_0 O$

$= \int D\varphi Z_{\text{CFT}}[\varphi] e^{i \int d^d x J \varphi}$

UV fine-tuning: $m^2 \varphi^2 \ldots$ adds $OO$ operator. Eg: $O = \bar{\psi} \psi$, $OO = \bar{\psi} \psi \bar{\psi} \psi$
AdS/CFT cont’d:

As with QM example, 2 different solutions \( \Rightarrow \) 2 different CFTs

\[ \varphi = \varphi_0 z^{\Delta^+} : \quad \mathcal{Z}_{\text{grav.}} \left|_{\varphi \to \varphi_0 z^{\Delta^+}} \right. = \mathcal{Z}_{\text{CFT}}[\varphi_0] \]

\[ \varphi = J z^{\Delta^-} : \quad \mathcal{Z}_{\text{grav.}} \left|_{\varphi \to J z^{\Delta^-}} \right. = \mathcal{Z}_{\text{CFT}}[J] \]

\[ S = S_{\text{CFT}} + \int d^d x \, \phi_0 \mathcal{O} \]

\[ = \int D\varphi \mathcal{Z}_{\text{CFT}}[\varphi] e^{i \int d^d x J \varphi} \]

UV fine-tuning: \( m^2 \varphi^2 \ldots \) adds \( \mathcal{O} \mathcal{O} \) operator. Eg: \( O = \bar{\psi} \psi, \mathcal{O} \mathcal{O} = \bar{\psi} \psi \bar{\psi} \psi \)

\( \Rightarrow \) analog of \( \delta(r) \) in QM example tuned to unstable UV fixed pt.
A relativistic example: defect Yang-Mills theory
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Charged relativistic fermions on a d-dimensional defect + 4D conformal gauge theory (eg, N=4 SYM)

$$S = \int d^{d+1}x \; i\bar{\psi} \gamma^\mu D_\mu \psi - \frac{1}{4g^2} \int d^4x \; F^a_{\mu\nu} F^{a,\mu\nu}$$
A relativistic example: defect Yang-Mills theory

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\[ S = \int d^{d+1}x \, i \bar{\psi} \gamma^\mu D_\mu \psi - \frac{1}{4g^2} \int d^4x \, F_{\mu\nu}^a F^{a,\mu\nu} \]

\text{g doesn’t run}
g doesn’t run by construction

Expect a phase transition as a function of g:

\[
\langle \bar{\psi} \psi \rangle = \begin{cases} 
0 & g < g_* \\
\Lambda_{d \text{IR}}^d & g > g_*
\end{cases}
\]
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Expect a phase transition as a function of g:

\[ \langle \bar{\psi} \psi \rangle = \begin{cases} 
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\end{cases} \]

Add a contact interaction to the theory (as in QM & AdS/CFT examples!) and study its running:

\[ \Delta S = \int d^{d+1}x \left( -\frac{c}{2} (\bar{\psi} \gamma_\mu T_a \psi)^2 \right) \]
g doesn’t run by construction

Expect a phase transition as a function of g:

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Add a contact interaction to the theory (as in QM & AdS/CFT examples!) and study its running:

\[ \Delta S = \int d^{d+1}x \left( -\frac{c}{2} (\bar{\psi}\gamma_\mu T_\alpha \psi)^2 \right) \]

Phase transition is in perturbative regime for \( d=1+\varepsilon \) (spatial dimensions of “defect”): compute \( \beta \)-function
$\beta(c): \quad \begin{array}{c}
\begin{aligned}
\quad & \quad \\
\end{aligned}
\end{array}$

\[ \frac{1}{\varepsilon} \text{ pole for } d = (1 + \varepsilon) \]
\[ \beta(c) = -\frac{g^2}{2\pi} - \epsilon c - \frac{N_c}{2\pi} c^2 \]

\[ = \frac{1}{2\pi} \left( \frac{\pi^2 \epsilon^2}{N_c} - g^2 \right) - \frac{N_c}{2\pi} \left( c - \frac{\epsilon \pi}{N_c} \right)^2 \]
\[ \beta(c) = -\frac{g^2}{2\pi} - \epsilon c - \frac{N_c}{2\pi} \epsilon^2 c^2 \]

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- Find BKT transition at \( g^2 = g_*^2 = (\epsilon \pi)^2 / N_c \)
  \( \Lambda_{IR} \sim \Lambda_{UV} \exp[-\pi/\sqrt{(g^2-g_*^2)}] \)
- Schwinger-Dyson gap eq (rainbow approx) gives qualitatively same results
Back to QCD at LARGE $N_c$ and $N_f$:

- asymptotic freedom
- conformal
- trivial

$\langle \bar{\psi} \psi \rangle \neq 0$

Transition at $x = x_c$?
Back to QCD at LARGE $N_c$ and $N_f$:

- **asymptotic freedom** → **conformal** → **trivial**

$0 \quad \langle \bar{\psi}\psi \rangle \neq 0 \quad x_c \quad 11/2 \quad x$

**gauge coupling:** $\alpha_*$

Transition at $x=x_c$?

Schwinger-Dyson (rainbow approximation):

- **Miransky 1985**
- **Appelquist, Terning, Wijerwardhana 1996**
Back to QCD at LARGE $N_c$ and $N_f$:

**asymptotic freedom** $\leftrightarrow$ **conformal** $\leftrightarrow$ **trivial**

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**gauge coupling**: $\alpha_*$

Transition at $x=x_c$?

**Schwinger-Dyson (rainbow approximation):**

Miransky 1985

Appelquist, Terning, Wijewardhana 1996

Found: BKT scaling for $\langle \bar{\psi} \psi \rangle$...not rigorous, but qualitatively correct?
**Conjecture:** loss of conformality for QCD at $x_c$ is of BKT type, due to fixed point merger.
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\[
\Delta \bar{\psi} \psi - 2 x^2 + 2 = 4?
\]

\[
\Delta^+ + \Delta^- = 4?
\]

Free boson

Free fermions

$X^* = \frac{11}{2}$
Conjecture: loss of conformality for QCD at $x_c$ is of BKT type, due to fixed point merger.
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Near Banks-Zaks (IR) fixed point:

$$\Delta^+ + \Delta^- = 4?$$

Free fermions

Free boson

$X_BZ = 11/2$

$\Delta^\Psi\Psi$

$\Delta^-$

$\Delta^+$

$X$
Conjecture: loss of conformality for QCD at $x_c$ is of BKT type, due to fixed point merger.

Near Banks-Zaks (IR) fixed point:

\[ \Delta_{\psi \bar{\psi}} = 3 - \# g^2 N_c \]

(almost free quarks)
Conjecture: loss of conformality for QCD at $x_c$ is of BKT type, due to fixed point merger.

Near Banks–Zaks (IR) fixed point:

**QCD:**

$$\Delta^+_{\psi\overline{\psi}} = 3 - \# g^2N_c$$

(almost free quarks)

**Partner theory QCD*:**

$$\Delta^-_{\psi\overline{\psi}} = d - \Delta^+_{\psi\overline{\psi}} = 1 + \# g^2N_c$$

(almost free scalar?)
WANTED

Conformal theory defined at nontrivial UV fixed point to merge with QCD at $x=x_c$

LAST SEEN WITH WEAKLY COUPLED SCALAR

Monday, February 22, 2010
Haven’t found a Lorentz invariant perturbative example with:

(i) weakly coupled scalar;

(ii) full $\text{SU}(N_f) \times \text{SU}(N_f)$ chiral symmetry

(iii) Matching anomalies
Haven’t found a Lorentz invariant perturbative example with:

(i) weakly coupled scalar;

(ii) full SU(Nf)xSU(Nf) chiral symmetry

(iii) Matching anomalies

Look for nonperturbative QCD* on the lattice?

One place to start: strong/weak transition for QCD with Nf in conformal window?

(A. Hasenfratz)
Conclusions:
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II. Leads to similar scaling as in the BKT transition:
\[ \Lambda_{\text{IR}} \sim \Lambda_{\text{UV}} \exp\left[-\pi/\sqrt{(-\kappa-\kappa_*)}\right] \]
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III. Both relativistic & non-relativistic examples
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IV. Analog in AdS/CFT; implications for AdS below the Breitenlohner-Freedman bound?
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IV. Analog in AdS/CFT; implications for AdS below the Breitenlohner-Freedman bound?

V. Implications for QCD with many flavors? Is there a pair of conformal QCD theories? What is QCD*?
Conclusions:

I. Fixed point annihilation appears to be a generic mechanism for the loss of conformality

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\[ \Lambda_{IR} \sim \Lambda_{UV} \ e^{-\pi/\sqrt{(-\kappa - \kappa_*)}} \]

III. Both relativistic & non-relativistic examples

IV. Analog in AdS/CFT; implications for AdS below the Breitenlohner-Freedman bound?

V. Implications for QCD with many flavors? Is there a pair of conformal QCD theories? What is QCD*?

VI. Finding QCD* should be on field theory / lattice QCD “to-do” list.