2 problems chosen to be graded are marked by a $\ast$ and are worth 40 points each; the tutorial questions are worth an additional 20 points total.

$\ast$ Gasiorowicz 2-12 Several ways to do this. Note that the question is dealing with standard wave properties, and not wave-particle duality, so there will be no $\hbar$ in the final answer. One way: a property of Fourier transforms is $\Delta x \Delta k \gtrsim 1$ (this is the same as $\Delta x \Delta p \gtrsim \hbar$, if you use the de Broglie relations). Here, $\Delta x = c \Delta t$, so $\Delta k \gtrsim 1/(c \Delta t)$. To relate to $\Delta \lambda$, note that $k = 2\pi/\lambda$, so a small change in $k$ by $dk$ and a small change in $\lambda$ by $d\lambda$ are related by $dk = \left(-\frac{2\pi}{\lambda}\right) d\lambda$. Taking the absolute value, and calling $|dk| = \Delta k$, $\Delta \lambda = |d\lambda|$ we get

$$\Delta \lambda = \frac{\lambda^2}{2\pi} \Delta k \gtrsim \frac{\lambda^2}{2\pi c \Delta t} = \frac{(6000 \text{ A})^2}{2\pi (3 \times 10^{18} \text{ A/s}) 10^{-9} \text{ s}} = 1.9 \times 10^{-3} \text{ A}.$$  

(1)

Note that since $\Delta \lambda \ll \lambda$, it was justified to use differentials $d\lambda$ to represent $\Delta \lambda$. If instead of this method you used the uncertainty relations for $\Delta x \Delta p$ or $\Delta E \Delta t$, that should have given you the same answer.

Gasiorowicz 3-6 Given:

$$\psi = \sqrt{\frac{2a^3}{\pi}} \frac{1}{x^2 + a^2}, \quad \int_{-\infty}^{\infty} dx \ |\psi|^2 = 1,$$  

(2)

$$\int_{-\infty}^{\infty} dx \ \frac{1}{x^2 + a^2} = \frac{\pi}{a}, \quad \phi(p) = \sqrt{\frac{a}{\hbar}} e^{-a|p|/\hbar}.$$  

(3)

Note that $|\psi(x)|^2$ and $|\phi(p)|^2$ are even functions of $x$ and $p$ respectively, so

$$\langle x \rangle = \int_{-\infty}^{\infty} dx \ x |\psi(x)|^2 = 0, \quad \langle p \rangle = \int_{-\infty}^{\infty} dp \ p |\phi(p)|^2 = 0.$$  

(4)

Then

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} dx \ x^2 |\psi|^2 = -a^2 \int_{-\infty}^{\infty} dx \ |\psi|^2 + \int_{-\infty}^{\infty} dx \ (x^2 + a^2) |\psi|^2$$

$$= -a^2 + \frac{2a^3}{\pi} \int_{-\infty}^{\infty} dx \ \frac{1}{x^2 + a^2} = -a^2 + \frac{2a^3}{\pi} \frac{\pi}{a} = a^2,$$  

(5)

and

1
\[
\langle p^2 \rangle = \int_{-\infty}^{\infty} dp \, p^2 |\phi(p)|^2 = 2a \hbar \int_{0}^{\infty} dp \, p^2 e^{-2ap/h} = \frac{2a}{\hbar} \left( \frac{\hbar}{-2} \right)^2 \frac{d^2}{da^2} \left( \int_{0}^{\infty} dp \, e^{-2ap/h} \right) = \frac{\hbar a}{2} \frac{d^2}{da^2} \left( \frac{\hbar}{2a} \right) = \frac{\hbar}{2a^2} . \tag{6}
\]

Therefore
\[
\Delta x = \sqrt{\langle x^2 \rangle} = a , \quad \Delta p = \sqrt{\langle p^2 \rangle} = \frac{\hbar}{a} \sqrt{2} , \tag{7}
\]
and \( \Delta x \Delta p = \hbar / \sqrt{2} > \hbar / 2 \), in keeping with the Heisenberg uncertainty relation.

Gasiorowicz 3-8 Again, there are many ways to solve this problem, some harder than others. I will show you a nice way and a very slick way.

First the nice way. In the momentum representation, \( \hat{p} = p \) and \( \hat{x} = -\frac{\hbar}{i} \frac{d}{dp} \), where \( \hat{p} \) and \( \hat{x} \) are operators. Therefore
\[
e^{i\frac{p}{\hbar}a} \hat{x} e^{-i\frac{p}{\hbar}a} = e^{i\frac{p}{\hbar}a} \left( -\frac{\hbar}{i} \frac{d}{dp} \right) e^{-i\frac{p}{\hbar}a} = e^{i\frac{p}{\hbar}a} \left( ae^{-i\frac{p}{\hbar}a} + e^{-i\frac{p}{\hbar}a} \left( -\frac{\hbar}{i} \frac{d}{dp} \right) \right) = a - \frac{\hbar}{i} \frac{d}{dp} = a + \hat{x} . \tag{8}
\]

The second, very slick, way is to define an operator which is a function of the number \( a \):
\[
\hat{F}(a) = e^{i\frac{p}{\hbar}a} \hat{x} e^{-i\frac{p}{\hbar}a} . \tag{9}
\]

Note that \( \hat{F}(0) = \hat{x} \). Now take the derivative with respect to \( a \):
\[
\frac{d\hat{F}(a)}{da} = e^{i\frac{p}{\hbar}a} \hat{x} \cdot e^{-i\frac{p}{\hbar}a} + e^{i\frac{p}{\hbar}a} \cdot \frac{d}{da} \left( \hat{x} \cdot e^{-i\frac{p}{\hbar}a} \right) = \frac{d}{da} \left( e^{i\frac{p}{\hbar}a} \cdot \hat{x} \cdot e^{-i\frac{p}{\hbar}a} \right) = 1 , \tag{10}
\]

where I used \( [\hat{p}, \hat{x}] = \hbar / i \) (eq. 3-38) and that fact that \( \hat{p} \) commutes with itself. So now we see that \( \hat{F}(a) \) satisfies
\[
\frac{d\hat{F}(a)}{da} = 1 , \quad \hat{F}(0) = \hat{x} \tag{11}
\]
which is a first order differential equation with boundary condition, with the unique solution
\[
\hat{F}(a) = \hat{x} + a . \tag{12}
\]
We have the wave function \( \psi(\theta), \theta \in [-\pi, \pi] \) with the boundary condition \( \psi(\pi) = \psi(-\pi) \). This could be the wave function for a particle living on a circle of wire, with \( \theta \) being the angular coordinate around the circle. The boundary condition on \( \psi \) then is equivalent to saying that \( \psi \) is continuous and single-valued on the circle.

We have \( \hat{L} = \frac{\hbar}{i} \frac{d}{d\theta} \) (\( L \) is the angular momentum around the circle, but you don’t need to know that for this problem). Then

\[
\langle L \rangle = \int_{-\pi}^{\pi} d\theta \, \psi^* \hat{L} \psi = \frac{\hbar}{i} \int_{-\pi}^{\pi} d\theta \, \psi^* \frac{d}{d\theta} \psi .
\]

It follows then that

\[
\langle L \rangle^* = -\frac{\hbar}{i} \int_{-\pi}^{\pi} d\theta \, \psi \frac{d}{d\theta} \psi^* .
\]

Integrating by parts, we get

\[
\langle L \rangle^* = -\frac{\hbar}{i} |\psi(\theta)|^2 \bigg|_{-\pi}^{\pi} + \frac{\hbar}{i} \int_{-\pi}^{\pi} d\theta \left( \frac{d}{d\theta} \psi \right) \psi^* = \langle L \rangle ,
\]

since the first term vanishes, given that \( \psi(\pi) = \psi(-\pi) \). That is the same as saying that the boundary term in the integration by parts vanishes, since a circle has no boundary. Therefore \( \hat{L} \) has a real expectation value, and is a “hermitian” operator.