Physics 324, Fall 2001

First Midterm Exam

Monday 10/29/01

Problem 1. (7 points)
In this problem, consider a particle confined to the region $0 \leq x \leq L$ with the wavefunction

$$
\psi(x, t) = N \sin \left( \frac{2\pi x}{L} \right) e^{-iEt}, \quad E = \frac{2\hbar^2}{mL^2}.
$$

1 a. Find a value for $N$ that normalizes $\psi(x, t)$.

$$
1 = \int_0^L |\psi(x, t)|^2 dx = |N|^2 \int_0^L \sin^2 \left( \frac{2\pi x}{L} \right) dx
$$

$$
= |N|^2 \left( \frac{L}{\pi} \right)^2 \int_0^{\pi} \sin^2(2y) dy
$$

$$
= |N|^2 \left( \frac{L}{\pi} \right)^2 \frac{\pi^2}{2} = |N|^2 \left( \frac{L}{2} \right)
$$

where I substituted variables $y = \pi x / L$ and used the first integral given on the front page of the exam. So we need $N = \sqrt{2/L}$ (times an arbitrary phase, which we can take to be one).

\[ N = \sqrt{2/L} \]

1 b. Sketch the probability density $P(x, t)$ for finding the particle at point $x$. Label your axes. What is the probability density $P(L/2, t)$ for the particle to be at $x = L/2$ at time $t$?

$$
P(x, t) = |\psi(x, t)|^2 = |N|^2 \sin^2(2\pi x / L) = (2/L) \sin^2(2\pi x / L).
$$

Note that $P(x, t)$ is time independent, and that $P(L/2, t) = 0$.

Problem 1 continued on next page ⇒
1 c. Compute the expectation value for the position, $\langle x \rangle$.  (You can express your answer in terms of $N$, so that if you got part (a) wrong, you can still get this right.)

As should be evident from the sketch of $P(x)$, the probability density is symmetric about $x = L/2$, and so $\langle x \rangle = L/2$. In equations:

$$
\langle x \rangle = \int_0^L x \, P(x) \, dx
= |N|^2 \int_0^L x \sin^2(2\pi x/L) \, dx
= |N|^2 \left( \frac{L}{\pi} \right)^2 \int_0^\pi y \sin^2(2y) \, dy
= |N|^2 \left( \frac{L}{\pi} \right)^2 \frac{\pi^2}{4} = |N|^2 \left( \frac{L}{2} \right)^2 = \frac{L}{2},
$$

where I substituted variables $y = \pi x/L$ and used the second integral given on the front page of the exam.

$$
\langle x \rangle = |N|^2 \left( \frac{L}{2} \right)^2 = \frac{L}{2}
$$

1 d. Compute the expectation value of the square of the momentum, $\langle p^2 \rangle$ (again, you needn’t substitute in the value for $N$ that you found in (a)).

$$
\langle p^2 \rangle = \int_0^L \psi^* \left( -\frac{\hbar}{i} \frac{\partial}{\partial x} \right)^2 \psi \, dx
= -\hbar^2 |N|^2 \int_0^L \sin(2\pi x/L) \frac{d^2}{dx^2} \sin(2\pi x/L) \, dx
= \hbar^2 \left( \frac{2\pi}{L} \right)^2 |N|^2 \int_0^L \sin^2(2\pi x/L) \, dx
= \hbar^2 \left( \frac{2\pi}{L} \right)^2 .
$$

where I took advantage of the fact that the last integral was the same as the normalization integral in (a).

$$
\langle p^2 \rangle = \hbar^2 \left( \frac{2\pi}{L} \right)^2 |N|^2 \frac{L}{2} = \hbar^2 \left( \frac{2\pi}{L} \right)^2
$$

Problem 2 begins on next page ⇒
Problem 2. (7 points) Consider a particle of mass $m$ in one dimension, moving in a potential $V(x) = -k/x^{3/2}$, where $k > 0$.

2 a. Sketch $V(x)$ versus $x$.

2 b. Use the uncertainty principle in the form $\Delta x \Delta p \sim \hbar$ to estimate the ground state energy $E_0$ for the particle in terms of $k$, $m$ and $\hbar$.

We take $p \sim \hbar/x$, so that $E = \frac{p^2}{2m} - \frac{k}{|x|^{3/2}} \sim \frac{\hbar^2}{2mx^2} - \frac{k}{|x|^{3/2}}$. Minimizing with respect to $x$ we get

$$0 = \frac{dE}{dx} \sim -\frac{\hbar^2}{mx^3} + \frac{3k}{2|x|^{5/2}} \implies \sqrt{x} \sim \frac{2\hbar^2}{3mk}$$

(4)

Plugging this value of $x$ back into $E$ we get an estimate for the groundstate energy,

$$E_0 \sim \frac{\hbar^2}{2m} \left( \frac{3mk}{2\hbar^2} \right)^4 - k \left( \frac{3mk}{2\hbar^2} \right)^3 = \left( \frac{k^4m^3}{\hbar^6} \right) \left( \frac{3^4}{2^5} - \frac{3^3}{2^3} \right) = -\frac{27}{32} \left( \frac{k^4m^3}{\hbar^6} \right).$$

(5)

$$E_0 \sim -\frac{27}{32} \left( \frac{k^4m^3}{\hbar^6} \right)$$

2 c. What is the qualitative behavior of your estimate for $E_0$ if you were to make $\hbar$ smaller? How do you make sense of your answer in the $\hbar \to 0$ limit?

We find $E_0 \to -\infty$ as $\hbar \to 0$. The limit $\hbar \to 0$ is supposed to give you the classical result, and in classical mechanics there is no minimum energy for a particle in a potential that is unbounded below.
Problem 3. (6 points) Look at the following normalized (real) wavefunctions, plotted as $\psi(x)$ versus $x$, for a particle in a box $-1 \leq x \leq 1$:

![Graphs A, B, and C]

You should be able to answer the following questions without any calculations:

3 a. Which wavefunction (A, B, or C) will yield the largest value for $\langle x \rangle$? Briefly give your reason.

Note that both A and C wave functions are odd in $x$: $\psi_{A,C}(x) = -\psi_{A,C}(-x)$. Thus both $|\psi_{A,C}|^2$ are even in $x$, and $\langle x \rangle_{A,C}$ will vanish in both cases. In contrast, $\psi_B$ is asymmetric, with $|\psi_B(x)|^2$ weighted more heavily for $x > 0$, and so $\langle x \rangle_B > 0$.

![Graph B]

3 b. Which wavefunction (A, B, or C) will yield the largest value for $\langle x^2 \rangle$? Briefly give your reason.

Both $|\psi_{A,B}|^2$ have most of their weight around $x \sim \pm .5$, while $|\psi_C|^2$ has most of its weight at $x \sim \pm 1$. Thus $\psi_C$ will yield the higher value for $\langle x^2 \rangle$, since $(\pm 1)^2 > (\pm .5)^2$.

![Graph C]

3 c. Which wavefunction (A, B, or C) will correspond to the highest kinetic energy? Briefly give your reason.

Kinetic energy $\hat{p}^2/2m$ measures curvature as $\hat{p}^2/2m = -(\hbar^2/2m)d^2/dx^2$, so the wiggliest wavefunction (shortest wavelength) corresponds to the highest momentum, and hence the highest energy. $\psi_C$ wins the prize.

![Graph C]
Problem 4. (5 points) Which is shorter: the wavelength of a photon with energy $E = 1$ eV, or the de Broglie wavelength of an electron with kinetic energy $E = 1$ eV? Justify your answer with some equations.

For both electrons and photons we have $\lambda = h/p$, where $p$ is the momentum.

Photons: $p_\gamma = E_\gamma/c \implies \lambda_\gamma = h/c/E_\gamma$.

Electrons: $p_e = \sqrt{2m_eE_e} = (\sqrt{2m_e^2E_e})/c \implies \lambda_e = h/c/\sqrt{2m_e^2E_e}$.

Therefore, with $E_\gamma = E_e \equiv E = 1$ eV, it follows that

$$\frac{\lambda_e}{\lambda_\gamma} = \frac{E_\gamma}{\sqrt{2m_e^2E_e}} = \sqrt{\frac{E}{2m_e^2}} \approx \sqrt{\frac{1 \text{ eV}}{1 \times 10^6 \text{ eV}}} = 1 \times 10^{-3}.$$ \hspace{1cm} (6)

So the electron’s wavelength is much shorter than the photon’s, even though they have the same energy. That is why electron microscopes can resolve much smaller objects than optical microscopes.