Problem 2. (10 points)

Consider two orthonormal energy eigenstates |1⟩ and |2⟩, where \( \hat{H}|1⟩ = E_1|1⟩ \) and \( \hat{H}|2⟩ = E_2|2⟩ \), \( \hat{H} \) being the Hamiltonian and \( E_1 \neq E_2 \). Let \( |A⟩ \) and \( |B⟩ \) define two different linear combinations of the states |1⟩ and |2⟩:

\[
|A⟩ ≡ \frac{|1⟩ + i|2⟩}{\sqrt{2}} , \quad |B⟩ ≡ \frac{|1⟩ - i|2⟩}{\sqrt{2}} .
\]

2 a. Compute \( \langle A|A \rangle \), \( \langle B|B \rangle \), \( \langle A|B \rangle \) and \( \langle B|A \rangle \).

\[
\langle A|A \rangle = \langle B|B \rangle = 1 , \quad \langle A|B \rangle = \langle B|A \rangle = 0 .
\]

2 b. If initially at time \( t = 0 \) the particle is in the state \( |ψ, 0⟩ = |A⟩ \), what is the wavefunction \( |ψ, t⟩ \) at later times?

\[
|ψ, t⟩ = \frac{e^{-iE_1t/\hbar}|1⟩ + ie^{-iE_2t/\hbar}|2⟩}{\sqrt{2}} .
\]

2 c.

Suppose you have a measuring device which can tell you whether the particle is in the state \( |A⟩ \) or the state \( |B⟩ \). For the above initial condition \( |ψ, 0⟩ = |A⟩ \), what are the probabilities \( P_A(t) \) and \( P_B(t) \) that a measurement at time \( t > 0 \) will find the particle in state \( |A⟩ \) or in state \( |B⟩ \) respectively? Sketch \( P_A(t) \) and \( P_B(t) \) as functions of \( t \) on the same graph. In the sketch, identify the coordinates \{\( t, P(t) \}\) of maxima and minima of the two functions \( P_A(t) \) and \( P_B(t) \).

\[
P_A(t) = |\langle A|ψ, t⟩|^2 = \frac{1}{2} \left( e^{-iE_1t/\hbar} + e^{-iE_2t/\hbar} \right)^2
\]

\[
= \frac{1}{2} e^{-i(E_1+E_2)t/2\hbar} \left( e^{-i(E_1-E_2)t/2\hbar} + e^{i(E_1-E_2)t/2\hbar} \right)^2
\]
\[ \cos^2[(E_1 - E_2)t/2\hbar] . \] 

Similarly, 
\[ P_B(t) = |\langle B|\psi, t\rangle|^2 = \sin^2[(E_1 - E_2)t/2\hbar] . \]

Note that \( P_A + P_B = 1 \). A plot looks like:

where the horizontal axis is in terms of \( t' \equiv t(E_1 - E_2)/2\hbar \).

Problem 3. (10 points)

Consider particles of mass \( m \) moving in 1-dimension in the presence of a \( \delta \)-function potential \( V(x) = g\delta(x) \). The time independent Schrödinger equation reads

\[ \left[ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + g\delta(x) \right] u(x) = E u(x) . \]

A beam of monoenergetic particles is incident from the left (negative \( x \)) with wavenumber \( k = \sqrt{2mE/\hbar^2} \).

- Compute the transmission probability for these particles, as a function of \( k \). Circle your final answer.

- Does your answer depend on whether the potential is attractive (\( g < 0 \)) or repulsive (\( g > 0 \))?

- What happens to the transmission probability as \( k \to \infty \)?

Defining \( x < 0 \) to be region I and \( x > 0 \) to region II, we have

\[ u_I(x) = I e^{ikx} + Re^{-ikx} , \quad u_{II}(x) = Te^{ikx} , \quad k = \sqrt{2mE/\hbar^2} . \]

I have discarded a solution incoming from the right in region II. The boundary conditions are:

\[ u_I(0) = u_{II}(0) \implies (I + R) = T ; \]
or \( R = (T-I) \); for the derivatives, as discussed in class, the \( \delta \)-function potential gives rise to

\[
-\frac{\hbar^2}{2m} (u''_I(0) - u'_I(0)) + gu_I(0) = 0
\]

or:

\[
-\frac{\hbar^2}{2m} ik(T - I + R) + gT = 0 \implies T = I \left( \frac{ik\hbar^2}{ik\hbar^2 - mg} \right).
\]

Therefore the transmission probability \( P_T \) is

\[
P_T = \left| \frac{T}{I} \right|^2 = \frac{k^2\hbar^4}{k^2\hbar^4 + g^2m^2}.
\]

Evidently the transmission probability does not depend on the sign of \( g \), and \( P_T \to 1 \) as \( k \to \infty \).

**Problem 4.** For which particle does the wavefunction have the largest magnitude for the frequency \( \nu \)?

(a) A photon with momentum \( p_\gamma = 1 \text{ eV}/c \) (the photon mass is \( m_\gamma = 0 \)).

(b) An electron with momentum \( p_e = 1 \text{ eV}/c \) (the electron mass is \( m_e = 0.5 \text{ MeV}/c^2 \)).

(c) An proton with momentum \( p_p = 1 \text{ eV}/c \) (the proton mass is \( m_p = 938 \text{ MeV}/c^2 \)).

(d) A neutron with momentum \( p_n = 1 \text{ eV}/c \) (the neutron mass is \( m_n = 940\), \text{ MeV}/c^2).

(e) An electron in the ground state of the hydrogen atom.

The answer is (e). From Planck and de Broglie we know that \( E = h\nu \), so the larger the energy, the larger \( \nu \). We have \( E_\gamma = c p_\gamma = 1 \text{ eV} \). For the electron, proton, and neutron in (b-d), \( E = p^2/(2m) \ll 1 \text{ eV} \). The binding energy of the H atom electron is 13.6 \text{ eV} and wins the prize.

**Problem 5.** Which two quantities describing a particle’s motion in a central potential in 3-dimensions can not be simultaneously measured with arbitrary accuracy, even in principle:

(a) \( y \) position and the \( z \)-component of angular momentum.

(b) Total angular momentum and the \( x \)-component of angular momentum.

(c) \( x \) position and the \( x \)-component of angular momentum.
(d) Energy and total angular momentum.

(e) Energy and the $x$-component of angular momentum.

The answer is (a). Since $\vec{L} = \vec{r} \times \vec{p}$, $p_z = (xp_y - yp_x)$, and $y$ doesn’t commute with $p_y$. Compare with (c) where $L_x = (yp_z - zp_y)$, all terms of which commute with $x$.

**Problem 6.** Suppose I have a particle of mass $m$ in some quantum state $|\psi, t\rangle$ of a 1-dimensional harmonic oscillator with potential $V(x) = \frac{1}{2}kx^2$. The corresponding classical frequency $\omega = \sqrt{k/m}$. Which one equation below is **false**?

(a) $\frac{d}{dt} \langle \psi, t | \dot{x} | \psi, t \rangle = \langle \psi, t | \dot{p} | \psi, t \rangle / m$

(b) $\frac{d}{dt} \langle \psi, t | \dot{p} | \psi, t \rangle = -k \langle \psi, t | \dot{x} | \psi, t \rangle$

(c) $\langle \psi, t | \dot{x} | \psi, t \rangle = x_0 \cos(2\omega t)$, for some nonzero constant $x_0$

(d) $\langle \psi, t | \dot{x}^2 | \psi, t \rangle = x_0^2$, for some nonzero constant $x_0$

(e) $\langle \psi, t | \hat{H} | \psi, t \rangle = 2\hbar \omega$

The answer is (c). You showed that (a) and (b) were generally true in a problem set. Putting them together yields $d^2 \langle x \rangle / dt^2 = -\omega^2 \langle x \rangle$, the classical equation for the position...the general solution is $\langle x \rangle = x_0 \cos(\omega t + \delta)$, and a frequency of $2\omega$ is impossible. Also easy to see using raising and lowering operators. As for (d) and (e): we can have $\langle x^2 \rangle = x_0^2$ for some $x_0$ (for example, in the ground state), and the expectation value of the energy can equal $2\hbar \omega$ even if there is no energy eigenstate with that energy.

**Problem 7.** Suppose at $t = 0$ a particle is in the state

$$|\psi, 0\rangle = N \left( |1\rangle - i |2\rangle + 2 |3\rangle + \sqrt{3} |4\rangle \right)$$

where $|n\rangle$ are the orthonormal eigenstates of some Hamiltonian with $\hat{H} |n\rangle = E_n |n\rangle$. The number $N$ is chosen to normalize the state so that $\langle \psi, 0 | \psi, 0 \rangle = 1$. Which of the following statements is **false**?

(a) At later times $t$, $\langle \psi, t | \psi, t \rangle = 1$

(b) The normalization constant can be taken to be $N = 1/3$.

(c) The expectation value of the Hamiltonian, $\langle \psi, t | \hat{H} | \psi, t \rangle$, is time independent.
(d) The probability of measuring the energy to be \( E_3 \) is twice the probability of measuring the energy to be \( E_1 \).

(e) A measurement of the particle’s energy can only yield the values \( E_1, E_2, E_3 \) or \( E_4 \).

The answer is (d), since the probability for measuring \( E_3 \) is \( 2^2 = 4 \) times more likely than measuring \( E_1 \). All the other statements are true.

**Problem 8.** The wavefunction of a diatomic molecule is described by \( \psi(\theta, \phi) \), where \( \theta \) and \( \phi \) are the polar and azimuthal angles respectively describing the orientation of the axis connecting the two atoms. Suppose that

\[
\psi(\theta, \phi) = N \sin^2 \theta e^{i\phi} .
\]

Which of one of the following combinations of values for \( \ell \) and \( m \) **could** result from a measurement of \( L^2 \) and \( L_z \) for this state?

(a) \( \ell = 0 \) and \( m = 0 \)
(b) \( \ell = 1 \) and \( m = 0 \)
(c) \( \ell = 2 \) and \( m = 0 \)
(d) \( \ell = 1 \) and \( m = 1 \)
(e) \( \ell = 2 \) and \( m = 1 \)
(f) \( \ell = 2 \) and \( m = 2 \)
(g) None of the above

For this problem, here is a list of relevant spherical harmonics \( Y_{\ell,m}(\theta, \phi) = \langle \theta, \phi | \ell, m \rangle \):

\[
\begin{align*}
Y_{0,0} &= \sqrt{\frac{1}{4\pi}} \\
Y_{1,0} &= \sqrt{\frac{3}{4\pi}} \cos \theta \\
Y_{1,1} &= -\sqrt{\frac{3}{8\pi}} e^{i\phi} \sin \theta \\
Y_{2,0} &= \sqrt{\frac{5}{16\pi}} (3 \cos^2 \theta - 1) \\
Y_{2,1} &= -\sqrt{\frac{15}{8\pi}} e^{i\phi} \sin \theta \cos \theta \\
Y_{2,2} &= \sqrt{\frac{15}{32\pi}} e^{2i\phi} \sin^2 \theta 
\end{align*}
\]

The answer is (d). The probability for measuring a particular \( \ell, m \) pair is \( P_{\ell,m} = |\langle \ell, m | \psi \rangle|^2 \), where

\[
\langle \ell, m | \psi \rangle = \int_0^\pi d\theta \int_0^{2\pi} d\phi \sin \theta \langle \ell, m | \theta, \phi \rangle \langle \theta, \phi | \psi \rangle = \int_0^\pi d\theta \int_0^{2\pi} d\phi \sin \theta Y_{\ell,m}^*(\theta, \phi) \psi(\theta, \phi) .
\]
In this particular case, to avoid having the $\phi$ integral vanish, we must have $Y_{\ell,m} \propto e^{i\phi}$; that leaves only (d), (e) or (g) as possibilities. Then the integration over $\phi$ yields $2\pi$, and we must look at the integration of $\theta$. For $\ell = m = 1$,

$$P_{1,1} = -2\pi N \sqrt{\frac{3}{8\pi}} \int_0^\pi d\theta \sin^4 \theta$$

which is clearly nonzero, as the integrand is positive everywhere. On the other hand

$$P_{2,1} = -2\pi N \sqrt{\frac{15}{8\pi}} \int_0^\pi d\theta \sin^3 \theta \cos \theta$$

which vanishes as $\sin \theta$ is even and $\cos \theta$ is odd over the interval $\theta = [0, \pi]$.

End of exam...have a good vacation
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