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Distribution of particles in Fermi systems

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We show that the number of particles within a subvolume of a finite Fermi system has a variance that scales with the number of particles as $N^{2/3} \log N$. This contrasts with the naive classical model, which gives a scaling proportional to N. It also contrasts with a certain collective model that produces a variance that scales as $N^{2/3}$.

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It is possible to imagine experiments in which the number of particles in a certain subvolume of an extended Fermi system are measured. For example, in high energy heavy ion collisions, the projectile would shear off a large chunk of the target, and the number of nucleons in the remaining piece could, in principle, be measured. The question arises as to the distribution of nucleon number to be expected. Bondorf *et al.*[1] have considered this question and calculated a theoretical variance within a simple collective model.

A first guess for the variance might be the independent particle classical limit. Here the variance σ^2 for a measurement of the number of particles within a certain subvolume would satisfy

$$\sigma^2 \equiv \langle n^2 \rangle - \langle n \rangle^2 = Np(1-p)\,, \tag{1}$$

where N is the total number of particles being considered, and $p = \langle n \rangle / N$ is the probability that a particle is in the volume. In Ref. 1 the variance found is considerably smaller.

The variance can be computed quantum mechanically in terms of the two-body density. It appears to be a remarkable feature of Fermi systems that the Pauli principle reduces the fluctuations drastically, changing the dependence on $\langle n \rangle$ to

$$\sigma^2 \sim \langle n \rangle^{2/3} \log \langle n \rangle\,. \tag{2}$$

We shall derive this result for a Fermi gas wave function. In Ref. 1 an approximation is made which results in a dependence of the variance as $\langle n \rangle^{2/3}$. These authors also suggest that a more rapid dependence than $\langle n \rangle^{2/3}$ might be closer to reality.[2]

It is first necessary to write down the expression for the variance in a measurement of single-particle projection operator, Θ. For a determinantal wave function built up out of single-particle orbits j, k the variance is given by

$$\sigma^2 \equiv \langle \Theta^2 \rangle - \langle \Theta \rangle^2 = \sum_{jk} \langle j | \Theta | k \rangle \langle k | (1-\Theta) | j \rangle\,. \tag{3}$$

We apply this to a Fermi gas in a box of side L, using plane waves and periodic boundary conditions. The single-particle wave functions have the form

$$\phi_j(\vec{r}) = \frac{1}{L^{3/2}} e^{i\vec{k}_j \cdot \vec{r}}\,, \quad -L/2 \le r_{x,y,z} \le L/2\,. \tag{4}$$

The momentum vector k_j is given in terms of Cartesian quantum numbers n^j by

$$\vec{k}_j = \frac{2\pi}{L}(n_x^j, n_y^j, n_z^j)\,. \tag{5}$$

For the ground state of the Fermi gas, these quantum numbers satisfy

$$n_x{}^2 + n_y{}^2 + n_z{}^2 \le n_F{}^2\,. \tag{6}$$

The number of particles of a given kind N is approximately

$$N = \frac{4\pi}{3} n_F{}^3\,. \tag{7}$$

For our projection operator, we consider the number of nucleons on one side of the xy plane. The single-particle matrix elements are given by

$$\langle j | n(z>0) | k \rangle = P_{jk} \delta_{n_x^j, n_x^k} \delta_{n_y^j, n_y^k}$$

with

$$P_{jk} = \frac{1}{L} \int_0^{L/2} dz \exp[-(2\pi i/L)(n_z^j - n_z^k) z]$$
$$= \begin{cases} \frac{1}{2} & n_z^j = n_z^k \\ \frac{1}{(n_z^j - n_z^k)\pi i} & n_z^j - n_z^k \text{ odd} \\ 0 & n_z^j - n_z^k \text{ even}, \ne 0\,. \end{cases} \tag{8}$$

Inserting the explicit value of the diagonal matrix elements into Eq. (3), the variance becomes

$$\sigma^2 = \sum_j \langle j|\mathcal{O}|j\rangle - \sum_{jk} \langle j|\mathcal{O}|k\rangle^2$$
$$= \frac{N}{4} - \sum_{j\neq k} \langle j|\mathcal{O}|k\rangle^2 . \qquad (9)$$

The first term is just the classical result, Eq. (1). The effect of the Pauli principle is contained in the second term. We now evaluate both terms together, breaking up the sum over orbits into sums over the Cartesian quantum numbers,

$$\sigma^2 = \sum_{n_x n_y} \sum_{n_z}^{\pm N_z} \left(\frac{1}{4} - \sum_{\substack{n_z' \\ n_z - n_z' \text{ odd}}}^{\pm N_z} \frac{1}{\pi^2 (n_z - n_z')^2} \right). \qquad (10)$$

The limit N_z on the z sum is given by

$$N_z^2 = n_F^2 - n_x^2 - n_y^2 . \qquad (11)$$

The z sum is now expressed as

$$\sum_{n_z} \{ \} = \frac{2N_z + 1}{4} - \frac{2}{\pi^2} \left(\frac{2N_z}{1^2} + \frac{2N_z - 2}{3^2} + \cdots + \frac{2}{(2N_z - 1)^2} \right)$$
$$= \frac{2N_z + 1}{4} - \frac{2}{\pi^2} \left[(2N_z + 1) \left(1 + \frac{1}{3^2} + \cdots \frac{1}{(2N_z - 1)^2} \right) - \left(1 + \tfrac{1}{3} + \cdots + \frac{1}{2N_z - 1} \right) \right]. \qquad (12)$$

The series in square brackets are evaluated[3] to an accuracy of order unity as

$$(2N_z + 1) \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \right) = \frac{\pi^2}{4} N_z + O(1) ,$$
$$1 + \tfrac{1}{3} + \cdots + \frac{1}{2N_z - 1} = \tfrac{1}{2} \log N_z + O(1) . \qquad (13)$$

The sum then reduces to

$$\sum_{n_z} \{ \} = \frac{1}{\pi^2} \log N_z + O(1) . \qquad (14)$$

The remaining sums over the x and y quantum numbers may be approximated by an integral

$$\sum_{n_x, n_y} \simeq \pi \int_0^{n_F^2} d(n_x^2 + n_y^2) . \qquad (15)$$

Then the variance becomes

$$\sigma^2 \equiv \frac{1}{2\pi} \int_0^{n_F^2} dx \log(n_F^2 - x)$$
$$= \frac{1}{2\pi} n_F^2 \log n_F^2 + O(n_F^2)$$
$$\approx \frac{1}{3\pi} \left(\frac{3N}{4\pi} \right)^{2/3} \log\left(\frac{3N}{4\pi} \right). \qquad (16)$$

In the last line, we used Eq. (7) to obtain our final result.

If there were a representation of the Fermi gas in which the particles were all localized to some definite cell volume, then σ^2 would be proportional to the number of cells cut by the dividing plane, which in turn would be proportional to the area of cut. Thus this localized orbit picture suggests that the variance should depend on the size of the system according to

$$\sigma \sim \sqrt{\text{area}} \sim N^{1/3} . \qquad (17)$$

In fact, the variance increases slightly more rapidly than this. The localized picture of the Fermi gas is not completely accurate, but is closer to the truth than the completely delocalized independent particle picture.

In Ref. 1 the variance is studied via the relationship to excitations of the system:

$$\sigma^2 = \sum_{\substack{\text{excitations} \\ n}} \langle 0|\mathcal{O}|n\rangle\langle n|\mathcal{O}|0\rangle . \qquad (18)$$

This approach has the advantage that the matrix elements of the density operator between the ground state and collective states are rather well understood. The form and magnitude of the matrix element can be fixed with sum rules and experimental data. Another advantage of the technique is that it includes the effect of interactions, which are neglected in the Fermi gas treatment.

If the sum in Eq. (18) is replaced by a single term, as was done in Ref. 1, the resulting mass dependence of the variance is

$$\sigma \sim N^{1/3} .$$

The inclusion of all multipoles and of the modes with more radial structure will increase the result by a log N factor.

There is one more important lesson from this study. Since the Fermi system is much more uniform than classical physics suggests, the variance in mass emerging from low energy heavy ion collisions cannot be inferred from the particle transfer rates by a classical argument. The classical models only work well for calculating expectation values of one-body operators. This has been pointed out by Randrup.[4]

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[1]J. Bondorf, G. Fai, and O. Nielsen, Phys. Rev. Lett. 41, 391 (1978).
[2]J. Bondorf, G. Fai, and O. Nielsen, Niels Bohr Institute Report No. NBI-78-16 (unpublished).
[3]H. Dwight, *Tables of Integrals* (MacMillan, New York, 1961), p. 13.
[4]J. Randrup, Nucl. Phys. A307, 319 (1978).